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On N-Wave and NLS Type Systems: Generating Operators and the Gauge Group Action: the $so(5)$ Case

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We first review some recent developments [2] of the N-wave equations and their gauge equivalent ones related to the $so(5)$ algebra. These include the form of the equations and the relevant recursion operators, the properties of the scattering data etc. Next we generalize these results to the multicomponent nonlinear Schrödinger equations and their gauge equivalent ones known as the Heisenberg ferromagnet type equations. The explicit form of the $so(5)$ HF equations and the relevant recursion operators are derived.

1 Introduction

We start by briefly reviewing our previous results on N-wave equations and their gauge equivalent ones [1, 2] and extend them for the multicomponent NLS-type equation and their gauge equivalent.

Both classes of the above-mentioned nonlinear evolution equations (NLEE) are solvable by applying the inverse scattering method [3, 4] to the generalized Zakharov–Shabat system [5]:

$$L(\lambda)\psi \equiv \left(i\frac{d}{dx} + q(x,t) - \lambda J\right)\psi(x,t,\lambda) = 0, \tag{1}$$

related to the simple Lie algebra $\mathfrak{g}$ of rank $r > 1$. The gauge in (1) is fixed up by choosing J to be a constant element of the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$. Then the potential $q(x,t)$ can be cast in the form $q(x,t) = [J, Q(x,t)]$, where $Q(x,t)$ is a generic element of $\mathfrak{g}$. We also assume that $q(x,t)$ tends to zero fast enough for $x \to \pm\infty$.

For the N-wave type equations J is chosen to be a regular element of $\mathfrak{h}$; as a consequence the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J coincides with $\mathfrak{h}$. For the multi-component NLS (MNLS) type equations J as a rule is **not** a regular element of $\mathfrak{h}$ and the corresponding subalgebra $\mathfrak{g}_J$ is a non-commutative one. This makes more difficult the derivations of: i) the fundamental analytic solutions (FAS) of (1); ii) the construction of the related recursion operators and iii) the application of the gauge transformation, i.e. the transition to the pole gauge in which:

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dx} - \lambda\mathcal{S}(x,t)\right)\tilde{\psi}(x,t,\lambda) = 0, \tag{2}$$

$$\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda), \qquad \mathcal{S}(x,t) \equiv \mathrm{Ad}_g \cdot J = g^{-1}(x,t)Jg(x,t),$$

$$g(x,t) = \psi(x,t,\lambda=0), \qquad \text{i.e.,} \qquad \left(i\frac{d}{dx} + q(x,t)\right)g(x,t) = 0. \tag{3}$$

To be more specific let us write down the explicit form of the above mentioned NLEE along with their Lax representations and dispersion laws. The N-wave equations and their gauge equivalent read:

$$i[J,Q_t] - i[I,Q_x] + [[I,Q],[J,Q]] = 0, \tag{4}$$

$$\mathcal{S}_t - \frac{d}{dx} f(\mathcal{S}) = 0, \tag{5}$$

where $Q(x,t)$ is related to the potential $q(x,t)$ in (1) by $q(x,t) = [J, Q(x,t)]$. The constant regular element $I \in \mathfrak{h}$ can be expressed as a function (polynomial) of J: $I = f(J)$, see [2]. The corresponding M-operators of (4) and (5) equal:

$$M(\lambda)\psi \equiv \left(i\frac{d}{dt} + [I, Q(x,t)] - \lambda I \right) \psi(x,t,\lambda) = 0, \tag{6}$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dt} - \lambda f(\mathcal{S}) \right) \tilde{\psi}(x,t,\lambda) = 0, \tag{7}$$

and the dispersion law for both equations is linear in λ: $f_{N-w}(\lambda) = \lambda I$. An important point here is that I and J are both regular and *linearly independent;* otherwise the corresponding equations (4) and (5) become linear.

As we mentioned above the Lax operator for the MNLS equations formally has the form (1) but now J is no more regular element of $\mathfrak{h}$. This means that the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J (i.e., the kernel of the operator ad_J) is a non-commutative one. The dispersion law of the MNLS equation is quadratic in λ: $f_{\mathrm{MNLS}} = 2\lambda^2 J$. The MNLS equation, its gauge equivalent MHF equation and their M-operators have the form:

$$i\frac{dq}{dt} + 2\,\mathrm{ad}_J^{-1}\frac{d^2 q}{dx^2} + [q, \pi_0[q, \mathrm{ad}_J^{-1} q]] - 2i(\mathbb{1} - \pi_0)[q, \mathrm{ad}_J^{-1} q_x] = 0, \tag{8}$$

$$i\frac{d\mathcal{S}}{dt} + 2\frac{d}{dx}\left(\mathrm{ad}_{\mathcal{S}}^{-1} \frac{d\mathcal{S}}{dx} \right) = 0, \tag{9}$$

$$M(\lambda)\psi \equiv \left(i\frac{d}{dt} - V_0^{\mathrm{d}} + 2i\,\mathrm{ad}_J^{-1} q_x(x,t) + 2\lambda q(x,t) - 2\lambda^2 J \right) \psi(x,t,\lambda) = 0, \tag{10}$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dt} - 2i\lambda\,\mathrm{ad}_{\mathcal{S}}^{-1}\mathcal{S}_x - 2\lambda^2 \mathcal{S} \right) \tilde{\psi}(x,t,\lambda) = 0, \tag{11}$$

where $V_0^d = \pi_0 \left([q, \mathrm{ad}_J^{-1} q_x] \right)$ and π_0 is the projector onto $\mathfrak{g}_J$; (see also Section 3 below).

The interpretation of the inverse scattering method (ISM) as a generalized Fourier transform and the expansions over the so called "squared solutions" (see e.g. [5] for regular J and [6] for non-regular J) allows one to study all the fundamental properties of the relevant NLEE's. These include: (i) the description of the whole class NLEE related to the Lax operator $L(\lambda)$ (1) solvable by the ISM; (ii) derivation of the infinite family of integrals of motion and (iii) the Hamiltonian formulation of the NLEE's.

The "squared solutions" that appeared first in [7,8] were later generalized in [5,6] to Lax operators of the type (1); they can also be viewed as natural generalizations of the usual exponentials and are introduced by:

$$e_\alpha^{\pm}(x,t,\lambda) = (\mathbb{1} - \pi_0) \left(\chi^{\pm}(x,t,\lambda) E_\alpha \hat{\chi}^{\pm}(x,t,\lambda) \right), \tag{12}$$

where $\chi^{\pm}(x,t,\lambda)$ is the fundamental analytic solution of the Lax operator L (1). In fact their completeness relations [5,6] provide us the spectral decompositions of the operators:

$$\Lambda_+ e_{\pm\alpha}^{\pm} = \lambda e_{\pm\alpha}^{\pm}, \qquad \Lambda_- e_{\mp\alpha}^{\pm} = \lambda e_{\pm\alpha}^{\pm}, \qquad \alpha \in \Delta_+, \tag{13}$$

known as the generating (or recursion) operators; they play crucial role in deriving the properties of the NLEE. The recursion operator appeared first in the AKNS-approach [7] as a tool to generate the class of all M-operators as well as the NLEE related to the given Lax operator. Next I.M. Gel'fand and L.A. Dickii [9] discovered that the class of these M-operators is contained in the diagonal of the resolvent of L which could be explicitly defined in terms of the FAS $\chi^{\pm}(x,\lambda)$.

In the present paper we outline the construction of the recursion operators $\Lambda_{\pm}$ related to L (1) both for regular and non-regular elements J. In order to avoid a number of technicalities we consider only the Lax operators (1) related to the algebra $so(5)$. Our ideas, however, can be generalized to any simple Lie algebra thus allowing one to treat the corresponding class of NLEE. Another important point in our considerations is the explicit gauge covariant approach, which allows one to reformulate everything also for the pole gauge, thus extending the results in [10,6].

In Section 2 we briefly summarize our results on N-wave type equations related to $so(5)$ and their gauge equivalent, see [2,11]. In Section 3 we resolve the technical problems in constructing the recursion operators $\Lambda_{\pm}$ for the $so(5)$ MNLS type equations by using the spectral decomposition of the operator ad_J. As a result we obtain explicit expressions for the recursion operators $\widetilde{\Lambda}_{\pm}$, gauge equivalent to $\Lambda_{\pm}$. In Section 4 we briefly show how these recursion operators generate the hierarchies of Hamiltonian structures of the MNLS equations and their gauge equivalent ones.

2 N-wave type systems and their gauge-equivalent: $so(5)$-case

The $so(5)$ algebra has two simple roots α_1, α_2 and four positive roots $\Delta_+ = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ with $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, $\alpha_1+\alpha_2 = e_1$, $\alpha_1 + 2\alpha_2 = e_1 + e_2$. Here we use the following parametrization for J and $q(x)$

$$q(x,t) \equiv \sum_{\alpha\in\Delta_+} (q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & q_{10} & q_{11} & q_{12} & 0 \\ p_{10} & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & q_{10} \\ 0 & p_{12} & -p_{11} & p_{10} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_2 & 0 \\ 0 & 0 & 0 & 0 & -J_1 \end{pmatrix}, \tag{14}$$

where q_{jk} and p_{jk} are related to the roots $j\alpha_1 + k\alpha_2$ and $-j\alpha_1 - k\alpha_2$ respectively. Since in this section J is regular we can assume that $J_1 > J_2 > 0$; then the positive roots satisfy $\alpha(J) > 0$. The corresponding 4-wave system has the form [11]:

$$\begin{aligned} &i(J_1 - J_2) q_{10,t} - i(I_1 - I_2) q_{10,x} + 2\kappa q_{11} q_{01}^* = 0, \\ &iJ_2 q_{01,t} - iI_2 q_{01,x} + \kappa (q_{11}^* q_{12} + q_{11} q_{10}^*) = 0, \\ &iJ_1 q_{11,t} - iI_1 q_{11,x} + \kappa (q_{12} q_{01}^* - q_{10} q_{01}) = 0, \\ &i(J_1 + J_2) q_{12,t} - i(I_1 + I_2) q_{12,x} - 2\kappa q_{11} q_{01} = 0, \end{aligned} \tag{15}$$

where we put $p_{jk} = q_{jk}^*$, I_k are the matrix elements of $I = \mathrm{diag}\,(I_1, I_2, 0, -I_2, -I_1)$ and $\kappa = J_1 I_2 - J_2 I_1$. The system (15) can be written down as: $i\,\mathrm{ad}_J^{-1}\frac{dq}{dt} - \Lambda[I, \mathrm{ad}_J^{-1} q(x,t)] = 0$, where the recursion operators $\Lambda_{\pm}$ satisfying (13) and $\Lambda = (\Lambda_+ + \Lambda_-)/2$ are defined by:

$$\Lambda_{\pm} Z(x) = \mathrm{ad}_J^{-1}(\mathbb{1} - \pi_0)\left(i\frac{dZ}{dx} + [q(x), Z(x)] + i\left[q(x), \pi_0 \int_{\pm\infty}^{x} dy\, [q(y), Z(y)] \right] \right). \tag{16}$$

The equation (15) find an applications in nonlinear optics [3,11], and their gauge equivalent – in differential geometry [13]. The equations, gauge equivalent to (15) are:

$$\frac{d\mathcal{S}}{dt} - f_1 \frac{d\mathcal{S}}{dx} - f_3 \frac{d\mathcal{S}^3}{dx} = 0, \qquad f_1 = \frac{I_2 J_1^3 - I_1 J_2^3}{J_1 J_2 \left(J_1^2 - J_2^2\right)}, \qquad f_3 = \frac{I_1 J_2 - I_2 J_1}{J_1 J_2 \left(J_1^2 - J_2^2\right)}, \tag{17}$$

where the matrix $\mathcal{S}(x,t) \in so(5)$ is constrained by:

$$\mathrm{tr}\,\mathcal{S}^2 = 2\left(J_1^2 + J_2^2\right), \qquad \mathrm{tr}\,\mathcal{S}^4 = 2\left(J_1^4 + J_2^4\right), \qquad \mathcal{S}\left(\mathcal{S}^2 - J_1^2\right)\left(\mathcal{S}^2 - J_2^2\right) = 0. \tag{18}$$

We are using explicitly gauge covariant approach which allows one to reformulate all fundamental properties of the NLEE to be easily reformulated from one gauge to another. In particular, the gauge-equivalent "squared solutions" and recursion operators are given by:

$$\widetilde{e}_{\alpha}^{\pm}(x,t,\lambda)=g^{-1}(x,t)e_{\alpha}^{\pm}(x,t,\lambda)g(x,t),\qquad \widetilde{\Lambda}_{\pm}\widetilde{Z}=(g^{-1}(x,t)\Lambda g(x,t))\widetilde{Z}(x,t), \tag{19}$$

where

$$\widetilde{\Lambda}_{\pm}\widetilde{Z}=i\,\mathrm{ad}_{\mathcal{S}(x)}^{-1}(\mathbb{1}-\widetilde{\pi}_0(x))\left\{\frac{d\widetilde{Z}}{dx}+\sum_{k=1}^{2}[\widetilde{h}_k(x),\mathrm{ad}_{\mathcal{S}(x)}^{-1}]\int_{\pm\infty}^{x}dy\left\langle[\widetilde{h}_k(y),\mathrm{ad}_{\mathcal{S}(y)}^{-1}\mathcal{S}_y],\widetilde{Z}(y)\right\rangle\right\}, \tag{20}$$

where $\widetilde{h}_k(x,t)=g^{-1}(x,t)H_kg(x,t)$, $\langle H_k,H_j\rangle=\langle\widetilde{h}_k(x,t),\widetilde{h}_j(x,t)\rangle=\delta_{jk}$. The main difficulty in deriving the explicit form of the gauge equivalent equations is in the need to express all relevant quantities, such as $g^{-1}q(x,t)g(x,t)$, $g^{-1}Ig(x,t)$, etc. in terms of $\mathcal{S}$ only. The starting point is that both I and J have a common set of eigensubspaces and also the fact that if $J\in\mathfrak{h}$ then also $J^3\in\mathfrak{h}$. In what follows we express the Cartan generators $H_1=\mathrm{diag}\,(1,0,0,0,-1)$ and $H_2=\mathrm{diag}\,(0,1,0,-1,0)$ as polynomials of J, see [11],

$$h_1=\frac{J}{J_1}\frac{J^2-J_2^2}{J_1^2-J_2^2},\qquad h_2=\frac{J}{J_2}\frac{J^2-J_1^2}{J_2^2-J_1^2},\qquad g^{-1}(x,t)q(x,t)g(x,t)=-i\,\mathrm{ad}_{\mathcal{S}(x)}^{-1}\mathcal{S}_x, \tag{21}$$

$$\widetilde{h}_1(x,t)=\frac{\mathcal{S}}{J_1}\frac{\mathcal{S}^2-J_2^2}{J_1^2-J_2^2},\qquad \widetilde{h}_2(x,t)=\frac{\mathcal{S}}{J_2}\frac{\mathcal{S}^2-J_1^2}{J_2^2-J_1^2}. \tag{22}$$

Thus we have $I=I_1h_1+I_2h_2$. Applying the gauge transformation to this formula allows us to find the desired expression for $g^{-1}Ig(x,t)$ in terms of $\mathcal{S}$:

$$f(\mathcal{S})=I_1\widetilde{h}_1(x,t)+I_2\widetilde{h}_2(x,t)=f_1\mathcal{S}+f_3\mathcal{S}^3,$$

compare with equation (17). The set of constraints on $\mathcal{S}$ come from the fact that it has constant eigenvalues equal to $\pm J_1$, $\pm J_2$ and 0. These formulas together with the set of nonlinear constraints mentioned above completely determine the NLEE's gauge equivalent to the N-wave ones. Note that the number of the independent coefficients in $\mathcal{S}$ is equal to the number of the roots of the algebra, equal to 4 in the case of $so(5)$.

One can also derive explicitly the operator inverse to $\widetilde{\Lambda}$ generalizing the result in [12]:

$$\widetilde{\Lambda}_{\pm}^{-1}\widetilde{Z}=\frac{1}{i}(\mathbb{1}-\widetilde{\pi}_0)\int_{\pm\infty}^{x}dy\,[S(y),\widetilde{Z}(y)]. \tag{23}$$

The direct scattering problem for the Lax operator (1) is based on the Jost solutions and the scattering matrix $T(\lambda)$:

$$\lim_{x\to\infty}\psi(x,\lambda)e^{i\lambda Jx}=\mathbb{1},\qquad \lim_{x\to-\infty}\phi(x,\lambda)e^{i\lambda Jx}=\mathbb{1},\quad T(\lambda)=(\psi(x,\lambda))^{-1}\phi(x,\lambda). \tag{24}$$

The FAS $\chi^{\pm}(x,\lambda)$ of $L(\lambda)$ are related to the Jost solutions by [3,5,14]

$$\chi^{\pm}(x,\lambda)=\phi(x,\lambda)S^{\pm}(\lambda)=\psi(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda), \tag{25}$$

where $T^{\pm}(\lambda)$ and $S^{\pm}(\lambda)$, $D^{\pm}(\lambda)$ are elements of the group $SO(5)$ and are factors in the Gauss decomposition of the scattering matrix: $T(\lambda)=T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda)=T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda)$. Here $\hat{S}\equiv S^{-1}$, $S^{\pm}(\lambda,t)=\exp\left(\sum_{\alpha\in\Delta_+}s_{\alpha}^{\pm}(\lambda,t)E_{\pm\alpha}\right)$, and similar expressions for $T^{\pm}(\lambda,t)$. In particular $D^{\pm}(\lambda)=\exp(\pm d_1^{\pm}(\lambda)H_1\pm d_2^{\pm}(\lambda)H_2)$ are analytic functions of λ for $\mathrm{Im}\,\lambda>0$ and $\mathrm{Im}\,\lambda<0$ respectively.

On the real axis $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ are related by $\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda)$, $G_0(\lambda) = \hat{S}^-(\lambda)S^+(\lambda)$, and the function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (1) [3, 14, 5].

If the potential $q(x,t)$ of $L(\lambda)$ (1) satisfies the N-wave equation (15) then $f_{N-w}(\lambda) = \lambda I$ and

$$i\frac{dS^\pm}{dt} - [f_{N-w}(\lambda), S^\pm(\lambda,t)] = 0, \qquad i\frac{dT^\pm}{dt} - [f_{N-w}(\lambda), T^\pm(\lambda,t)] = 0, \qquad \frac{dD^\pm}{dt} = 0, \quad (26)$$

i.e., $S^\pm(\lambda)$ and $T^\pm(\lambda)$ satisfy linear ODE while the coefficients $d_j^\pm(\lambda)$ are generating functional of the local integrals of motion of the NLEE.

The scattering data for the gauge equivalent equations $\tilde{S}^\pm(\lambda)$, $\tilde{T}^\pm(\lambda)$ and $\tilde{D}^\pm(\lambda)$ are related to the scattering data of the "canonical" systems as follows:

$$\tilde{S}^\pm(\lambda) = T(0)S^\pm(\lambda)\hat{T}(0), \qquad \tilde{T}^\pm(\lambda) = T^\pm(\lambda), \qquad \tilde{D}^\pm(\lambda) = D^\pm(\lambda)\hat{T}(0). \quad (27)$$

$T(0)$ is an element of the Cartan subgroup of $SO(5)$. On the real λ axis again $\tilde{G}_0(\lambda) = \hat{\tilde{S}}^-(\lambda)\tilde{S}^+(\lambda)$ can be considered as a *minimal set of scattering data* for the gauge equivalent systems.

In order to obtain the soliton solutions for the gauge equivalent N-wave systems one needs to apply the Zakharov–Shabat dressing method to a regular FAS $\tilde{\chi}^\pm_{(0)}(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)}$. Thus one gets a new singular solution $\tilde{\chi}^\pm_{(1)}(x,\lambda)$ of the Riemann–Hilbert problem with singularities located at prescribed positions $\lambda_1^\pm$. It is related to the regular one by the dressing Zakharov–Shabat factors [3] $\tilde{u}(x,\lambda)$:

$$\tilde{\chi}^\pm_{(1)}(x,\lambda) = \tilde{u}(x,\lambda)\tilde{\chi}^\pm_{(0)}(x,\lambda)\tilde{u}_-^{-1}(\lambda), \qquad \tilde{u}_-(\lambda) = \lim_{x\to-\infty}\tilde{u}(x,\lambda), \quad (28)$$

and the dressing factors for the gauge equivalent equations $\tilde{u}(x,\lambda)$ are related to $u(x,\lambda)$ by

$$\tilde{u}(x,\lambda) = g_{(0)}^{-1}(x,t)u^{-1}(x,\lambda=0)u(x,\lambda)g_{(0)}. \quad (29)$$

Then the new solutions $\tilde{\chi}^\pm_{(1)}(x,\lambda)$ will correspond to a potential $\mathcal{S}_{(1)}$ of $\tilde{L}$ with two discrete eigenvalues $\lambda_1^\pm$. The explicit construction of the dressing factors is discussed in [11] for the "canonical" N-wave systems and in [2] for their gauge equivalent. Note only that getting the explicit form of the dressing factors one gets the 1-soliton solution as well. The n-soliton solutions could be obtained by multiple applying the dressing method (n-times).

3 Generating (recursion) operators for MNLS type models

The MNLS type system for has a dispersion law $f(\lambda) = 2\lambda^2 J$. Our choice for J is a degenerate one, because $\alpha_1(J) = 0$; the set of roots $\Delta_1^+ = \{\alpha_2, \alpha_3, \alpha_4\}$ of $so(5)$, for which $\alpha(J) \neq 0$ labels the coefficients of the potential $q(x,t)$:

$$q(x,t) \equiv \sum_{\alpha\in\Delta_1^+}(q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & 0 & q_{11} & q_{12} & 0 \\ 0 & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & 0 \\ 0 & p_{12} & -p_{11} & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & -a \end{pmatrix}, \quad (30)$$

where we assumed $p_{jk} = q_{jk}^*$. The corresponding MNLS type system is of the form:

$$i\frac{dq_{12}}{dt} + \frac{1}{2a}\frac{d^2q_{12}}{dx^2} - \frac{1}{a}q_{12}(|q_1|^2 + |q_{11}|^2 + |q_{12}|^2) + \frac{i}{a}q_1 q_{11,x} - \frac{i}{a}q_{11}q_{1,x} = 0,$$

$$
\begin{aligned}
&i\frac{dq_{11}}{dt}+\frac{1}{a}\frac{d^2q_{11}}{dx^2}-\frac{1}{a}q_{11}\left(|q_1|^2+|q_{11}|^2+\frac{1}{2}|q_{12}|^2\right)+\frac{i}{a}q_{12}q^*_{1,x}+\frac{i}{2a}q_{12,x}q^*_1=0, \qquad (31)\\
&i\frac{dq_1}{dt}+\frac{1}{a}\frac{d^2q_1}{dx^2}-\frac{1}{a}q_1\left(|q_1|^2+|q_{11}|^2+\frac{1}{2}|q_{12}|^2\right)-\frac{i}{a}q_{12}q^*_{11,x}-\frac{i}{2a}q_{12,x}q^*_{11}=0.
\end{aligned}
$$

Our choice of J means that $K=\mathrm{ad}_J$ has five different eigenvalues: $-2a$, $-a$, 0, a and $2a$. Then the minimal characteristic polynomial for K is:

$$
K\left(K^2-a^2\right)\left(K^2-4a^2\right)=0. \qquad (32)
$$

Let us also introduce the projectors onto the eigensubspaces of K as follows:

$$
\begin{aligned}
&\pi_{\pm 2}=\frac{K\left(K^2-a^2\right)(K\pm 2a)}{24a^4}, \qquad \pi_{\pm 1}=-\frac{K\left(K^2-4a^2\right)(K\pm a)}{6a^4},\\
&\pi_0=\frac{\left(K^2-a^2\right)\left(K^2-4a^2\right)}{4a^4}.
\end{aligned}
$$

Using the characteristic equation (32) it is easy to check that π_j are orthogonal projectors; i.e. they satisfy: $\pi_j\pi_k=\delta_{jk}\pi_j$ for all $j,k=\pm 2$, ± 1, 0 and that $K\pi_{\pm 2}=\pm 2a\pi_{\pm 2}$, $K\pi_{\pm 1}=\pm a\pi_{\pm 1}$, $K\pi_0=0$. Thus any function $f(K)$ can be expressed in terms of these projectors: $f(K)=f(2a)\pi_2+f(a)\pi_1+f(0)\pi_0+f(-a)\pi_{-1}f(-2a)\pi_{-2}$ provided $f(\lambda)$ is regular for $\lambda=\pm 2a$, $\pm a$ and 0.

The phase space $\mathcal{M}_J$ of the MNLS equations is the co-adjoint orbit of the $so(5)$ determined by J; its elements are matrices $q(x,t)$ satisfying $\pi_0 q(x,t)=0$. Note also that $\mathrm{ad}_J=K$ introduces a grading on $\mathfrak{g}=\bigoplus_{j=-2}^{2}\mathfrak{g}_j$ and the projectors π_j project precisely onto $\mathfrak{g}_j$. Obviously $\mathfrak{g}_j=\pi_j\mathfrak{g}$, $\mathfrak{g}_0\equiv\mathfrak{g}_J$ and $\mathcal{M}_J\simeq\mathfrak{g}\backslash\mathfrak{g}_J$. The "squared solutions" again have the form (12), but now the corresponding FAS and the projectors π_0 are different due to the choice of J:

$$
\chi^{\pm}(x,t,\lambda)=\phi(x,t,\lambda)S_J^{\pm}(t,\lambda)=\psi(x,t,\lambda)T_J^{\mp}(t,\lambda)D_J^{\pm}(\lambda). \qquad (33)
$$

Here $S_J^{\pm}$, $T_J^{\pm}$ and $D_J^{\pm}$ are the factors in the generalized Gauss decompositions of $T(t,\lambda)$ [6]:

$$
T(t,\lambda)=T_J^{-}(t,\lambda)D_J^{+}(\lambda)\hat{S}_J^{+}(t,\lambda)=T_J^{+}(t,\lambda)D_J^{-}(\lambda)\hat{S}_J^{-}(t,\lambda), \qquad (34)
$$

$$
S_J^{\pm}(t,\lambda)=\exp\left(\sum_{\alpha\in\Delta_1^+}s_{J,\alpha}^{\pm}(t,\lambda)E_{\pm\alpha}\right), \qquad T_J^{\pm}(t,\lambda)=\exp\left(\sum_{\alpha\in\Delta_1^+}t_{J,\alpha}^{\pm}(t,\lambda)E_{\pm\alpha}\right), \qquad (35)
$$

$$
D_J^{\pm}(\lambda)=\exp\left(\pm d_1^{\pm}(\lambda)H_1\pm 2d_2^{\pm}(\lambda)H_2+d_{\alpha_1}^{\pm}(\lambda)E_{\alpha_1}+d_{-\alpha_1}^{\pm}(\lambda)E_{-\alpha_1}\right). \qquad (36)
$$

If $q(x,t)$ evolves according to the MNLS (3) then

$$
i\frac{dS_J^{\pm}}{dt}-2\lambda^2[J,S_J^{\pm}(t,\lambda)]=0, \qquad i\frac{dT_J^{\pm}}{dt}-2\lambda^2[J,T_J^{\pm}(t,\lambda)]=0, \qquad \frac{dD_J^{\pm}}{dt}=0. \qquad (37)
$$

This means that the MNLS equation (3) has four series of integrals of motion as compared to the two series of integrals for the $so(5)$ 4-wave system. This is due to the special (degenerate) choice of the dispersion law $f_{\mathrm{MNLS}}=2\lambda^2 J$. We have to remember, however, that only two of these four series are in involution, which in turn is related to the non-commutativity of the subalgebra $\mathfrak{g}_J$. As a result we get the following expression for the recursion operator:

$$
\Lambda_{\pm}Z=K^{-1}(\mathbb{1}-\pi_0)\left(i\frac{dZ}{dx}+[q(x),Z(x)]+i\left[q(x),\int_{\pm\infty}^{x}dy\,\pi_0[q(y),Z(y)]\right]\right), \qquad (38)
$$

where we assume that $Z\in\mathcal{M}_J$, i.e. $\pi_0 Z(x)=0$.

In order to evaluate the gauge equivalent recursion operator we again make use of the gauge covariant approach. First it allows one easily to recalculate the projectors on the eigensubspaces of $\mathrm{ad}_{S(x)} \equiv \widetilde{K}(x) = g_0^{-1} K g_0(x,t)$:

$$\widetilde{\pi}_{\pm 2} = \frac{\widetilde{K}(\widetilde{K}^2 - a^2)(\widetilde{K} \pm 2a)}{24a^4}, \qquad \widetilde{\pi}_{\pm 1} = -\frac{\widetilde{K}(\widetilde{K}^2 - 4a^2)(\widetilde{K} \pm a)}{6a^4},$$
$$\widetilde{\pi}_0 = \frac{(\widetilde{K}^2 - a^2)(\widetilde{K}^2 - 4a^2)}{4a^4}.$$

In particular, these formulae allow us to cast the MHF system (9) in the form:

$$iS_t - \frac{5}{4a^2}[S, S_{xx}] + \frac{1}{4a^4}\left((\mathrm{ad}_S)^3 S_x\right)_x = 0, \tag{39}$$

where $\mathcal{S}$ is constrained by $\mathcal{S}\left(\mathcal{S}^2 - a^2\right)^2 = 0$. In addition the operator $\widetilde{K}(x,t)$ satisfy the equation (32). One way to derive the explicit form of the recursion operators $\widetilde{\Lambda}_\pm$ is to apply the gauge transformation to $\Lambda_\pm$ (38). For degenerate choice of J this is more difficult due to the fact that $\mathfrak{g}_J$ is non-Abelian. Here we only provide the explicit form of the operators inverse to $\widetilde{\Lambda}_\pm$

$$\tilde{\Lambda}_\pm^{-1}\tilde{Z} = \frac{1}{i}(\mathbb{1} - \widetilde{\pi}_0)\int_{\pm\infty}^{x} dy\, [S(y), \widetilde{Z}(y)], \qquad \widetilde{\pi}_0 \widetilde{Z}(x) = 0. \tag{40}$$

The scattering data for the gauge equivalent MHF equations $\tilde{S}_J^\pm(\lambda)$, $\tilde{T}_J^\pm(\lambda)$ and $\tilde{D}_J^\pm(\lambda)$ are related to the scattering data of the "canonical" systems as follows: $\tilde{S}_J^\pm(\lambda) = T_J(0) S_J^\pm(\lambda)\hat{T}_J(0)$, $\tilde{T}_J^\pm(\lambda) = T_J^\pm(\lambda)$, and $\tilde{D}_J^\pm(\lambda) = D_J^\pm(\lambda)\hat{T}_J(0)$ and $\tilde{T}_J(0)$ is an element of the subgroup $\mathcal{G}_J$ of $SO(5)$. On the real λ axis again $\tilde{G}_{J,0}(\lambda) = \hat{\tilde{S}}_J^-(\lambda)\tilde{S}_J^+(\lambda)$ can be considered as a *minimal set of scattering data* for the gauge equivalent systems.

In order to obtain the soliton solutions for the gauge equivalent MHF systems one needs to apply the Zakharov–Shabat dressing method to a regular FAS $\tilde{\chi}_{(0)}^\pm(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)}$. Thus one gets a new singular solution $\tilde{\chi}_{(1)}^\pm(x,\lambda)$ of the Riemann–Hilbert problem with singularities located at prescribed positions $\lambda_1^\pm$. It is related to the regular one by the dressing factors $\tilde{u}(x,\lambda)$ whose structure is analogous to the ones presented in Section 2 above.

4 Hierarchy of Hamiltonian structures for MNLS type models

It is well known that both classes of NLEE possess hierarchies of Hamiltonian structures. The phase spaces $\mathcal{M}_J$ and $\widetilde{\mathcal{M}}_{\mathcal{S}}$ corresponding to the standard and the pole gauge

$$\mathcal{M}_J \equiv \{q(x,t),\ \pi_0 q(x,t) = 0\}, \qquad \widetilde{\mathcal{M}}_{\mathcal{S}} \equiv \{S(x,t),\ S(x,t) = g^{-1} J g(x,t)\}, \tag{41}$$

where in addition we assume that $q(x,t)$ and $S(x,t) - J$ are smooth functions tending to zero fast enough for $|x| \to \infty$.

The hierarchies of symplectic structures defined on $\mathcal{M}_J$ and $\widetilde{\mathcal{M}}_{\mathcal{S}}$ are generated by the corresponding recursion operators and are given by the following families of compatible two-forms:

$$\Omega_{\mathrm{q}}^{(k)} = i\int_{-\infty}^{\infty} dx \left\langle \delta q \mathop{\wedge}_{,} \Lambda^k \mathrm{ad}_J^{-1} \delta q(x,t)\right\rangle, \qquad \tilde{\Omega}_{\mathcal{S}}^{(k)} = i\int_{-\infty}^{\infty} dx \left\langle \delta\mathcal{S} \mathop{\wedge}_{,} \tilde{\Lambda}^k \mathrm{ad}_{\mathcal{S}}^{-1} \delta\mathcal{S}(x,t)\right\rangle.$$

Note also that the gauge transformation relates nontrivialy the symplectic structures, i.e. $\Omega_{\mathrm{q}}^{(k)} \simeq \tilde{\Omega}_{\mathcal{S}}^{(k+2)}$ (for NLS–HFE equivalence, see [16, 10]). These two hierarchies are dynamically equivalent.

5 Discussion

Here we derived new results about the construction of the recursion operators for MNLS models and their gauge equivalent MHF systems by using the gauge covariant approach. These results can be naturally extended in several directions. First, one can apply the reduction group method [17] in order to investigate the $\mathbb{Z}_2$ and other finite order reduction of the MNLS systems thus extending the results of [11, 1] for the N-wave systems. Second, it is natural to extend these reductions also to the gauge equivalent systems (5) and (9). Such results may display additional properties of the class isoparametric hypersurfaces [13] described by the equations (17) in the framework of the $so(5)$ algebra. A third open problem is to study reductions of the gauge equivalent systems and the spectral decompositions for the relevant recursion operators.

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