Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 774–779

Infinitesimal Affinely-Rigid Bodies in Riemann Spaces

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In general Riemannian spaces, as rule, there are no isometries, except the identity transformation. For that reason one cannot consider extended rigid bodies in the usual sense. But one can consider approximately rigid motion of small bodies. More rigorously, they are material points with attached orthonormal bases. These bases describe internal degrees of freedom and provide the symbolic description of relative motion degrees of freedom after the limit transition, when the body diameter tend to zero. Similarly, there is no extended affinely-rigid body concept in a general Riemann space. Affine degrees of freedom may be considered merely as internal ones, represented by a general linear basis attached at the material point.

1 Introduction

We shall consider some models of an infinitesimal affinely- or metrically-rigid body. Because we cannot describe extended bodies, we use material points with attached linear bases. We have the following generalized coordinates:

1. The spatial coordinates x^i in a manifold M, $\dim M = n$.
2. The attached basis components $e^i{}_A$, where $A = 1, \ldots, n$ are co-moving indices (referring to the material spaces), and $i = 1, \ldots, n$ are spatial indices. In the metrically-rigid case e_A are orthonormal: $g_{ij} e^i{}_A e^j{}_B = \delta_{AB}$.

E_A denotes some field of frames fixed on M once for all. In affinely-rigid case this field is not indispensable, but description based on it is more convenient and "natural". At any time instant t, the vectors $e_A(t) \in T_{x(t)} M$ of the co-moving frame are given by $e_A(t) = E_B(x(t)) U^B{}_A(t)$. In the metrically-rigid case, for any t, the matrix U is orthogonal; in the affine case it is general. The choice of E strictly depends on geometry of M. In this article we discuss two cases. First is when our infinitesimal affinelly-rigid bodies will move on the sphere. The second case is pseudo-spherical, explicitly:

$$\text{in the first case}: \quad E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R \sin\left(\frac{r}{R}\right)} \frac{\partial}{\partial \varphi}, \quad E^r_{(\varphi)} = E^{\varphi}_{(r)} = 0,$$

$$\text{in the second case}: \quad E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R \operatorname{sh}\left(\frac{r}{R}\right)} \frac{\partial}{\partial \varphi}, \quad E^r_{(\varphi)} = E^{\varphi}_{(r)} = 0.$$

Affine velocity in the co-moving representation is given by:

$$\frac{De^i{}_B}{Dt} := e^i{}_A \Omega^A{}_B. \tag{1}$$

With the help of the standard Kronecker delta metric we can trivially shift the co-moving indices, e.g. $\Omega_{AB} := \delta_{AC} \Omega^C{}_B$. The above quantity is skew-symmetric, i.e. $\Omega_{AB} = -\Omega_{BA}$, when the motion is metrically-rigid (gyroscopic). The spatial components of affine velocity are given by: $\Omega^i{}_j = e^i{}_A \Omega^A{}_B e^B{}_j$. In the metrically-rigid motion we can shift indices with the help of g_{ij}: $\Omega_{ij} = g_{ik} \Omega^k{}_j = -\Omega_{ji}$.

2 Hamiltonian description

The previously described models of kinetic energy [2, 5–8] were invariant under the g-isometry groups acting in tangent spaces and under the orthogonal group acting in the micromaterial space $\mathbb{R}^n$. However, in spite of the affine symmetry of degrees of freedom they were not affinely invariant. It may be interesting, at least for "academic" purpose, to investigate the affinely invariant models of kinetic energy. They have the Casimir–Killing forms:

$$T_{\mathrm{int}} = \frac{A}{2}\mathrm{Tr}\left[\Omega^2\right] + \frac{B}{2}\left(\mathrm{Tr}\left[\Omega\right]\right)^2, \tag{2}$$

where A, B are constants. The full kinetic energy is postulated in the form: $T = T_{\mathrm{tr}} + T_{\mathrm{int}}$, where T_{tr} denotes translational and T_{int} the internal part of kinetic energy, explicitly:

1. on sphere: $T_{\mathrm{tr}} = \dfrac{m}{2}\left(\left(\dfrac{dr}{dt}\right)^2 + R^2\sin^2\left(\dfrac{r}{R}\right)\left(\dfrac{d\varphi}{dt}\right)^2\right)$, where $r \in [0, \pi R)$,
2. on pseudo-sphere: $T_{\mathrm{tr}} = \dfrac{m}{2}\left(\left(\dfrac{dr}{dt}\right)^2 + R^2\,\mathrm{sh}^2\left(\dfrac{r}{R}\right)\left(\dfrac{d\varphi}{dt}\right)^2\right)$, where $r \in [0, \infty]$.

Let us write $U^B{}_A$ in the two-polar decomposition $\left(U^B{}_A = R^B{}_C D^C{}_K \left(V^{-1}\right)^K{}_A\right)$, explicitly:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad V = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}, \qquad D = \begin{bmatrix} e^{\lambda-\mu} & 0 \\ 0 & e^{\lambda+\mu} \end{bmatrix}.$$

The co-moving affine velocity Ω, defined in (1), is given by: $\Omega^A{}_B = \left(U^{-1}\right)^A{}_F \Gamma^F{}_{DC} U^D{}_B U^C{}_G v^G + \left(U^{-1}\right)^A{}_C \dot{U}^C{}_B$. After some calculations we can obtain Ω in the form:

$$\begin{bmatrix} -\chi\sin(2\beta)\,\mathrm{sh}(2\mu) + \dot\lambda + \cos(2\beta)\dot\mu & -\chi(\mathrm{ch}(2\mu) - \cos(2\beta)\,\mathrm{sh}(2\mu)) + \dot\beta + \sin(2\beta)\dot\mu \\ \chi(\mathrm{ch}(2\mu) - \cos(2\beta)\,\mathrm{sh}(2\mu)) - \dot\beta + \sin(2\beta)\dot\mu & -\chi\sin(2\beta)\,\mathrm{sh}(2\mu) + \dot\lambda - \cos(2\beta)\dot\mu \end{bmatrix},$$

and the internal kinetic energy has the form:

$$T_{\mathrm{int}} = -a\chi^2 + 2a\,\mathrm{ch}(2\mu)\chi\dot\beta - a\dot\beta^2 + (a+2b)\dot\lambda^2 + a\dot\mu^2,$$

where χ are given by:

1. on the sphere: $\chi = \dot\alpha + \cos\left(\dfrac{r}{R}\right)\dot\varphi$,
2. on the pseudo-sphere: $\chi = \dot\alpha + \mathrm{ch}\left(\dfrac{r}{R}\right)\dot\varphi$.

3 Action-angle description

3.1 Action variables

Let us write the kinetic energy in the concise form: $T = \frac{m}{2}G_{ij}\dot q^i \dot q^j$, where $\left\{q^i\right\} = \{r, \varphi, \alpha, \beta, \lambda, \mu\}$. Now we can introduce the canonical formalism: $H = T + V$, where $T = \frac{1}{2m}G^{ij}p_i p_j$. The matrix G^{jk} is reciprocal to G_{ij}, i.e. $G_{ij}G^{jk} = \delta_i{}^k$.

In the both discussed cases the stationary Hamilton–Jacobi equation

$$H\left(q^a, \frac{\partial S}{\partial q^a}\right) = \frac{1}{2m}G^{ij}\frac{\partial S}{\partial q^i}\frac{\partial S}{\partial q^j} + V(q) = E$$

is separable for potentials of the form: $V(q) = V_r(r) + V_\lambda(\lambda) + V_\mu(\mu)$. Then the action variables may be explicitly calculated, at least in principle (in quadrature sense), and the degeneracy problem may be analyzed [3]. The kinetic term in the Hamiltonian has the form:

1. sphere:

$$T = \frac{p_r^2}{2m} + \frac{\left(p_\varphi - \cos\left(\frac{r}{R}\right) p_\alpha\right)^2}{2mR^2 \sin^2\left(\frac{r}{R}\right)} + \frac{p_\alpha^2 + 2\operatorname{ch}(2\mu)\, p_\alpha p_\beta + p_\beta^2}{4A \operatorname{sh}(2\mu)} + \frac{p_\lambda^2}{4(A+2B)} + \frac{p_\mu^2}{4A},$$

2. pseudo-sphere:

$$T = \frac{p_r^2}{2m} + \frac{\left(p_\varphi - \operatorname{ch}\left(\frac{r}{R}\right) p_\alpha\right)^2}{2mR^2 \operatorname{sh}^2\left(\frac{r}{R}\right)} + \frac{p_\alpha^2 + 2\operatorname{ch}(2\mu)\, p_\alpha p_\beta + p_\beta^2}{4A \operatorname{sh}(2\mu)} + \frac{p_\lambda^2}{4(A+2B)} + \frac{p_\mu^2}{4A}.$$

The time-independent Hamilton–Jacobi equation has the form:

$$H\left(q, \frac{\partial S_0}{\partial q}\right) = E,$$

where

$$\begin{aligned} S_0 &= S_r(r) + S_\varphi(\varphi) + S_\alpha(\alpha) + S_\beta(\beta) + S_\lambda(\lambda) + S_\mu(\mu) \\ &= S_r(r) + l\varphi + C_\alpha\alpha + C_\beta\beta + S_\lambda(\lambda) + S_\mu(\mu). \end{aligned}$$

We can separate these equations if φ, α and β are cyclic variables.

Remark 1. The form of kinetic term in the Hamiltonian suggested us that perhaps λ should be cyclic too. But $\det[D] = \exp(2\lambda)$, so the size could freely expand to infinity or contract to the point (although in the infinite time). Of course, if at some initial moment the dilatations velocity $\dot{\lambda}$ had the vanishing value, λ would be constant; however, such a solution is exponentially nonstable. Because of this, for physical reasons one should use some model of $V_\lambda(\lambda)$ stabilizing the size.

In the spherical case the Hamilton–Jacobi equation has the form:

$$\begin{aligned} E = {} & \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{\left(l - C_\alpha \cos\left(\frac{r}{R}\right)\right)^2}{2mR^2 \sin^2\left(\frac{r}{R}\right)} + V(r) + \frac{C_\alpha^2 + 2\operatorname{ch}(2\mu)\, C_\alpha C_\beta + C_\beta^2}{4A \operatorname{sh}(2\mu)} \\ & + \frac{1}{4(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial \lambda}\right)^2 + \frac{1}{4A}\left(\frac{\partial S_\mu(\mu)}{\partial \mu}\right)^2 + V_\mu(\mu) + V_\lambda(\lambda), \end{aligned} \tag{3}$$

and in the pseudo-spherical case:

$$\begin{aligned} E = {} & \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{\left(l - C_\alpha \operatorname{ch}\left(\frac{r}{R}\right)\right)^2}{2mR^2 \operatorname{sh}^2\left(\frac{r}{R}\right)} + V(r) + \frac{C_\alpha^2 + 2\operatorname{ch}(2\mu)\, C_\alpha C_\beta + C_\beta^2}{4A \operatorname{sh}(2\mu)} \\ & + \frac{1}{4(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial \lambda}\right)^2 + \frac{1}{4A}\left(\frac{\partial S_\mu(\mu)}{\partial \mu}\right)^2 + V_\mu(\mu) + V_\lambda(\lambda). \end{aligned} \tag{4}$$

The previous equations may be separated and some explicit calculations are possible. Our action variables have the form:

$$\begin{aligned} J_\varphi &= \oint p_\varphi d\varphi = \oint \frac{\partial S_\varphi(\varphi)}{\partial \varphi} d\varphi = \int_0^{2\pi} l d\varphi = 2\pi l \Rightarrow l = \frac{J_\varphi}{2\pi}, \\ J_\alpha &= \oint p_\alpha d\alpha = \oint \frac{\partial S_\alpha(\alpha)}{\partial \alpha} d\alpha = C_\alpha \int_0^{2\pi} d\alpha = 2\pi C_\alpha \Rightarrow C_\alpha = \frac{J_\alpha}{2\pi}, \\ J_\beta &= \oint p_\beta d\beta = \oint \frac{\partial S_\beta(\beta)}{\partial \beta} d\beta = C_\beta \int_0^{2\pi} d\beta = 2\pi C_\beta \Rightarrow C_\beta = \frac{J_\beta}{2\pi}. \end{aligned} \tag{5}$$

When we have J_φ, J_α and J_β, we can explicitly calculate J_r, J_λ and J_μ. Let us split the energy equation into the part depending only on r and the one depending only on λ, μ:

$$K = \frac{J_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{16\pi^2 A\,\mathrm{sh}^2\,(2\mu)} + N(\lambda) + \frac{1}{4A}\left(\frac{\partial S_\mu\,(\mu)}{\partial\mu}\right)^2 + V_\mu(\mu), \tag{6}$$

where

$$N(\lambda) = \frac{1}{(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial\lambda}\right)^2 + V_\lambda(\lambda), \tag{7}$$

$$\text{on sphere:} \qquad E = \frac{1}{2m}\left(\frac{\partial S_r\,(r)}{\partial r}\right)^2 + \frac{\left(l - J_\alpha\cos\left(\frac{r}{R}\right)\right)^2}{8\pi^2 m R^2 \sin^2\left(\frac{r}{R}\right)} + V(r) + K, \tag{8}$$

$$\text{and on pseudo-sphere:} \qquad E = \frac{1}{2m}\left(\frac{\partial S_r\,(r)}{\partial r}\right)^2 + \frac{\left(l - J_\alpha\,\mathrm{ch}\left(\frac{r}{R}\right)\right)^2}{8\pi^2 m R^2\,\mathrm{sh}^2\left(\frac{r}{R}\right)} + V(r) + K. \tag{9}$$

Action variables have the following form:

$$J_\mu = \oint p_\mu d\mu = \oint\sqrt{4A\left(K - N(\lambda) - V_\mu(\mu)\right) - \frac{J_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{4\pi^2\,\mathrm{sh}^2\,(2\mu)}}d\mu,$$
$$J_\lambda = \oint p_\lambda d\lambda = \oint\sqrt{4(A+2B)\left(N(\lambda) - V_\lambda(\lambda)\right)}d\lambda,$$

for both cases;

$$\text{on sphere:} \qquad J_r = \oint p_r dr = \oint\sqrt{2m\left(E - K - V_r(r)\right) - \frac{\left(J_\varphi - J_\alpha\cos\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2\sin^2\left(\frac{r}{R}\right)}}dr,$$

$$\text{on pseudo-sphere:} \qquad J_r = \oint p_r dr = \oint\sqrt{2m\left(E - K - V_r(r)\right) - \frac{\left(J_\varphi - J_\alpha\,\mathrm{ch}\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2\,\mathrm{sh}^2\left(\frac{r}{R}\right)}}dr.$$

3.2 Some convenient models of potentials

In elastic models of internal degrees of freedom the potentials should have a local minimum at the non-deformed configuration (usually, although not necessarily, identified with $\lambda = 0$, $\mu = 0$). For example, $V_\lambda(\lambda) = \frac{\kappa}{2}\lambda^2$. In non-elastic models this is no necessary. But the potentials should prevent the body to expand to infinity or contract to a point. Thus, e.g., $V_\lambda(\lambda)$ must tend to plus infinity, when λ tends to minus infinity. Similarly, it should tend to plus infinity, or to some limit higher than its infimum, when λ tends to plus infinity. In some non-elastic problems, for certain reason connected with the theory of integrable lattices, something similar to the inverse-square rule may be convenient, e.g.

$$V_\lambda(\lambda) = \frac{\kappa}{2}\lambda^2, \quad V_\mu(\mu) = G\,\mathrm{cth}^2(2\mu), \quad \text{where} \quad \kappa,\ G \text{ are constants.} \tag{10}$$

On $\mathrm{SL}\,(2,\mathbb{R})$ the geodetic model is interesting, when V_μ vanishes at all.

Let us take into account potentials $V(\mu)$ depending only on μ like, e.g. (10). For this potential J_μ is given by:

$$J_\mu = \frac{1}{4}\left(\sqrt{\left(J_\alpha + J_\beta\right)^2 + 16AG\pi^2} + \sqrt{\left(J_\alpha - J_\beta\right)^2 + 16AG\pi^2}\right) - 2\pi\sqrt{A(G - K + N(\lambda))}.$$

Now let us consider a potential V depending only on λ. For all reasons presented in the Remark 1 we propose $V(\lambda)$ like, e.g., in (10) and for that potential J_λ is given by:

$$J_\lambda = 2\pi N(\lambda)\sqrt{\frac{2(A+2B)}{\kappa}}.$$

It is interesting to consider classical and quantum problems without V_μ at all. Then the sign of constant of separation K depends on the effective potential [4], which is given by (in a consequence of (7) and without $V(\mu)$, but not without $V(\lambda)$):

$$V_{\text{eff}}(\mu,\lambda,J_\alpha,J_\beta) = \frac{J_\alpha^2 + 2\,\text{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{4\pi^2\,\text{sh}^2\,(2\mu)} + 4AN(\lambda).$$

The orbits are bounded, when $K \leqslant 0$, an this is just the case we consider. Then J_μ is given by:

$$J_\mu = \frac{1}{2}J_\alpha - 2\pi\sqrt{A(N(\lambda)-K)} \qquad \text{and} \qquad N(\lambda) = \frac{J_\lambda}{2\pi}\sqrt{\frac{\kappa}{2(A+2B)}}.$$

Now let us consider some potential $V(r)$. At the beginning we consider the spherical case. We suggest the following model:

$$V(r) = \eta\,\text{ctg}^2\left(\frac{r}{R}\right),$$

where η is constant, and for that potential J_r is given by:

$$J_r = \frac{1}{2}\left(\sqrt{(J_\varphi - J_\alpha)^2 + 8m\pi^2\eta R^2} - \sqrt{(J_\varphi + J_\alpha)^2 + 8m\pi^2\eta R^2}\right) - \sqrt{J_\alpha^2 + 8m\pi^2R^2(E+\eta-K)}.$$

In the pseudo-spherical case we take potential in the form:

$$V(r) = \frac{\gamma}{R^2}\,\text{cth}^2\left(\frac{r}{R}\right),$$

where γ is constant. For this potential J_r is given by:

$$J_r = \frac{1}{2}\left(\sqrt{(J_\varphi - J_\alpha)^2 + 8m\gamma\pi^2} - \sqrt{(J_\varphi + J_\alpha)^2 + 8m\gamma\pi^2}\right) - \sqrt{J_\alpha^2 + 8m\pi^2R(RE+\gamma-RK)}.$$

Then for the spherical and pseudo-spherical cases, substituting this to the J_r formula and solving it with respect to the energy parameter E we find in principle the explicit dependence of Hamiltonian on the action variables. As the resulting expression is rather complicated and obscure, we do not quote explicit formulae. The J_r's without potentials look more interesting and have more lucid structure. Description of the dependence of H on J's is more clear:

$$J_r = J_\alpha - 2\pi\sqrt{2mR^2(E-K) + \frac{J_\alpha^2}{4\pi^2}}.$$

3.3 Bohr–Sommerfeld quantization

In 1915 W. Wilson and in 1916 A. Sommerfeld discovered the rules of quantization known today as the old quantum theory. Now we call them the Bohr–Sommerfeld rules of quantization. We can use these rules when our Hamiltonians are cyclic in some variables. Then we have:

$$J_a = \oint p_a da = 2\pi n_a \hbar.$$

Then E_n without potentials for both cases is given by:

$$E_n = \frac{1}{2mR^2} n_r(n_r + 2n_\alpha)\hbar^2 + \frac{1}{4A}(n_\mu - 2n_\alpha)^2\hbar^2 + \sqrt{\frac{\kappa}{4(A+2B)}} n_\lambda \hbar.$$

We can do the same for the cases with potentials, but the formulae we obtain are very complicated.

Remark 2. Affinely-rigid body is the simplest generalization of metrically- rigid body with nontrivial deformable degrees of freedom. The model of internal degrees of freedom, which we present in this paper, is thought on as a preliminary step towards relativistic theory and mechanics of continua with microstructure. One can realize some physical applications like the motion of continental plates or oil pollution on oceans. It is very interesting to consider some special cases of incompressible affinely-rigid body, like fat spots on a water surface. It is just a two-dimensional analogue of three-dimensional incompressible objects, like fluid droplet (e.g. a “droplet” of nuclear matter).

Acknowledgements

I would like to express my gratitude to professor J.J. Sławianowski for his introducing me into the topic and for encouragement to prepare this report and to all organizers of the Fifth International Conference “Symmetry in Nonlinear Mathematical Physics” for warm hospitality. This work was performed within the framework of the grants 5T07A04824 and 8T07A04720.

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