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On Computational Aspects of the Fourier–Mukai Transform

Nikolaj M. GLAZUNOV

Glushkov Institute of Cybernetics, 40 Glushkov Ave., Kyiv-187, Ukraine
E-mail: *glanm@d105.icyb.kiev.ua*

We survey and investigate some computational aspects of the Fourier–Mukai transform.

1 Introduction

Let $f(x)$ be a function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$, $n \in \mathbb{N}$. Then its Fourier transform from the time domain into its frequency domain is given by the discrete Fourier transform

$$\hat{f}(\omega) = \sum_{x=0}^{n-1} f(x) \exp\left(2\pi i \frac{\omega x}{n}\right).$$

Here ωx is the perfect pairing (in the case the scalar product) on the product of the time and of the frequency domains. It is well-known also that the solution of a linear differential equation with constant coefficients is related by the Fourier transform to a solution of the polynomial equation. The Fourier–Mukai transform is a strong generalization of mentioned and some another approaches. Let A be an Abelian variety, $\hat{A}$ its dual Abelian variety and P the Poincaré divisor on $A \times \hat{A}$. Let $D^b(A)$ and $D^b(\hat{A})$ be derived categories of bounded complexes of sheaves on A and $\hat{A}$ respectively. A Fourier–Mukai transform was defined by Mukai as an exact equivalence

$$\mathcal{FM} :\ D^b(\hat{A}) \to D^b(A)$$

between derived categories of above mentioned bounded complexes [1, 2]. For this transform analogies of the Fourier Inversion Theorem and the Parseval Theorem are valid. Works by B. Bartocci, U. Bruzzo, D. Ruipérez [3] and A. Maciocia [4] generalized this approach to another classes of sheaves and varieties. Now by a Fourier–Mukai transform

$$\mathcal{FM} :\ D^b(Y) \to D^b(X)$$

an exact equivalence between bounded derived categories of coherent sheaves on two smooth projective varieties X and Y is understood. It is known that the derived categories of coherent sheaves on some projective varieties are equivalent to the derived categories of representations of n-vertices quivers. For instance, the derived category of coherent sheaves on $\mathbb{P}^2$ is equivalent to the derived category of representations of 3-vertices quivers with relations [5]. There is a very interesting connection between Fourier–Mukai transforms and mirror symmetry [8]. A quiver can be interpreted as a directed graph [6, 7]. We investigate some computational aspects of Fourier–Mukai transforms. This paper is a continuation of [9, 10].

The organization of the article is as follows. In Section 2 we recall from the computational point of view some facts about Abelian varieties. In Section 3 we very shortly consider parameterization of Abelian varieties and some moduli spaces. In Section 4 we recall follow to Mukai the definition of the Fourier–Mukai transform. In Section 5 we present our computer algebraic method for computation of cutsets of some quivers and its implementation.

2 Abelian varieties

Let $\mathbb{C}^g$ be a complex vector space of the dimension g and Λ a lattice in $\mathbb{C}^g$ with a bases $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2g}\}$. For a lattice vector $\boldsymbol{a} = (a_1, \dots, a_{2g}) \in \Lambda$ its length is denoted by $|\boldsymbol{a}|$. The factor $\mathbb{C}^g/\Lambda = A$ is the commutative compact topological group (complex torus). Recall the case $g = 1$.

Example 1 (Elliptic curves). Let $\Lambda = \{\boldsymbol{a}, \boldsymbol{b}\}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}$, $\boldsymbol{a}/\boldsymbol{b} \notin \mathbb{R}$. Then $E = \mathbb{C}/\Lambda$ is an elliptic curve. In the case every such lattice defines an elliptic curve. What is the simplest representation of Λ?

Lemma 1 (Basis reduction in $\boldsymbol{\Lambda}$). *In any such lattice Λ there exists (under the rotations) one and only one basis with conditions:*

(i) $|\boldsymbol{b}| \geq |\boldsymbol{a}|$;

(ii) *length of the projection $\boldsymbol{b}$ on $\boldsymbol{a} \leq \frac{1}{2}|\boldsymbol{a}|$;*

(iii) *the angle $(\boldsymbol{a}, \boldsymbol{b}) < \pi/2$ (acute angle).*

So we can reduce a basis of Λ to the form $\Lambda = \{1, \tau\}$, where $\operatorname{Im}\tau > 0$, $-\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2}$, $|\tau| \geq 1$.

Let

$$h = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \qquad \det h = 1, \qquad a, b, c, d \in \mathbb{Z}.$$

Two lattices $\Lambda = \{1, \tau\}$, $\Lambda' = \{1, \tau'\}$ are equivalent if $\tau' = \frac{a\tau + b}{c\tau + d}$. In the case there is a birational isomorphism between elliptic curves $E = \mathbb{C}/\Lambda$ and $E' = \mathbb{C}/\Lambda'$.

If $g > 1$ then there are groups $A = \mathbb{C}^g/\Lambda$ which are not Abelian varieties [11]. The complex torus A is an Abelian variety if and only if there exist $\mathbb{R}$-bilinear antisymmetric form $F(x, y)$ such that the form $F(x, ix)$ is a positive definite Hermitian form that takes integer values at points of Λ. The form $F(x, y)$ is called the *polarization* on Λ and a pair (Λ, F) is called the polarized Abelian variety. For computational purposes a good idea is to reduce the bases of the lattice to a some simple form. The bases reduction of Λ in the case $g = 1$ is well-known (Example 1).

On basis reduction in lattices (the case $\boldsymbol{g > 1}$). Let a lattice $\Lambda = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2g}\}$, $\boldsymbol{a}_i = (a_{i1}, \dots, a_{i1})$ have coordinates with the condition $a_{ij} \in \mathbb{Q}$.

There are *Minimum Basis problem*, *Shortest Lattice Vector problem* and *LLL Lattice Reduction* [12, 13].

The method of a solution of the Minimum basis problem is based on the theory of successive minima, developed by Minkowski [12]. By this method we try to find short linear independent vectors one after one. The solution of the Shortest Lattice Vector problem is based on Minkowski's theorem on convex body [12]. Probably both the methods are NP-hard.

In some applications the concept "*reduces*" means something like "nearly orthogonal". The L^3 algorithm [14] accept as input any basis $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2g}$ of Λ, and it computes a reduced basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{2g}$ of Λ. In general, a lattice may have more then one reduced basis. The ordering of the basis vectors is not arbitrary. The algorithm has a very good theoretical complexity (polynomial-time in the length of the input parameters).

3 Parameterization of Abelian varieties and moduli spaces

Let (Λ, F) be a polarized Abelian variety. Denote by M^{tr} the transposition of a matrix M. There is a canonical form of Abelian varieties. In the form the lattice of the Abelian variety has the representation

$$\Omega = (E_g, Z),$$

where E_g is the identity matrix and $Z = X + iY$ is a complex $g \times g$ matrix with conditions: a) $Z = Z^{\mathrm{tr}}$, $X = X^{\mathrm{tr}}$, $Y = Y^{\mathrm{tr}}$ (X, Y are reals); b) $Y > 0$ is the matrix of a positive quadratic form. The set $\{Z\} = \mathbf{H}_g$ of the matrices is called the *Siegel upper half-space*. Let $J = \begin{vmatrix} 0 & E_g \\ -E_g & 0 \end{vmatrix}$ be the symplectic matrix. Let $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $M \in GL(2g, \mathbb{Z})$, $M^{\mathrm{tr}} J M = J$, be the modular group. The modular group acts on $\mathbf{H}_g$ by the formula

$$Z' = (aZ + b)(cZ + d)^{-1}$$

(here a, b, c, d are $g \times g$ matrices with integer coefficients). If matrices X and Y have integer coefficients, then all data with the exception of complex unit are defined over $\mathbb{Z}$. Therefore they and their factors mod p for prime p are well-adapted for computers. Moduli spaces [15,16] are used for specification and investigation of classes of objects which could be algebraic curves (or, more generally, schemes), sheaves, vector bundles, morphisms and others. Parameter varieties is a class of moduli spaces. These varieties is a very convenient tool for computer algebra investigation of objects that are parameterized by the parameter varieties. If we investigate a class of algebraic varieties over integers, then the first step of the investigation in many cases is the analysis of the varieties over prime finite fields. We have used the approach for investigation of rational points on families of hyperelliptic curves [17]. Expansion of the group law of an Abelian variety near 0 defines a formal group of the dimension one in the case of elliptic curves [18] and of the dimension $g > 1$ for $g-$dimensional Abelian varieties [19]. Results about formal group over finite fields and over ρ-adic fields [20] can be useful for investigation of the groups over $\mathbb{Z}$. The paper [21] demonstrates computational advantages of the localization under computation of cohomologies.

Let A, B be two Abelian varieties of the dimension g over a field k, $k(A)$, $k(B)$ their fields of functions, $\lambda : A \to B$ some surjective morphism with a finite kernel (an *isogeny* of the varieties). Any isogeny $\lambda : A \to B$ defines a field extension $[k(A) : k(B)]$. The degree of the extension is called the *degree* of the isogeny. The polarization F defines an isogeny of degree d^2. Follow to [22] it is possible to construct for a prime finite field $\mathbf{F}_p$ the moduli space $X_{g,d} \otimes \mathbf{F}_p$ of all Abelian varieties of dimension g with a polarization F of degree d^2.

4 Elements of category theory and the Fourier–Mukai transform

4.1 Equivalence of categories

Let $\mathcal{C}$ be a category [23], $\mathrm{Ob}\,\mathcal{C}$ its class of objects, and for $a, b \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}(a, b)$ the class of morphisms (arrows) from a to b. An arrow $u : a \to b$ is *the equivalence* in $\mathcal{C}$ if there is an arrow $u' : b \to a$ such that $u'u = 1_a$ and $uu' = 1_b$. A *functor* $\mathcal{F}$ from a category $\mathcal{C}$ to a category $\mathcal{K}$ is a function which maps $\mathrm{Ob}\,(\mathcal{C}) \to \mathrm{Ob}\,(\mathcal{K})$, and which for each pair a, b of objects of $\mathcal{C}$ maps $\mathcal{C}(a, b) \to \mathcal{C}(\mathcal{F}(a), \mathcal{F}(b))$, while satisfying the two conditions: $\mathcal{F}\,\mathrm{id}_a = \mathrm{id}_{\mathcal{F}a}$ for every $a \in \mathrm{Ob}\,(\mathcal{C})$, $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

Let $\mathrm{Funct}\,(\mathcal{C}, \mathcal{D})$ be the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations as morphisms. An equivalence in $\mathrm{Funct}\,(\mathcal{C}, \mathcal{D})$ is *the equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$. The following theorem is a particular case of the theorem from [23].

Theorem 1. *A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is the equivalence if, and only if*

a) *$\mathcal{F}$ is fully faithful, and*

b) *any $Y \in \mathrm{Ob}\,\mathcal{D}$ is isomorphic to an object of the form $\mathcal{F}(X)$ for an object $X \in \mathrm{Ob}\,\mathcal{C}$.*

4.2 The Fourier–Mukai transform

Let A be an Abelian variety and $\hat{A}$ the dual Abelian variety which is by definition a moduli space of line bundles of degree zero on A [11]. The *Poincaré bundle* $\mathcal{P}$ is a line bundle of degree zero on the product $A \times \hat{A}$, defined in such a way that for all $a \in \hat{A}$ the restriction of $\mathcal{P}$ on $A \times \{a\}$ is isomorphic to the line bundle corresponding to the point $a \in \hat{A}$. This line bundle is also called *the universal bundle.* Let

$$\pi_A : \ A \times \hat{A} \to A,$$
$$\pi_{\hat{A}} : \ \hat{A} \times A \to \hat{A},$$

and $\mathcal{C}_A$ be the category of O_A-modules over $A, M \in \mathrm{Ob}\,\mathcal{C}_A$,

$$\hat{S}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

Then, by definition, *the Fourier–Mukai transform* $\mathcal{FM}$ is the derived functor $R\hat{S}$ of the functor $\hat{S}$. Let $\mathcal{D}(A)$, $\mathcal{D}(\hat{A})$ be bounded derived categories of coherent sheaves on A and $\hat{A}$ respectively.

Theorem 2 (Mukai). *The derived functor $\mathcal{FM} = R\hat{S}$ induces an equivalence of categories between two derived categories $\mathcal{D}(\hat{A})$ and $\mathcal{D}(A)$.*

5 Cutset method and its implementation

A quiver can be interpreted as a directed graph. Here we consider quivers without multiple edges and loops. The solution of some problems requires to cut all cycles or construct a spanning-tree of a directed graph. Rooted graph G is the graph that each node of G is reachable from a node $r \in G$. By the *cutset* of a graph we shall understand an appropriated subset of nodes (called cutpoints) such that any cycle of the graph contains at least one cutpoint. DFS is the abbreviation of Depth First Search method. During DFS we are numbering nodes and label (mark) edges. By Tarjan [24] the DFS method has linear complexity. The main function Cutsetdg of the package CUTSETDG computes Cutset (a subset of vertices which cut all cycles in the graph) of arbitrary rooted directed graph. It uses 3 functions: Adjarcn, Cutpoints and Unicut. For this program the author developed rather simple and efficient algorithm that is based on DFS-method. The function is the base for Cutset methods for systems of procedures. The package is implemented by the Allegro Common Lisp language [25, 26].

Note 1. It is not difficult to modify the Cutset algorithm for processing directed graphs with multiple edges.

Package CUTSETDG
Title: Cutset of a rooted directed graph
Summary: This package implements computation of a cutset of a rooted directed graph. The graph have to defined by adjacency list.
Author: Nikolaj M. Glazunov
Example of Call: (Cutsetdg 'a) where "a" is a root of exploring graph.
The Common Lisp text of the package CUTSETDG

```
;; Allegro CL 3.0.2
;;

(DEFUN Cutsetdg (startnode)

;; startnode is a root of exploring graph
;; DFS - Depth First Search method for connected graph
```

```
;; st - stack
;; v - exploring node
;; sv -  son of the exploring node
;; inl - inverse edges list
;; df -  DFS-numbering of the node (property)
;; Outarcs - adjacency list of Output arcs of the node (property)
;; 1, 2  - lables

    (SETF (GET startnode 'df ) 1)   ;DFS-numbering is equal 1
   (SETF (GET startnode 'Outarcs)
            (Adjarcn  (LIST startnode))) ;;startnode obtained the a-list
                                 ;; property Outarcn (Output arcs of the
                                 ;; node)
    (PROG ( st v sv inl  cuts)
         (PUSH startnode st)
   1   (SETQ v (CAR st)) ;;explored node received value from stack of nodes
   2      (COND ((NOT (EQ  NIL (GET v 'Outarcs)))
                    (SETQ sv (CAAR (GET v 'Outarcs)))
                        (COND ((EQ NIL (GET sv 'df))
                            (SETF (GET sv 'df) (+ 1 (GET v 'df)))
       ;; modification of the edge of son's adj-list
       (SETF (GET sv 'Outarcs)
            (Adjarcn  (LIST sv))) ;; node sv obtained the adj-list property
                                 ;; Outarcs (adjacency Outarcs)
          (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                              :TEST 'EQUAL))
                           (PUSH sv st)
                            (GO 1)
            )
              (T
                 (COND ((AND (> (GET v 'df) (GET sv 'df))
                            (NOT (EQUAL v sv)))
              (SETQ inl (CONS (CAR (GET v 'Outarcs)) inl))
              (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                              :TEST 'EQUAL))
                        (GO 2)
                        )
              (T
               (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                              :TEST 'EQUAL))
                (GO 2)
                )
                       )
                )
                   )
                 )
                              (T
                (POP st)
         (COND     ((NOT (NULL st))
                      (GO 1)
              )
                     (T                  ;; end
                   (SETQ cuts (Unicut (Cutpoints inl)))
                (RETURN cuts)
                )
              )
```

```
                ))
     )
   )
(DEFUN Adjarcn (PATH)
      (MAPCAN #'(LAMBDA (E)
              (COND ((MEMBER (CAR E) (CDR E)) NIL)
                   (T (LIST E))))
          (MAPCAR #'(LAMBDA (E)
                            (CONS E PATH))
             (CAR  (GET (CAR PATH) 'NEIGHBORS)))))
(DEFUN Cutpoints (inl)
        (MAPCAR 'CAR inl))
(DEFUN Unicut (cpl)
     (COND ((NULL cpl) NIL)
           (T (CONS (CAR cpl)
                (Unicut (REMOVE (CAR cpl) cpl)))
             )
     )
     )
```

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[1] Mukai S., Duality between $D(X)$ and $D(X)$ with its application to Picard sheaves, *Nagoya Math. J.*, 1981, V.81, 153–175.

[2] Mukai S., Fourier functor and its application to the moduli of bundles on an Abelian variety, *Adv. Pure Math.*, 1987, V.10, 515–550.

[3] Bartocci C., Bruzzo U. and Hernández-Ruipérez D., A Fourier–Mukai transform for stable bundles on $K3$ surface, alg-geom/9405006.

[4] Maciocia A., Generalized Fourier–Mukai transforms, *J. reine. angew. Math.*, 1996, V.480, 197–211.

[5] Bondal A., Representation of associative algebras and coherent sheaves, *Izvestiya AN SSSR, Ser. Math.*, 1989, V.53, 25–44.

[6] Dräxler P. and Nörenberg R., Classification problems in the representation theory of finite-dimensional algebras, Preprint pre.97008, Bielefeld, University of Bielefeld, 1997.

[7] Roiter A.V., Representations of marked quivers, math.RT/0112158.

[8] Thomas R., Mirror symmetry and actions of braid groups on derived categories, math.AG/0001044.

[9] Glazunov N., Mirror symmetry: algebraic geometric and special Lagrangian fibrations aspects, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 2, 623–628.

[10] Glazunov N., On algebraic geometric and computer algebra aspects of mirror symmetry, hep-th/0112250.

[11] Mumford D., Abelian varieties, Bombay, Tata Institute, 1968.

[12] Cassels J., An introduction to the geometry of numbers, Berlin, Springer-Verlag, 1971.

[13] Grötschel M., Schrÿver A. and Lovász L., Geometric algorithms and combinatorial optimization, Berlin, Springer-Verlag, 1988.

[14] Lenstra A., Lenstra H.Jr. and Lovász L., Factoring polynomials with rational coefficients, *Math. Ann.*, 1982, V.261, 515–534.

[15] Mamford D., Towards an enumerative geometry of the moduli space of curves, in Progress in Math., Arithmetic and Geometry, Boston, Birkhauser, 1983, Vol. II, 271–328.

[16] Harris J. and Morrison J., Moduli of curves, Berlin – New York, Springer, 1998.

[17] Glazunov N., On moduli spaces, equidistribution, bounds and rational points of algebraic curves, *Ukr. Math. J.*, 2001, V.53, N 9, 1174–1183.

[18] Tate J., The arithmetic of elliptic curves, *Inventiones Math.*, 1974, V.23, 179–206.

[19] Honda T., On the theory of commutative formal groups, *J. Math. Soc. Japan*, 1972, V.22, N 2, 213–246.

[20] Glazunov N., Remarks on n-dimensional commutative formal groups over integer ring of ρ-adic field, *Ukr. Math. J.*, 1973, V.25, N 3, 352–355.

[21] Kornyak V., Modular algorithm for computing cohomology, math.RT/0305155.

[22] Deligne P. and Mumford D., On irreducibility of the space of curves of given genus, *Pub. Math. IHES*, 1969, N 36, 75–109.

[23] MacLane S., Categories for the working mathematicians, Berlin, Springer-Verlag, 1971.

[24] Tarjan R., Depth-first search and linear graph algorithms, *SIAM J. Comput.*, 1972, V.1, N 2, 146–160.

[25] Guy L. and Steele Jr., Common Lisp, The language, Second edition, 1990.

[26] Allegro CL for Windows, Franz Inc, 1997.