Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 110–117

Group Foliation of Euler Equations in Nonstationary Rotationally Symmetrical Case

Sergey V. GOLOVIN

Lavrentyev Institute of Hydrodynamics, 15 Lavrentyev Ave., 630090 Novosibirsk, Russia
E-mail: *sergey@hydro.nsc.ru*

Euler equations for rotationally symmetrical motions of ideal fluid are considered. Basis of differential invariants for infinite-dimensional part of admitted group is calculated. The basis is used for construction of group foliation of Euler equations. Both automorphic and resolving systems are completed to involution. The resolving part of group foliation inherits finite-dimensional part of group, admitted by Euler equations. It allows us to construct invariant and partially invariant solutions of resolving system. The original functions are restored then by means of integration of automorphic system. Example of such construction is provided.

1 Introduction

The present paper can be regarded as development of the author's previous work published in [1]. Objects of investigations are differential invariants of infinite-dimensional groups, which appear as admissible groups of continuous transformations for various hydrodynamical systems. The main goal is utilization of infinite-dimensional part of admitted groups for construction of new exact solutions and obtaining new information about the observed systems. Here we use an approach based on so-called group foliation (or group stratification) of the system of PDEs with respect to admitted group. This approach proved to be useful in several examples [2, 4–7]. Here we construct a group foliation of Euler equations describing rotationally symmetrical motions of ideal fluid. The base of group foliation is an infinite-dimensional group, which involves two arbitrary functions of time. In comparison with cited articles we observe not a single equation but a system of equations with 4 unknown functions. It causes the main complexity of present work.

Description of classical theory of differential invariants one can find in the books [2, 3]. All the necessary information concerned automorphic system and algorithm for construction of group foliation are presented there also. The problem of group foliation construction of given system of equations was first set by Sophus Lie [8]. The group G, admitted by some system of equations E transforms any solution U: $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ into solution again. Thus, in the set of all solutions of system E the equivalence relation appears: two solutions U and U' are equivalent if they are connected by the group G transformation: $U' = TU$, $T \in G$. Each equivalence class is the orbit of one of the solutions under all possible transformations of group G. The problem of group foliation is formulated as follows: for a given system of equations E and a given group G, being admitted by system E, it is required to form a system of equations, which would describe an orbit of any solution (system AG) and a system, which would give an assemblage of all orbit of different solutions (system RE). System AG is named as *automorphic* and has a property that any its solution belongs to the orbit of one solution, i.e. any solution obtained from any other by the action of group G. On the contrary, the *resolving* system RE does not admit the original group and, thus, distinguishes the orbits of different solutions.

In present work we use Ovsiannikov's algorithm [2] for construction of group foliation. This algorithm sufficiently uses the basis of differential invariants for investigated group.

2 Preliminary information

We consider the Euler equations, which describe rotationally symmetrical motions of ideal fluid. Rotational symmetry means that in the cylindrical coordinate system (r, θ, z) all functions are assumed to be independent of the polar angle θ. Thus the system under consideration is the following:

$$\begin{aligned} &u_t + uu_r + wu_z + p_r = \frac{v^2}{r}, \\ &v_t + uv_r + wv_z = -\frac{uv}{r}, \\ &w_t + uw_r + ww_z + p_z = 0, \\ &u_r + \frac{1}{r}u + w_z = 0. \end{aligned} \tag{1}$$

Here u, v and w are velocity components which correspond to (r, θ, z) axes; p is a pressure. The admitted group for system (1) is known [9]. Its finite-dimensional part L_4 is generated by the following operators

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = -\frac{1}{r^2 v}\partial_v + \frac{1}{r^2}\partial_p, \qquad X_3 = t\partial_t + r\partial_r + z\partial_z, \\ &X_4 = 2t\partial_t + r\partial_r + z\partial_z - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p. \end{aligned} \tag{2}$$

Operators of infinite-dimensional part L_∞ of admitted group depend on two arbitrary functions of time:

$$Z_f = f(t)\partial_z + \dot{f}(t)\partial_w - \ddot{f}(t)z\partial_p, \qquad H_\varphi = \varphi(t)\partial_p. \tag{3}$$

Further we construct the group foliation of the system (1) with respect to an infinite-dimensional group L_∞ (3).

3 The representation of the solution

The first step is the calculation of the basis of differential invariants and operators of invariant differentiation of the group L_∞.

Lemma 1. *A basis of differential invariants of the group L_∞ can be chosen as follows:*

$$t, \quad r, \quad u, \quad v, \quad W_0 = w_t + ww_z + p_z, \quad W_1 = w_r, \quad W_2 = w_z, \quad P = p_r. \tag{4}$$

Operators of invariant differentiation are

$$\delta_0 = D_t + wD_z, \qquad \delta_1 = D_r, \qquad \delta_2 = D_z. \tag{5}$$

The commutative relations between δ_i are the following:

$$[\delta_0, \delta_1] = -W_1\delta_2, \qquad [\delta_0, \delta_2] = -W_2\delta_2, \qquad [\delta_1, \delta_2] = 0. \tag{6}$$

Proof. The proof is the same as in [1]. ■

As it follows from general theory the Euler equations (1) can be rewritten as a relation between differential invariants only. Since action of operators of invariant differentiation on differential invariant provides differential invariants of higher order we have the following system

$$\begin{aligned} &\delta_0 u + u\delta_1 u + P = \frac{v^2}{r}, \qquad \delta_0 v + u\delta_1 v = -\frac{uv}{r}, \\ &W_0 + uW_1 = 0, \qquad \delta_1 u + \frac{u}{r} + W_2 = 0, \end{aligned} \tag{7}$$

which is equivalent to the original equations (1). The system (7) provides the first part of relations between differential invariants.

According to the general algorithm [2] we have to choose 3 (the same number as the number of original independent variables) of the differential invariants (4) to be the new independent variables. The remaining 5 invariants from the basis further will be regarded as functions of the 3 chosen independent variables.

It is convenient to choose invariants t, r, u as the new independent variables. This choice is valid for all solutions where $u_z \neq 0$. The case $u_z = 0$ will be investigated separately. Due to the third equation of (7) the representation of solution for the group foliation has the following form:

$$w_t + ww_z + p_z = -uW_1(t,r,u), \qquad w_r = W_1(t,r,u), \qquad w_z = W_2(t,r,u),$$
$$p_r = P(t,r,u), \qquad v = V(t,r,u). \tag{8}$$

For given functions V, W_1, W_2 and P equations (7) and (8) provide an overdetermined system of nonlinear PDEs. It is a part of automorphic system AS of group foliation. Compatibility conditions of system (8) which are equations only for invariant functions V, W_1, W_2 and P give a resolving system RS of the group foliation. For convenience we use further the terminology of partially differential equations. Namely, we call functions V, W_1, W_2, P 'invariant' and function u a 'superfluous'.

4 Compatibility conditions

Here we investigate compatibility conditions of the overdetermined system (7), (8). Note that first and fourth equations (7) give the expression for derivatives of the superfluous function u:

$$\delta_0 u = \frac{u^2+V^2}{r} + uW_2 - P,$$
$$u_r = -\frac{u}{r} - W_2. \tag{9}$$

Substituting a representation of v from (8) into the second equation (7) by virtue of (9) we obtain an equation only for invariant functions

$$V_t + uV_r + \left(\frac{1}{r}V^2 - P\right)V_u = -\frac{u}{r}V. \tag{10}$$

Cross-differentiating of the second and third equations of (8) allows us to eliminate function w providing

$$W_{1u}u_z = W_{2r} - W_{2u}\left(\frac{u}{r} + W_2\right). \tag{11}$$

Then we differentiate the first equation of (8) with respect to r and subtract $\delta_0(w_r)$ and $(p_r)_z$, obtained from the second and fourth equations (8). Using equations (9) and commutative relations (6) we find

$$P_u u_z = -W_{1t} + \frac{u}{r}W_1 - uW_{1r} - W_{1u}\left(\frac{1}{r}V^2 - P\right). \tag{12}$$

Elimination of the derivative u_z from relations (11), (12) provides an additional equation for invariant functions only

$$W_{2r}P_u - W_{2u}\left(\frac{u}{r} + W_2\right)P_u + W_{1t}W_{1u} - \frac{u}{r}W_1W_{1u}$$
$$+ W_{1u}^2\left(\frac{1}{r}V^2 - P\right) + uW_{1r}W_{1u} = 0. \tag{13}$$

Next we have to derive a compatibility conditions for the superfluous function u. Hereafter we suppose that $W_{1u} \neq 0$. Case $W_{1u} = 0$ by virtue of (8) corresponds to separation of variables $w = f(t, r) + g(t, z)$. This case is not considered here. Note only that analysis of the particular case $w = f(t, r)$ serves as a topic for paper [10].

According to the above propositions all derivatives of the superfluous function u are determined from relations (9) and (11). Its cross-differentiation taking into consideration commutative relations (6) gives three equations, which involve first-order derivatives of u. Elimination of these derivatives gives three more equations for invariant functions only. Due to a big amount of calculation we present only the result:

$$
\begin{aligned}
W_1 W_{1u}^{-1}\left(W_{2r} - \left(\frac{u}{r} + W_2\right) W_{2u}\right) &= W_{2t} + uW_{2r} + \frac{2}{r}VV_r - P_r - \frac{1}{r}P \\
&+ \left(\frac{1}{r}V^2 - P\right) W_{2u} - \frac{2u}{r}W_2 + W_2 P_u - \frac{2}{r}W_2 V V_u - \frac{2u}{r^2}VV_u + \frac{u}{r}P_u - W_2^2 - \frac{2u^2}{r^2},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&-W_{2tr} + \left(\frac{u}{r} + W_2\right) W_{2tu} - \left(\frac{u^2 + V^2}{r} + uW_2 - P\right) W_{2ru} \\
&\quad + \left(\frac{u}{r} + W_2\right)\left(\frac{u^2 + V^2}{r} + uW_2 - P\right) W_{2uu} \\
&\quad + \left(W_{2t} + uW_{2r} - \frac{2}{r}V\left(\frac{u}{r} + W_2\right) V_u - \left(\frac{u}{r} + W_2\right)\left(\frac{u}{r} - P_u\right)\right) W_{2u} \\
&\quad + \left(\frac{2u}{r} + \frac{2}{r}VV_u - P_u\right) W_{2r} + \left(\frac{1}{r}V^2 - P\right)\left(W_{2u}^2 + \frac{1}{r}W_{2u}\right) \\
&\quad = W_{1u}^{-1}\left(-W_{2r} + \left(\frac{u}{r} + W_2\right) W_{2u}\right)\left(W_{1tu} + \left(\frac{u^2 + V^2}{r} + uW_2 - P\right) W_{1uu}\right),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&W_{2rr} - 2\left(\frac{u}{r} + W_2\right) W_{2ru} + \left(\frac{u}{r} + W_2\right)^2 W_{2uu} + \frac{1}{r}W_{2r} + \frac{u}{r^2}W_{2u} \\
&\quad = W_{1u}^{-1}\left(-W_{2r} + \left(\frac{u}{r} + W_2\right) W_{2u}\right)\left(-W_{1ru} + \left(\frac{u}{r} + W_2\right) W_{1uu}\right).
\end{aligned} \tag{16}
$$

Note that equation (16) involves only functions W_i and their derivatives. System of equations (10), (13), (14)–(16) forms an overdetermined resolving system of 5 equations for 4 functions V, W_1, W_2, P. It is necessary to investigate its compatibility.

Lemma 2. *System of equations* (10), (13), (14)–(16) *is consistent and in involution. Arbitrariness of its general solution is 4 functions of 2 arguments and 9 functions of 1 argument.*

Proof. To proof the lemma we check the Cartan criterion [11]. Having the goal to obtain second-order system of equations we differentiate all first-order equations (10), (13), (14) with respect to t, r, u. The complete second-order system (together with equations (15), (16)) consists of 11 equations including $4C_4^2 = 24$ second-order derivatives of invariant functions. Let us denote that system as E_1. The matrix of coefficients of the second derivatives in E_1 happens to be non-degenerate. We calculate the auxiliary constant $\tau_0 = 24 - 11 = 13$. Next we add the following relations to E_1

$$
V_{ti} = 0, \qquad W_{1ti} = 0, \qquad W_{2ti} = 0, \qquad P_{ti} = 0, \qquad i = t, r, u \tag{17}
$$

(all second derivatives involving t-differentiation vanish). An extended system E_2 included 23 equations. Rank of matrix constructed from coefficients of second derivatives equals 20. Thus $\tau_1 = 24 - 20 = 4$. Further we add to E_2 the relations

$$
V_{ri} = 0, \qquad W_{1ri} = 0, \qquad W_{2ri} = 0, \qquad P_{ri} = 0, \qquad i = t, r, u \tag{18}
$$

(all second derivatives containing r derivatives vanish). The obtained system contains 31 equations. Rank of matrix of coefficients of the second derivatives equals 24. Thus $\tau_2 = 24 - 24 = 0$.

The data obtained allows to calculate Cartan characters $\sigma_1 = \tau_0 - \tau_1 = 9$, $\sigma_2 = \tau_1 - \tau_2 = 4$, $\sigma_3 = \tau_2 = 0$. The Cartan number is $Q = \tau_0 + \tau_1 + \tau_2 = 17$. To check the Cartan criterion the third prolongation of the initial second-order system E_1 is required. Cartan criterion is satisfied if the number of 'free' third-order derivatives in prolonged system coincides with Cartan number Q. Actually, the number of all third derivatives of invariant functions is $4C_5^3 = 40$. Prolongation of E_1 allows to calculate the rank of matrix of coefficients of third derivatives. It is equal to 23. Hence $40 - 23 = 17 = Q$, i.e. the Cartan criterion for system (10), (13), (14)–(16) is satisfied. General solution of that system is determined with arbitrariness in σ_i functions of i variables. ■

Summarizing all obtained above we can formulate the following theorem.

Theorem 1. *The group foliation of equations* (1) *with respect to infinite-dimensional group* (3) *in the regular case $u_z \neq 0$, $w_{rz} \neq 0$ consists of resolving system* (10), (13), (14)–(16) *for invariant functions V, W_1, W_2, P and automorphic system* (8), (9), (11).

5 Special case $u_z = 0$

Here we observe the special case specified in the title of the paragraph. In this case variables t, r, u cannot be chosen as new independent variables for group foliation as it was done above. However it is possible to describe all 'lost' solutions. Actually, the consequence of stated condition is $u = u(t, r)$. From the last equation (1) by integration we obtain that function w linearly depends on z

$$w = -\left(u_r + \frac{u}{r}\right) z + w_0(t, r). \tag{19}$$

We substitute representation (19) into the third equation (1). After integration with respect to z we obtain pressure p to be a quadratic polynomial of variable z:

$$\begin{aligned}
&p = a(t, r)z^2 - 2b(t, r)z + p_0(t, r),\\
&a(t, r) = u_{tr} + uu_{rr} + \frac{1}{r}u_t - u_r^2 - \frac{1}{r}uu_r - \frac{2}{r^2}u^2,\\
&b(t, r) = w_{0t} + uw_{0r} - \left(u_r + \frac{1}{r}u\right) w_0.
\end{aligned} \tag{20}$$

Substitution of (20) into the first equation (1) allows us to express v^2 as a quadratic polynomial of variable z:

$$v^2 = r\left(u_t + uu_r + a_r z^2 - 2b_r z + p_{0r}\right). \tag{21}$$

Finally, we rewrite the second equation of (1) in the form

$$\left(v^2\right)_t + u\left(v^2\right)_r + w\left(v^2\right)_z = -\frac{2uv^2}{r}. \tag{22}$$

Substituting (19), (21) into (22) we obtain the equation which has a special form: quadratic polynomial of z equals zero. Splitting with respect to z provides a system of 3 equations for 3 unknown functions: u, w_0 and p_0. Solutions of that well-defined system completely determine an investigated class of solutions of Euler equations. It should be noted that these solutions with $u_z = 0$ were retrieved in Pukhnachov's work [12] on the base of the so-called 'heuristic' method.

Remark 1. The only unexamined case left is $w_{rz} = 0$. Its investigation is complicated by the fact that equations (1) do not split after substitution of function w. The higher-order prolongation of the obtained system is required for its completion to involution.

6 Admitted group

The resolving system RS inherits finite-dimensional part of L_4 of the group, admitted by equations (1). Action of L_4 in the space of differential invariants (4) is generated by the following operators:

$$Y_1 = \partial_t, \qquad Y_2 = -\frac{1}{r^2 V}\partial_V - \frac{2}{r^3}\partial_P, \qquad Y_3 = t\partial_t + r\partial_r - W_1\partial_{W_1} - W_2\partial_{W_2} - P\partial_P,$$
$$Y_4 = 2t\partial_t + r\partial_r - u\partial_u - v\partial_v - 2W_1\partial_{W_1} - 2W_2\partial_{W_2} - 3P\partial_P. \tag{23}$$

Group (23) can be used for construction of invariant and partially invariant solutions of the resolving system RS. The velocity components u, v, w and pressure p are restored then by means of integration of automorphic system AS. The optimal system of subalgebras [13, 14] ΘL_4 can be easily constructed. Its representatives are written below.

<u>dim = 4</u>

1. $\{Y_1,\ Y_2,\ Y_3,\ Y_4\}$.

<u>dim = 3</u>

1. $\{Y_1,\ Y_3,\ Y_4\}$, 2. $\{Y_2,\ Y_3,\ Y_4\}$, 3. $\{Y_1,\ Y_2,\ \alpha Y_3 + \beta Y_4;\ \alpha^2 + \beta^2 = 1\}$.

<u>dim = 2</u>

1. $\{Y_1,\ Y_2\}$, 2. $\{Y_3,\ Y_4\}$, 3. $\{Y_1,\ Y_3 + \alpha Y_4\}$, 4. $\{Y_1 + \alpha Y_2,\ 2Y_3 + Y_4\}$,
5. $\{Y_2,\ Y_3 + \alpha Y_4\}$, 6. $\{Y_1,\ \alpha Y_2 + Y_4\}$, 7. $\{Y_2,\ Y_4\}$.

<u>dim = 1</u>

1. $\{Y_1 + \alpha Y_2\}$, 2. $\{Y_2\}$, 3. $\{Y_3 + \alpha Y_4\}$, 4. $\{Y_1 + 2Y_3 - Y_4\}$, 5. $\{\alpha Y_2 + Y_4\}$.

Here $\{Y_1, \ldots, Y_k\}$ denotes a Lie algebra generated by basic operators $Y_1, \ldots, Y_k$; α, β are arbitrary constants.

7 Example of a partially invariant solution

Here we provide an example of a partially invariant solution for resolving system RS. We use the whole 4-dimensional group with generators (23) admitted by the resolving system RS. Invariants of L_4 in the space $\mathbb{R}^7(t, r, u, V, W_1, W_2, P)$ are

$$\frac{rW_1}{u}, \qquad \frac{rW_2}{u}, \qquad \frac{r}{u^2}\left(P - \frac{V^2}{r}\right). \tag{24}$$

All the invariant functions except for V could be expressed from invariants (24). One can construct partially invariant solution of rank 0 and defect 1. The representation of solution of resolving system RS is

$$W_1 = \frac{au}{r}, \qquad W_2 = \frac{bu}{r}, \qquad P = \frac{V^2}{r} + \frac{p_0 u^2}{r}, \qquad a, b, p_0 = \text{const}. \tag{25}$$

The non-invariant superfluous function V supposed to depend on all the variables

$$V = V(t, r, u). \tag{26}$$

The representation of solution (25), (26) is to be substituted into resolving system RS. In accordance with general theory an overdetermined system for function $V(t,r,u)$ and constants a, b, p_0 will appear. This system of 5 nonlinear equations should be completed to involution. It is consistent since it has a trivial solution. However the careful investigation of all cases, which arise during this system's compatibility analysis, is quite complicated.

To avoid this intricate process we start directly from the automorphic system AS given by formulas (8), (9), (11). We substitute the representation of solution (25), (26) into automorphic system AS and repeat all steps which were previously done to obtain the resolving system RS. Now it is simpler since invariant functions have a specific form.

Remark 2. The described algorithm provides a so-called differentially invariant solution. It can be used without any prior information about group foliation of the system (1) starting directly from optimal system of subgroups for admitted group.

Implementation of the algorithm gives the following. The solution is reduced to

$$\begin{aligned} &u = ru(t,z), \qquad v^2 = \frac{k}{r^2} - r^2u^2(t,z), \\ &w = w(t,z), \qquad p = -\frac{k}{2r^2} + p(t,z), \qquad k = \text{const}. \end{aligned} \tag{27}$$

Note that $k = 0$ modulo to Y_2 transformation. Functions $u(t,z)$, $w(t,z)$, $p(t,z)$ satisfy the system

$$u_t + wu_z = -2u^2, \qquad w_z = -2u, \qquad w_t + ww_z + p_z = 0. \tag{28}$$

Equations (28) are reduced to one key equation for function $w(t,z)$ only

$$w_{tz} + ww_{zz} = w_z^2. \tag{29}$$

Equation (29) can be an object of separate investigation. It admits infinite-dimensional group of contact transformation with the following generators:

$$\begin{aligned} &\partial_t, \qquad t\partial_t + z\partial_z, \qquad z\partial_z + w\partial_w, \qquad t^2\partial_t - tz\partial_z - (z+3tw)\partial_w, \\ &h_{w_z}\partial_z + (w_zh_{w_z} - h)\partial_w - h_t\partial_{w_t}. \end{aligned} \tag{30}$$

Here $h = h(t, w_z)$ is an arbitrary function, which satisfies the linear equation

$$h_{tw_z} + (w_z)^2h_{w_zw_z} - w_zh_{w_z} + h = 0. \tag{31}$$

Apparently it indicates that equation (29) is linearizable.

8 Conclusion

We constructed a group foliation of Euler ideal liquid equations in the rotationally symmetrical case. For group foliation we used an infinite-dimensional part of the admitted group. The resolving system RS of group foliation consists of 5 equations (two of them are second-order) for 4 sought functions of 3 variables. This system is in involution. The arbitrariness of its general solution is determined. For any solution of resolving system one can restore original functions by solving a consistent automorphic system AS.

The resolving system inherits finite-dimensional part of initial admitted group. We construct a partially invariant solution for resolving system constructed on the base of inherited transformations. This solution can be observed as a differentially invariant solution for the original Euler equations.

Acknowledgements

The work was supported by RFBR grant 02-01-00550 and by Council of Support of the Leading Scientific Schools, grant Sc.Sch.-440.2003.1.

[1] Golovin S.V., Application of differential invariants of infinite-dimensional groups in hydrodynamics, *Comm. in Nonlin. Sci. and Num. Simul.*, 2004, V.9, N 1, 35–51.

[2] Ovsiannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[3] Olver P., Equivalence, invariance and symmetry, Cambridge, Cambridge University Press, 1995.

[4] Vereshchagina L.I., Group separation of the spatial nonstationary boundary layer equations, *Vestnik Leningr. Univ.*, 1974, V.3, N 13, 82–86 (in Russian).

[5] Nutku Y. and Sheftel' M.B., Differential invariants and group foliation for the complex Monge–Ampère equation, *J. Phys. A: Math. Gen.*, 2001, V.34, N 1, 137–156.

[6] Martina L., Sheftel' M.B. and Winternitz P., Group foliation and non-invariant solutions of the heavently equation, *J. Phys. A: Math. Gen.*, 2001, V.34, 9243–9263.

[7] Golovin S.V., Group stratification and exact solutions of the equation of transonic gas motions, *J. App. Mech. Tech. Phys.*, 2003, V.44, N 3, 344–354.

[8] Lie S., Über Differentialinvarianten, *Math. Ann.*, 1884, V.24, N 1, 52–89.

[9] Andreev V.K., Kaptsov O.V., Pukhnachov V.V. and Rodionov A.A., Applications of group-theoretical methods in hydrodynamics, Dordrecht, Kluwer Academic Press, 1998.

[10] Pukhnachov V.V., An integrable model of nonstationary rotationally symmetrical motion of ideal incompressible liquid, *Nonlinear Dynamics*, 2000, V.22, 101–109.

[11] Pommaret J.F., Systems of partial differential equations and Lie pseudogroups, New York, Gordon and Breach, 1978.

[12] Pukhnachov V.V., Exact solutions of the hydrodynamic equations derived from partially invariant solutions, *J. Appl. Mech. Tech. Phys.*, 2003, V.44, N 3, 317–323.

[13] Ovsiannikov L.V., On optimal systems of subalgebras, *Russian Acad. Sci. Dokl. Math.*, 1994, V.48, N 3, 645–649.

[14] Patera J. and Winternitz P., Subalgebras of real three- and four-dimensional Lie algebras, *J. Math. Phys.*, 1977, V.18, N 4, 1449–1455.