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Invariant and Conditionally Invariant Solutions of Magnetohydrodynamic Equations in $(3+1)$ Dimensions

A.M. GRUNDLAND † *and P. PICARD* ‡

† *Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*
E-mail: *grundlan@crm.umontreal.ca*

‡ *Département de Physique, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*
E-mail: *picard@crm.umontreal.ca*

The symmetry reduction method is used to obtain invariant and conditionally invariant solutions of magnetohydrodynamic equations in (3+1) dimensions. Detail description of the procedure for constructing such solutions is presented. These solutions represent simple and double Riemann waves.

1 Introduction

We consider the magnetohydrodynamic (MHD) equations in (3+1) dimensions which can be written in the matrix evolutionary form

$$\mathrm{u}_t + A^1(\mathrm{u})\mathrm{u}_x + A^2(\mathrm{u})\mathrm{u}_y + A^3(\mathrm{u})\mathrm{u}_z = 0, \qquad \nabla\cdot\vec{H} = 0, \tag{1}$$

where the 8 by 8 matrix functions A^1, A^2 and A^3 have the form

$$A^1 = \begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & u & 0 & 0 & 0 & H_2/\rho & H_3/\rho \\ 0 & 0 & 0 & u & 0 & 0 & -H_1/\rho & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & H_1/\rho \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & H_2 & -H_1 & 0 & 0 & u & 0 \\ 0 & 0 & H_3 & 0 & -H_1 & 0 & 0 & u \end{pmatrix}, \tag{2}$$

$$A^2 = \begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & -H_2/\rho & 0 & 0 \\ 0 & 1/\rho & 0 & v & 0 & H_1/\rho & 0 & H_3/\rho \\ 0 & 0 & 0 & 0 & v & 0 & 0 & -H_2/\rho \\ 0 & 0 & -H_2 & H_1 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & H_3 & -H_2 & 0 & 0 & v \end{pmatrix},$$

and

$$A^3 = \begin{pmatrix} w & 0 & 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & w & 0 & 0 & \kappa p & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 \\ 0 & 1/\rho & 0 & 0 & w & H_1/\rho & H_2/\rho & 0 \\ 0 & 0 & -H_3 & 0 & H_1 & w & 0 & 0 \\ 0 & 0 & 0 & -H_3 & H_2 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix}. \tag{3}$$

Here ρ and p represent the density and the pressure of the fluid, respectively; κ is an adiabatic exponent; $\vec{\mathrm{v}} = (u, v, w)$ and $\vec{H} = (H_1, H_2, H_3)$ represent the velocity of the fluid and the magnetic field, respectively. This system describes an ideal nonstationary isentropic flow of a compressible conductive fluid (with negligible dissipative effect, and the electrical conductivity of the fluid is assumed to be infinitely large) placed in magnetic fluid $\vec{H}$. The MHD system of equations (1) is composed of nine equations involving eight unknown functions $\mathrm{u} = (\rho, p, \vec{\mathrm{v}}, \vec{H}) \in \mathbb{R}^8$.

The aim of this paper is to analyse two special cases of point symmetry of MHD equations (1) that describe the nonlinear propagation and superposition of waves. The methodological approach accepted here is based on the Lie group methods of infinitesimal transformations [1, 2]. For a comprehensive review of this subject see references [3–5]. Examples of invariant and conditionally invariant solutions of system (1) are presented, and some physical and mathematical consequences are discussed.

2 Conditionally invariant solutions and multiples waves

Consider a properly determined homogeneous quasilinear hyperbolic system of first order partial differential equations in p independent variable $\mathrm{x} = (x^1, \dots, x^p)$ and q dependent variables $\mathrm{u} = (u^1, \dots, u^q)$ of the form

$$A^i\,(\mathrm{u})\mathrm{u}_i = 0, \qquad i = 1, \dots, p, \tag{4}$$

where $A^1, \dots, A^p$ are $q \times q$ matrix functions of u. A vector λ is said to be a *wave vector* of the system (4) if there exists a non zero function $\lambda\,(\mathrm{u}) = (\lambda_1(\mathrm{u}), \dots, \lambda_p(\mathrm{u}))$ such that $\ker(\lambda_i A^i) \neq 0$. The function r of x and u is said to be a *Riemann invariant* associated to the wave vector λ if the following relation holds

$$r(\mathrm{x}, \mathrm{u}) = \lambda_i(\mathrm{u})x^i. \tag{5}$$

Now let us fix $1 \leq k \leq p$ linearly independent wave vectors $\lambda^1, \dots, \lambda^k$ with corresponding Riemann invariants

$$r^1(\mathrm{x}, \mathrm{u}) = \lambda^1_i(\mathrm{u})x^i, \dots, r^k(\mathrm{x}, \mathrm{u}) = \lambda^k_i(\mathrm{u})x^i. \tag{6}$$

The equation

$$\mathrm{u} = f\big(r^1(\mathrm{x}, \mathrm{u}), \dots, r^k(\mathrm{x}, \mathrm{u})\big) \tag{7}$$

defines a unique function u(x) on a neighborhood of $\mathrm{x} = 0$, for any function $f : \mathbb{R}^k \to \mathbb{R}^q$, and the Jacobian matrix is given by

$$\frac{\partial u^\alpha}{\partial x^i} = \big(\phi^{-1}(\mathrm{x})\big)^l_j \lambda^j_i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial f^\alpha}{\partial r^l}\big(r(\mathrm{x}, \mathrm{u})\big), \qquad l, j = 1, \dots, k,\ i = 1, \dots, p,\ \alpha = 1, \dots, q, \tag{8}$$

where the $k \times k$ matrix ϕ takes the form

$$\big(\phi\,(\mathrm{x})\big)^l_j = \delta^l_j - \frac{\partial r^l}{\partial u^\alpha}\,\frac{\partial f^\alpha}{\partial r^j}\,\big(r\,(\mathrm{x},\mathrm{u})\big). \tag{9}$$

Note that the rank of u(x) is at most equal to k. If the vector-valued function

$$\xi_a(\mathrm{u}) = \big(\xi^1_a(\mathrm{u}),\ldots,\xi^p_a(\mathrm{u})\big)^T, \qquad a = 1,\ldots,p-k \tag{10}$$

satisfies the orthogonality condition

$$\lambda^s_i \cdot \xi^i_a = 0, \qquad s = 1,\ldots,k, \quad a = 1,\ldots,p-k \tag{11}$$

then we get

$$\xi^i_a\big(u^\alpha(\mathrm{x})\big)\frac{\partial u^\alpha}{\partial x^i} = 0. \tag{12}$$

Therefore $u^1(\mathrm{x}),\ldots,u^q(\mathrm{x})$ are invariants of the vector field

$$\xi^i_a\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial}{\partial x^i}, \qquad a = 1,\ldots,p-k \tag{13}$$

on $\mathbb{R}^p$. Hence the graph of a solution Γ_f is invariant under the vector field

$$X_a = \xi^i_a(\mathrm{u})\frac{\partial}{\partial x^i}, \qquad a = 1,\ldots,p-k \tag{14}$$

on $\mathbb{R}^p \times \mathbb{R}^q$. Conversely, if u(x) is a q-component function defined on a neighborhood of $\mathrm{x} = 0$ such that the graph Γ_f is invariant under all vector fields (14) with the property (11), then u(x) is the solution of (7) for some function f, because $r^1,\ldots,r^k,u^1,\ldots,u^q$ constitute a complete set of invariants of the Abelian algebra of such vector fields (14). This geometrically characterizes the solutions of the equations of the form (7). If u(x) is the solution of (7) then

$$A^i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial \mathrm{u}}{\partial x^i}(\mathrm{x}) = \big(\phi^{-1}(\mathrm{x})\big)^l_j\lambda^j_i\big(\mathrm{u}(\mathrm{x})\big)A^i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial f}{\partial r^l}\big(r\big(\mathrm{x},\mathrm{u}(\mathrm{x})\big)\big) = 0. \tag{15}$$

Now we choose an appropriate coordinate system on space $\mathbb{R}^p \times \mathbb{R}^q$ in order to rectify the vector fields (14). We assume that the k by k matrix

$$\Lambda = \big(\lambda^i_j\big), \qquad 1 \le i,\, j \le k \tag{16}$$

is invertible. The independent vector fields

$$X_{k+1} = \frac{\partial}{\partial x^{k+1}} - \sum_{i,j=1}^{k}\,(\Lambda^{-1})^j_i\lambda^i_{k+1}\frac{\partial}{\partial x^j}, \quad \ldots, \quad X_p = \frac{\partial}{\partial x^p} - \sum_{i,j=1}^{k}\,(\Lambda^{-1})^j_i\lambda^i_p\frac{\partial}{\partial x^j} \tag{17}$$

have the required form (14) with the orthogonality property (7). If we change the independent and dependent variables as follows

$$\overline{x}^1 = r^1(\mathrm{x},\mathrm{u}),\ \ldots,\ \overline{x}^k = r^k(\mathrm{x},\mathrm{u}),\ \overline{x}^{k+1} = x^{k+1},\ \ldots,\ \overline{x}^p = r^p,\ \overline{u}^1 = u^1,\ \ldots,\ \overline{u}^q = u^q \tag{18}$$

then the vector fields (17) take rectified form

$$X_{k+1} = \frac{\partial}{\partial \overline{x}^{k+1}}, \quad \ldots, \quad X_p = \frac{\partial}{\partial \overline{x}^p}. \tag{19}$$

The graph of the solution Γ_f invariant under $X_{\kappa+1},\ldots,X_p$ is defined by equations of the form

$$\overline{\mathrm{u}}(\overline{\mathrm{x}}) = f\big(\overline{x}^1,\ldots,\overline{x}^k\big), \tag{20}$$

where $f:\mathbb{R}^k\to\mathbb{R}^q$ is an arbitrary function. This implies that expression (20) is the general solution of the invariance conditions

$$\overline{\mathrm{u}}_{\overline{k+1}} = 0, \quad \ldots, \quad \overline{\mathrm{u}}_{\overline{p}} = 0. \tag{21}$$

The system (4) is described in the coordinates $(\overline{\mathrm{x}},\overline{\mathrm{u}})$ by

$$\overline{A}^i\big(\overline{\mathrm{x}},\overline{\mathrm{u}},\overline{\mathrm{u}}_{\overline{\mathrm{x}}}\big)\overline{\mathrm{u}}_{\overline{i}} = 0, \tag{22}$$

where,

$$\overline{A}^1 = \frac{Dr^1}{Dx^i}A^i,\ \ldots,\ \overline{A}^k = \frac{Dr^k}{Dx^i}A^i, \quad \overline{A}^{k+1} = A^{k+1},\ \ldots,\ \overline{A}^p = A^p. \tag{23}$$

Now, we have

$$\frac{Dr^i}{Dx^i}\bigg|_{\overline{\mathrm{u}}_{\overline{k+1}},\ldots,\overline{\mathrm{u}}_{\overline{p}}=0} = \big(\Phi^{-1}\big)^i_j\lambda^j_i, \quad \text{where} \quad \big(\Phi^{-1}\big)^i_j = \delta^i_j - \frac{\partial r^i}{\partial\overline{u}^\alpha}\overline{u}^\alpha_j, \quad i,j=1,\ldots,k. \tag{24}$$

Adding the invariance conditions (21) to the system (22) we obtain the reduced system of the form

$$\sum_{i,j=1}^{k}\sum_{i=1}^{p}\big(\Phi^{-1}\big)^i_j\lambda^j_i A^i\overline{\mathrm{u}}_{\overline{i}} = 0, \qquad \overline{\mathrm{u}}_{\overline{k+1}} = 0,\ \ldots,\ \overline{\mathrm{u}}_{\overline{p}} = 0, \tag{25}$$

which must then be solved.

3 Double Alfvén-entropic wave

Using a version of the conditional symmetry method presented above we construct an example of a double Alfvén-entropic wave. According to the proposed method we consider the case where two wave vectors λ^1 and λ^2 corresponding to the Alfvén and entropic wave vectors are given by

$$\lambda^1 = \big(\delta_A - (\vec{\lambda}^1\cdot\vec{\mathrm{v}}),\vec{\lambda}^1\big), \quad \lambda^2 = \big(-(\vec{\lambda}^2\cdot\vec{\mathrm{v}}),\vec{\lambda}^2\big), \quad \delta_A = \varepsilon\frac{(\vec{H}\cdot\vec{\lambda}^1)}{\sqrt{\rho}}, \quad \varepsilon=\pm 1, \tag{26}$$

$$\vec{\lambda}^1 = (\cos\varphi,\sin\varphi,0), \quad \vec{\lambda}^2 = (\cos\theta,\sin\theta,0). \tag{27}$$

We consider additionally two vectors ξ_1, ξ_2 which are orthogonal to the given vectors λ^1 and λ^2. We look for the graph of the double wave solution which is invariant under these vector fields

$$X_1 = \sin(\theta-\varphi)\frac{\partial}{\partial t} - [\delta_A\sin\theta + u\sin(\varphi-\theta)]\frac{\partial}{\partial x} + [\delta_A\cos\theta + v\sin(\theta-\varphi)]\frac{\partial}{\partial y},$$

$$X_2 = \frac{\partial}{\partial z}. \tag{28}$$

Note that the vector fields X_1 and X_2 form an Abelian distribution on the space $\mathcal{M}\subset\mathbb{R}^4\times\mathbb{R}^8$, i.e. $[X_1,X_2]=0$. This requirement constitutes the necessary and sufficient conditions for the existence of a local curvilinear system of coordinates on the integral surface spanned by the fields

X_1 and X_2 in the space $\mathcal{M}$. Hence the unknown functions can be written in the parametric form $\mathrm{u} = f(r,s)$ where r and s are the Riemann invariants of the vector fields X_1 and X_2

$$r = \vec{\lambda}^2 \cdot \vec{\mathrm{x}} - (\vec{\lambda}^2 \cdot \vec{\mathrm{v}}), \qquad s = \vec{\lambda}^1 \cdot \vec{\mathrm{x}} + \big(\delta_A - (\vec{\lambda}^1 \cdot \vec{\mathrm{v}})\big)t. \tag{29}$$

Substituting the unknown functions in term of r and s into the MHD equations (1) we obtain the following solution

$$\begin{aligned} &\rho = \rho(r), \qquad p(r) + \frac{1}{2}\mathcal{H}^2(r) = p_0, \qquad \vec{\mathrm{v}} = \frac{\varepsilon \vec{H}}{\sqrt{\rho(r)}} + w(r)\vec{e}_3, \\ &\vec{H} = \mathcal{H}(r)[\cos\varphi(s)\vec{e}_1 + \sin\varphi(s)\vec{e}_2], \qquad p_0 \in \mathbb{R}, \qquad \varepsilon = \pm 1, \end{aligned} \tag{30}$$

where ρ, p, w, $\mathcal{H}$ and φ are arbitrary functions of their arguments

$$r = y\cos\varphi - x\sin\varphi, \qquad s = x\cos\varphi + y\sin\varphi, \tag{31}$$

and $\vec{e}_i$, $i = 1,2,3$ denote the constant unit vectors along x, y, z axes of $\mathbb{R}^3$. Note that the rank of the solution determined by equations (30) and (31) is equal two. The Alfvén-entropic double wave describes a stationary incompressible fluid flow. We have an equilibrium between hydrodynamic and magnetic pressures resulting from the entropic wave. We notice that the density ρ and the pressure p are unchanged by the Alfvén wave. They do however depend on the Riemann invariant r, which is related to the entropic wave.

4 Spherically invariant solution

We intend now to discuss the construction of nonstationary spherically symmetric solution of the MHD equations (1). For the purpose of this investigation we limit ourselves to the reduction obtained from the subalgebra

$$g_1 = \mathrm{span}\,\{J_1, J_2, J_3\}, \tag{32}$$

where the infinitesimal generators are of the form

$$\begin{aligned} J_1 &= z\partial_y - y\partial_z + w\partial_v - v\partial_w + H_3\partial_{H_2} - H_2\partial_{H_3}, \\ J_2 &= x\partial_z - z\partial_x + u\partial_w - w\partial_u + H_1\partial_{H_3} - H_3\partial_{H_1}, \\ J_3 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v + H_2\partial_{H_1} - H_1\partial_{H_2}. \end{aligned} \tag{33}$$

Note that for the subalgebra g_1 the transversality condition is not satisfied

$$\mathrm{rank}\,\{\Omega_1\} < \mathrm{rank}\,\{\Omega_2\}, \tag{34}$$

where matrices Ω_1 and Ω_2 are of the form

$$\Omega_1 = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & z & -y & 0 & w & -v & 0 & H_3 & -H_2 \\ -z & 0 & x & -w & 0 & u & -H_3 & 0 & H_1 \\ y & -x & 0 & v & -u & 0 & H_2 & -H_1 & 0 \end{pmatrix}, \tag{35}$$

and in principle the classical symmetry reduction method cannot be applied. However, there exists a certain domain $\mathcal{D}$ given by

$$u^j = x^j f(t,x,y,z), \qquad H_j = x^j g(t,x,y,z), \qquad j = 1,2,3 \tag{36}$$

for which the rank condition is fulfilled, i.e.,

$$\operatorname{rank}\{\Omega_1\}\big|_{\mathcal{D}} = \operatorname{rank}\{\Omega_2\}\big|_{\mathcal{D}}. \tag{37}$$

In this case the subgroup G_1 acts regularly and transversally and invariant solutions can be constructed [2]. The corresponding invariants of subalgebra g_1 are

$$\begin{aligned} &t, \quad s = \sqrt{x^2+y^2+z^2}, \quad \Phi^1 = \sqrt{u^2+v^2+w^2}, \quad \Phi^2 = xu+yv+zw, \\ &\Phi^3 = \sqrt{H_1^2+H_2^2+H_3^2}, \quad \Phi^4 = xH_1+yH_2+zH_3, \quad \Phi^5 = \rho, \quad \Phi^6 = p. \end{aligned} \tag{38}$$

Note that the condition

$$\operatorname{rank}\left(\frac{\partial \Phi^i}{\partial u^j}\right) = 7, \qquad i = 1,\dots,6, \quad j = 1,\dots,8 \tag{39}$$

implies that the dimension of the graph of the solution Γ_f is not preserved by the action of the symmetry group G_1. Therefore, we cannot obtain invariant solutions from this set of invariants by the classical symmetry reduction method. This is due to the fact that the defect structure δ of the solution with respect the the group G_1 is equal to one, because the rank of the characteristic matrix Q associated to the subalgebra g_1 is equal to one

$$\delta = \operatorname{rank}\left(\mathrm{Q}\left(\mathrm{x}, u^{(1)}\right)\right) = 1. \tag{40}$$

The graph of the solution sweeps out an orbit of dimension $\dim(G_1\Gamma_f) = p+\delta = 5$. The equations determining the orbit of the graph of the solution are given by

$$\begin{aligned} &\rho = R(t,s), \qquad p = A(t,s), \qquad v = \frac{(V-xu)-zw}{y}, \\ &w = \frac{z(V-xu)+\varepsilon y\sqrt{[U^2-u^2](y^2+z^2)-(xu-V)^2}}{y^2+z^2}, \qquad H_2 = \frac{(X-xH_1)-zH_3}{y^2+z^2}, \\ &H_3 = \frac{z(Y-xH_1)+\varepsilon y\sqrt{[X^2-H_1^2](y^2+z^2)-(xH_1-Y)^2}}{y^2+z^2}, \qquad \varepsilon = \pm 1, \end{aligned} \tag{41}$$

where u and H_1 are arbitrary functions of (t,x,y,z); $\Phi^2 = V(t,s)$, $\Phi^3 = X(t,s)$, $\Phi^4 = Y(t,s)$, $\Phi^5 = R(t,s)$ and $\Phi^6 = A(t,s)$ are arbitrary functions of their arguments. Assuming that conditions (36) and $p = A_0\rho^\kappa$ hold and substituting (41) into the MHD equations (1) we find the reduced system

$$\begin{aligned} &\frac{\partial R}{\partial s} + sf\frac{\partial R}{\partial s} + \left(3f + s\frac{\partial f}{\partial s}\right)R = 0, \\ &\frac{\partial f}{\partial t} + \left(f + s\frac{\partial f}{\partial s}\right)f + \kappa\frac{A_0}{s}R^{(\kappa-2)}\frac{\partial R}{\partial s} = 0, \\ &3g + s\frac{\partial g}{\partial s} = 0, \qquad A_0 \in \mathbb{R}. \end{aligned} \tag{42}$$

Using the method of separation of variables we can solve system (42). We obtain the following invariant solution

$$\begin{aligned} &\rho = \rho_o\left[\frac{s^2}{2\alpha}\frac{d^2\alpha}{dt^2} - \frac{s^2}{\alpha^2}\left(\frac{d\alpha}{dt}\right)^2 + C_1\alpha^{3(\kappa-1)}\right]^{1/(\kappa-1)}, \\ &p = A_o\rho^\kappa, \qquad u^j = -\frac{x^j}{\alpha}\frac{d\alpha}{dt}, \qquad H_j = \frac{\alpha}{s^3}x^j, \end{aligned} \tag{43}$$

where

$$\rho_0 = \left(\frac{\kappa - 1}{\kappa A_0}\right)^{1/(\kappa-1)}, \qquad \alpha(t) = C_2 \exp\left[\int \eta(t')\, dt'\right], \qquad C_1, C_2 \in \mathbb{R}, \quad j = 1, 2, 3. \tag{44}$$

The function η satisfies the ordinary differential equation

$$\frac{d^2\eta}{dt^2} - (1 + 3\kappa)\eta\frac{d\eta}{dt} + (3\kappa - 1)\eta^3 = 0 \tag{45}$$

which can be solved as follows. First, we introduce the change of variable

$$\zeta(\eta) = \frac{d\eta}{dt}, \tag{46}$$

so that the equation (45) can be rewritten as

$$\zeta\frac{d\zeta}{d\eta} - (1 + 3\kappa)\eta\zeta + (3\kappa - 1)\eta^3 = 0. \tag{47}$$

Using the second change of variable

$$\zeta(\eta) = \eta^2\chi(\tau), \qquad \text{where} \quad \tau = \ln\eta, \tag{48}$$

we solve (47) to finally obtain

$$\tau = -\int \frac{\chi d\chi}{2\chi^2 - (1 + 3\kappa)\chi + (3\kappa - 1)} + \tau_0, \qquad \tau_0 \in \mathbb{R}. \tag{49}$$

The solution (43) represents a nonstationary compressible irrotational fluid flow for which, according Kelvin's Theorem, the circulation around a fluid element is preserved. Note that $\vec{F}_m \cdot \vec{\mathrm{v}} = 0$ which indicates that the interaction effect between the hydrodynamic and magnetic expressions in the MHD equations can be decoupled.

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