Nonlocal Brackets and Integrable String Models

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The closed string model in the background gravity field is considered as a bi-Hamiltonian system in assumption that string model is the integrable model for particular kind of the background fields. The dual nonlocal Poisson brackets (PB), depending of the background fields and of their derivatives, are obtained. The integrability condition is formulated as compatibility of the bi-Hamiltonity condition and the Jacobi identity of the dual Poisson bracket. It is shown that the dual brackets and dual Hamiltonians can be obtained from the canonical PB and from the initial Hamiltonian by imposing the second kind constraints on the initial dynamical system, on the closed string model in the constant background fields, as example. Two types of the nonlocal brackets are introduced.Constant curvature and time-dependent metrics are considered, as examples. It is shown, that the Jacobi identities for the nonlocal brackets have particular solution for the space-time coordinates, as matrix representation of the simple Lie group.

1 Introduction

The bi-Hamiltonian approach [1–3] to the integrable systems was initiated by Magri [4] for investigation of the integrability of the KdV equation.

Definition 1. A finite dimensional dynamical system with $2N$ degrees of freedom x^a, $a = 1,\dots,2N$ is integrable, if it is described by the set of the n integrals of motion $F_1,\dots,F_n$ in involution under some Poisson bracket (PB)

$$\{F_i, F_k\}_{\mathrm{PB}} = 0.$$

The dynamical system is completely solvable, if $n = N$. Any integral of motion (or any linear combination of them) can be considered as the Hamiltonian $H_k = F_k$.

Definition 2. The bi-Hamiltonity condition has following form:

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = \cdots = \{x^a, H_N\}_N. \tag{1}$$

The hierarchy of new PB arises in this connection:

$$\{,\}_1,\ \{,\}_2,\ \dots,\ \{,\}_N.$$

The hierarchy of new dynamical systems arises under the new time coordinates t_k.

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1} = \{x^a, H_k\}_{n+1}. \tag{2}$$

The new equations of motion describe the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion. The dual set of the integrals of motion can be obtained from the original one by the mirror transformations and by contraction of the integrals of motion algebra. Contraction of the integrals of motion algebra means that the dynamical system belongs to the orbits of corresponding generators and describes the invariant subspace. The set of the commuting integrals of motion belongs to Cartan subalgebra of this algebra. KdV equation is one of the most interesting examples of the infinite-dimensional integrable mechanical systems with soliton solutions. We consider the dynamical systems with constraints. In this case, first-kind constraints are generators of the gauge transformations and they are integrals of motion. First-kind constraints $F_k(x^a) \approx 0$, $k = 1, 2, \ldots$ form the algebra of constraints under some PB

$$\{F_i, F_k\}_{\text{PB}} = C^l_{ik} F_l \approx 0.$$

The structure functions C^l_{ik} may be functions of the phase space coordinates in general case. The second kind constraints $f_k(x^a) \approx 0$ are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition

$$\{f_i, f_k\} = C_{ik} \neq 0.$$

The reversible matrix C_{ik} is not a constraint, and also it is a function of phase space coordinates. The second-kind constraints take part in deformation of the $\{,\}_{\text{PB}}$ to the Dirac bracket $\{,\}_D$. As a rule, such deformation leads to nonlinear and to nonlocal brackets. The bi-Hamiltonity condition leads to the dual PB that are nonlinear and nonlocal brackets as a rule. We suppose that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We have applied [5–9] bi-Hamiltonity approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bi-Hamiltonity condition and the Jacobi identities for the dual brackets were considered as the integrability condition for a closed string model. They led to some restrictions on the background fields. B.A. Dubrovin and S.P. Novikov have considered the local dual PB of the similar type [10] in the application to the Hamiltonian hydrodynamical models. The PB of the hydrodynamical type for the phase coordinate functions $u^i(x,t)$ is defined by the formula

$$\{u^i(x), u^k(y)\} = g^{ik}(u(x))\frac{\partial}{\partial x}\delta(x-y) - g^{ij}\Gamma^k_{jl}(u(x))u^l_x\delta(x-y).$$

There $g_{ik}(u)$, $\Gamma^k_{jl}(u)$ are the arbitrary functions of the phase space coordinates and $u_x = \partial_x u$. The Jacobi identity is satisfied if g_{ik} is the Riemannian metric without torsion,the curvature tensor is equal to zero. The metric tensor is constant, locally. O.I. Mokhov and E.V. Ferapontov introduced the nonlocal PB [11, 12]. The Ferapontov's nonlocal PB is:

$$\begin{aligned}\{u^i(x), u^k(y)\} = {} & g^{ik}(u)\frac{\partial}{\partial x}\delta(x-y) - g^{ij}\Gamma^k_{jl}u^l_x\delta(x-y) \\ & + \omega^{(s)i}_k(u(x))u^j_x\nu(x-y)\omega^{(s)k}_d(u(y))u^d_y,\end{aligned}$$

where $\nu(x-y) = \operatorname{sgn}(x-y) = (\frac{d}{dx})^{-1}\delta(x-y)$. This PB was used for description of the Hamiltonian system of the hydrodynamical type. There are systems with functionals of the hydrodynamical type. The density of this functionals does not depend on the derivatives u^k_x, u^k_{xx}, ..., and Hamiltonian is also a functional of the hydrodynamical type. On the contrary to these models, the functionals of the closed string model is depended of the derivatives of the string coordinates. The plan of the paper is the following. In the second section we considered closed string model in the arbitrary background gravity field. We suppose that this model is an integrable model for some configurations of the background fields. The bi-Hamiltonity condition and the Jacobi identities for the dual PB must be a result of the integrability condition, which is restricted the possible configurations of the background fields.

2 Closed string in the background fields

The string model in the background gravity field is described by the system of equations:

$$\ddot{x}^a - x''^a + \Gamma^a_{bc}(x)(\dot{x}^b\dot{x}^c - x'^b x'^c) = 0,$$
$$g_{ab}(x)(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0, \qquad g_{ab}(x)\dot{x}^a x'^b = 0,$$

where $\dot{x}^a = \frac{dx^a}{d\tau}$, $x'^a = \frac{dx^a}{d\sigma}$. We will consider the Hamiltonian formalism. The closed string in the background gravity field is described by first-kind constraints in the Hamiltonian formalism:

$$h_1 = \frac{1}{2}g^{ab}(x)p_a p_b + \frac{1}{2}g_{ab}(x)x'^a x'^b \approx 0, \qquad h_2 = p_a x'^a \approx 0, \tag{3}$$

where $a, b = 0, 1, \ldots, D-1$, $x^a(\tau,\sigma)$, $p_a(\tau,\sigma)$ are the periodical functions on σ with the period on π. The original PB are the symplectic PB

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta^a_b\delta(\sigma-\sigma'), \{x^a(\sigma), x^b(\sigma')\}_1 = \{p_a(\sigma), p_b(\sigma')\}_1 = 0.$$

The Hamiltonian equations of motion of the closed string, in the arbitrary background gravity field under the Hamiltonian $H_1 = \int_0^\pi h_1 d\sigma$ and PB $\{\,,\,\}_1$, are

$$\dot{x}^a = g^{ab}p_b, \qquad \dot{p}_a = g_{ab}x''^b - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b}.$$

The dual PB are obtained from the bi-Hamiltonity condition

$$\dot{x}^a = \left\{x^a, \int_0^\pi h_1(\sigma')d\sigma'\right\}_1 = \left\{x^a, \int_0^\pi h_2(\sigma')d\sigma'\right\}_2,$$
$$\dot{p}_a = \left\{p_a, \int_0^\pi h_1(\sigma')d\sigma'\right\}_1 = \left\{p_a, \int_0^\pi h_2(\sigma')d\sigma'\right\}_2. \tag{4}$$

They have the following form:

Proposition 1.

$$\begin{aligned}\{A(\sigma), B(\sigma')\}_2 = {}& \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial x^b}[[\omega^{ab}(\sigma)+\omega^{ab}(\sigma')]\nu(\sigma'-\sigma) + [\Phi^{ab}(\sigma)+\Phi^{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)\\ &+ [\Omega^{ab}(\sigma)+\Omega^{ab}(\sigma')]\delta(\sigma'-\sigma)] + \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial p_b}[[\omega_{ab}(\sigma)+\omega_{ab}(\sigma')]\nu(\sigma'-\sigma)\\ &+ [\Phi_{ab}(\sigma)+\Phi_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma) + [\Omega_{ab}(\sigma)+\Omega_{ab}(\sigma')]\delta(\sigma'-\sigma)]\\ &+ \left[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} + \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}\right][[\omega^a_b(\sigma)+\omega^a_b(\sigma')]\nu(\sigma'-\sigma) + [\Phi^a_b(\sigma)+\Phi^a_b(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)]\\ &+ \left[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} - \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}\right][\Omega^a_b(\sigma)+\Omega^a_b(\sigma')]\delta(\sigma'-\sigma).\end{aligned}$$

The arbitrary functions A, B, ω, Φ, Ω are the functions of the $x^a(\sigma)$, $p_a(\sigma)$. The functions ω^{ab}, ω_{ab}, Φ^{ab}, Φ_{ab} are the symmetric functions on a, b and Ω^{ab}, Ω_{ab} are the antisymmetric functions to satisfy the condition $\{A,B\}_2 = -\{B,A\}_2$.

The equations of motion under the Hamiltonian $H_2 = \int_0^\pi h_2(\sigma')d\sigma'$ and PB $\{\,,\,\}_2$ are

$$\begin{aligned}\dot{x}^a = {}& -\omega^a_b x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p''_b - 2\Phi^a_b x''^b + 2\Omega^a_b x'^b - 2\Omega^{ab}p'_b\\ &+ \int_0^\pi d\sigma'\left[\omega^a_b x'^a + \frac{d\omega^{ab}}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p'_b - \frac{d\Phi^a_b}{d\sigma}x'^b,\end{aligned}$$

$$\dot{p}_a = -\omega_{ab}x^b - 2\Phi_{ab}x''^b + 2\Omega_{ab}x'^b + 2\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' \left[\omega_{ab}x'^b + \frac{d\omega_a^b}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x'^b + \frac{d\Phi_a^b}{d\sigma}p_b'.$$

The bi-Hamiltonity condition (4) is led to the two constraints

$$-\omega_b^a x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p_b'' - -2\Phi_b^a x''^b + 2\Omega_b^a x'^b - 2\Omega^{ab}p_b' + \int_0^\pi d\sigma' \left[\omega_b^a x'^a + \frac{d\omega^{ab}}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p_b' - \frac{d\Phi_b^a}{d\sigma}x'^b = g^{ab}p_b,$$

$$-\omega_{ab}x^b - 2\Phi_{ab}x''^b + 2\Omega_{ab}x'^b + 2\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' \left[\omega_{ab}x'^b + \frac{d\omega_a^b}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x'^b + \frac{d\Phi_a^b}{d\sigma}p_b'$$
$$= g_{ab}x''^b - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}x'^b x'^c + \frac{\partial g_{ac}}{\partial x^b}x'^b x'^c.$$

In reality, there is the list of the constraints depending on the possible choice of the unknown functions ω, Ω, Φ. In the general case, there are both the first-kind constraints and the second-kind constraints too. Also it is possible to solve the constraints equations as the equations for the definition of the functions ω, Φ, Ω. We considered the latter possibility and we obtained the following consistent solution of the bi-Hamiltonity condition:

$$\Phi^{ab} = 0, \qquad \Omega^{ab} = 0, \qquad \Phi_b^a = 0, \qquad \Omega_b^a = 0, \qquad \frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab},$$
$$\omega_{ab} = \frac{1}{2}\frac{\partial^2\omega^{cd}}{\partial x^a\partial x^b}p_c p_d, \qquad \omega_b^a = -\frac{\partial\omega^{ac}}{\partial x^b}p_c,$$
$$\Phi_{ab} = -\frac{1}{2}g_{ab}, \qquad \Omega_{ab} = \frac{1}{2}\left(\frac{\partial\Phi_{bc}}{\partial x^a} - \frac{\partial\Phi_{ac}}{\partial x^b}\right)x'^c, \qquad \frac{\partial\omega^{ab}}{\partial p_c} = 0.$$

Remark 1. In distinct from the PB of the hydrodynamical type, we need to introduce the separate PB for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as $f(x^a, p_a) \approx 0$.

Consequently, the dual PB for the phase space coordinates are

$$\{x^a(\sigma), x^b(\sigma')\}_2 = [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')]\nu(\sigma'-\sigma),$$
$$\{p_a(\sigma), p_b(\sigma')\}_2 = \left[\frac{\partial^2\omega_{cd}(\sigma)}{\partial x^a\partial x^b}p_c p_d + \frac{\partial^2\omega_{cd}(\sigma')}{\partial x^a\partial x^b}p_c p_d\right]\nu(\sigma'-\sigma) - \frac{1}{2}[g_{ab}(\sigma) + g_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma) + \left[\frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^a}\right]x'^c(\sigma)\delta(\sigma'-\sigma),$$
$$\{x^a(\sigma), p_b(\sigma')\}_2 = -\left[\frac{\partial\omega^{ac}(\sigma)}{\partial x^b}p_c + \frac{\partial\omega^{ac}(\sigma')}{\partial x^b}p_c\right]\nu(\sigma'-\sigma),$$
$$\{p_a(\sigma), x^b(\sigma')\}_2 = -\left[\frac{\partial\omega^{bc}(\sigma)}{\partial x^a}p_c + \frac{\partial\omega^{bc}(\sigma')}{\partial x^c}p_c\right]\nu(\sigma'-\sigma). \tag{5}$$

The function $\omega^{ab}(x)$ is satisfied on the equation:

$$\frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab}. \tag{6}$$

The Jacobi identities for the PB $\{\,,\,\}_2$ are led to the nonlocal consistence conditions on the unknown function $\omega^{ab}(\sigma)$. We can calculate unknown metric tensor $g^{ab}(\sigma)$ by substitution of the solution of the consistence condition for ω^{ab} to the equation (6). The Jacobi identity

$$\{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\}_J \\ \equiv \{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\} + \{x^c(\sigma''), x^a(\sigma)\}x^b(\sigma')\} + \{x^b(\sigma'), x^c(\sigma'')\}x^a(\sigma)\} = 0 \quad (7)$$

is led to the following nonlocal analogy of the WDVV consistence condition:

$$\left[\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma) + \omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma) + \omega^{db}(\sigma')]\right]\nu(\sigma' - \sigma)\nu(\sigma'' - \sigma) \quad (8)$$
$$+\left[\frac{\partial\omega^{cb}(\sigma')}{\partial x^d}[\omega^{da}(\sigma') + \omega^{da}(\sigma)] - \frac{\partial\omega^{ab}(\sigma')}{\partial x^d}[\omega^{dc}(\sigma') + \omega^{dc}(\sigma'')]\right]\nu(\sigma - \sigma')\nu(\sigma'' - \sigma')$$
$$+\left[\frac{\partial\omega^{ac}(\sigma'')}{\partial x^d}[\omega^{db}(\sigma'') + \omega^{db}(\sigma')] - \frac{\partial\omega^{cb}(\sigma'')}{\partial x^d}[\omega^{da}(\sigma'') + \omega^{da}(\sigma)]\right]\nu(\sigma - \sigma'')\nu(\sigma' - \sigma'') = 0.$$

This equation has the particular solution of the following form:

$$\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma) + \omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma) + \omega^{db}(\sigma')] \\ = [T^b, T^c]T^a]f(\sigma, \sigma', \sigma'')\nu(\sigma'' - \sigma)\nu(\sigma' - \sigma). \quad (9)$$

T^a, $a = 0, 1, \ldots, D-1$, is the matrix representation of the simple Lie algebra and $f(\sigma, \sigma', \sigma'')$ is arbitrary function. The Jacobi identity is satisfied on the Jacobi identity of the simple Lie algebra in this case:

$$([T^a, T^b]T^c] + [T^c, T^a]T^b] + [T^b, T^c]T^a])f(\sigma, \sigma', \sigma'') = 0$$

and we used the relation $\nu^2(\sigma - \sigma') = 1$. The local solution of the Jacobi identities is led to the constant metric tensor. The rest Jacobi identities are cumbrous and we do not reduce this expressions here. The symmetric factor of σ, σ' of the antisymmetric functions $\nu(\sigma' - \sigma)$, $\frac{\partial}{\partial\sigma}\delta(\sigma - \sigma')$ in the right side of the PB can be both sum of the functions of σ and σ', and production of them. Last possibility can be used in the vielbein formalism.

Proposition 2. *The bi-Hamiltonity condition can be solved in the terms PB $\{\,,\,\}_2$, which have the following form:*

$$\{x^a(\sigma), x^b(\sigma')\}_2 = e^a_\mu(\sigma)e^b_\mu(\sigma')\nu(\sigma' - \sigma),$$
$$\{x^a(\sigma), p_b(\sigma')\}_2 = -e^a_\mu(\sigma)\frac{\partial e^c_\mu(\sigma'}{\partial x^b}p_c(\sigma')\nu(\sigma' - \sigma),$$
$$\{p_a(\sigma), p_b(\sigma')\}_2 = \frac{\partial e^c_\mu(\sigma)}{\partial x^a}p_c(\sigma)\frac{\partial e^d_\mu(\sigma')}{\partial x^b}p_d(\sigma')\nu(\sigma' - \sigma) - e^\mu_a(\sigma)e^\mu_b(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma) \\ + \left[\frac{\partial e^\mu_a}{\partial x^c}e^\mu_b - \frac{\partial e^\mu_b}{\partial x^c}e^\mu_a - \frac{\partial e^\mu_c}{\partial x^a}e^\mu_b + \frac{\partial e^\mu_c}{\partial x^b}e^\mu_a\right]x'^c(\sigma)\delta(\sigma' - \sigma), \quad (10)$$

where veilbein e^a_μ is satisfied on the additional conditions:

$$g^{ab} = \eta^{\mu\nu}e^a_\mu e^b_\nu, \qquad g_{ab} = \eta_{\mu\nu}e^\mu_a e^\nu_b$$

and $\eta^{\mu\nu}$ is the metric tensor of the flat space.

The particular solution of the Jacobi identity is

$$\frac{\partial e^a_\mu(\sigma)}{\partial x^d} e^b_\mu(\sigma') e^d_\nu(\sigma) e^c_\nu(\sigma'') - \frac{\partial e^a_\mu(\sigma)}{\partial x^d} e^c_{mu}(\sigma'') e^d_\nu(\sigma) e^b_\nu(\sigma')$$
$$= [T^b, T^c]T^a] f(\sigma, \sigma', \sigma'') \nu(\sigma'' - \sigma) \nu(\sigma' - \sigma). \tag{11}$$

As example let me consider the the constant curvature space.

Example 1. The constant curvature space is described by the metric tensor $g_{ab}(x(\sigma))$ and by it inverse tensor g^{-1}_{ab}:

$$g_{ab} = \eta_{ab} + \frac{k x_a x_b}{1 - kx^2}, \qquad g^{ab} \equiv g^{-1}_{ab} = \eta_{ab} - k x_a x_b.$$

Proposition 3. *Dual PB* $\{\,,\,\}_2$ *are:*

$$\{x_a(\sigma), x_b(\sigma')\} = [\eta_{ab} - k x_a(\sigma) x_b(\sigma')] \nu(\sigma' - \sigma),$$
$$\{x_a(\sigma), p_b(\sigma')\} = k x_a(\sigma) p_b(\sigma') \nu(\sigma' - \sigma),$$
$$\{p_a(\sigma), p_b(\sigma')\} = -k p_a(\sigma) p_b(\sigma') \nu(\sigma' - \sigma) \tag{12}$$
$$- \frac{1}{2}\left[2\eta_{ab} + \frac{k x_a x_b}{1 - kx^2}(\sigma) + \frac{k x_a x_b}{1 - kx^2}(\sigma')\right] \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) + \frac{x_a x'_b - x_b x'_a}{2(1 - kx^2)} \delta(\sigma' - \sigma).$$

The Jacobi identity (7) is led to the equation

$$[\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma')] \nu(\sigma' - \sigma) \nu(\sigma - \sigma'') + [\eta_{bc} x_a(\sigma) - \eta_{ba} x_c(\sigma'')] \nu(\sigma - \sigma') \nu(\sigma' - \sigma'')$$
$$+ [\eta_{ca} x_b(\sigma') - \eta_{cb} x_a(\sigma)] \nu(\sigma' - \sigma'') \nu(\sigma'' - \sigma) = 0.$$

The particular solution of this equation is:

$$\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma') = [T_b, T_c] T_a] f(\sigma, \sigma', \sigma'') \nu(\sigma'' - \sigma) \nu(\sigma' - \sigma). \tag{13}$$

Consequently, the space-time coordinate $x_a(\sigma)$ is the matrix representation of the simple Lie algebra. The Jacobi identity $\{x_a(\sigma), x_b(\sigma')\} p_c(\sigma'')\}_J$ is led to the equation

$$k \eta_{ab} p_c(\sigma'') \nu(\sigma' - \sigma) [\nu(\sigma'' - \sigma) + \nu(\sigma'' - \sigma')] = 0. \tag{14}$$

These results can be obtained from the veilbein formalism under the following ansatz for the veilbein of the constant curvature space:

$$e^{a(s)}_\mu = n_\mu \big(m^{(s)}_1 n^a + \sqrt{k} m^{(s)}_2 x^a\big), \qquad e^{\mu(s)}_a = n^\mu g_{ab} \big(m^{(s)}_1 n^b + \sqrt{k} m^{(s)}_2 x^b\big),$$

where $n^2_\mu = 1$, $m^{(s)}_1 m^{(s)}_1 = 1$, $m^{(s)}_2 m^{(s)}_2 = -1$, $m^{(s)}_1 m^{(s)}_2 = 0$, $n^a n^b = \delta^{ab}$ and (s) is number of the solution of the equations

$$e^a_\mu e^b_\mu = g^{ab}, \qquad e^\mu_a e^\mu_b = g_{ab}, \qquad e^a_\mu e^\mu_b = \delta^a_b.$$

The following example is time-dependent metric space.

Example 2. The time-dependent metric in the light-cone variables has form:

$$ds^2 = g_{ik}(x^+) dx^i dx^k + g_{++}(x^+) dx^+ dx^+ + 2 g_{+-} dx^+ dx^-.$$

We are used Poisson brackets (5) for the space coordinates $x^a = \{x^i, x^+, x^-\}$, $i = 1, 2, \ldots, D-2$. We introduced the light-cone gauge as two first kind constraints:

$$F_1(\sigma) = x'^+ \approx 0, \qquad F_2(\sigma) = p'_- \approx 0,$$

and we imposed them on the equations of motion and on the Jacobi identities. The Jacobi identities are reduced to the simple equation

$$\frac{\partial\omega^{ab}}{\partial x^+}\omega^{+c} - \frac{\partial\omega^{ac}}{\partial x^+}\omega^{+b} = 0.$$

We obtained following result from this equation and additional condition (6): there is constant background gravity field only for the non-degenerate metric.

2.1 Constant background fields

In this subsection we are supplemented the bi-Hamiltonity condition (4) by the mirror transformations of the integrals of motion.

$$\dot{x}^a = \left\{x^a, \int_0^\pi h_1 d\sigma\right\}_1 = \left\{x^a, \int_0^\pi \pm h_2 d\sigma'\right\}_{\pm 2}.$$

The dual PB are

$$\{x^a(\sigma), x^b(\sigma')\}_{\pm 2} = \pm g^{ab}\nu(\sigma' - \sigma), \qquad \{x^a(\sigma), p_b(\sigma')\}_{\pm 2} = 0,$$
$$\{p_a(\sigma), p_b(\sigma')\}_{\pm 2} = \mp\frac{1}{2}g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma).$$

The dual dynamical system

$$\dot{x}^a = \{x^a, \pm H_2\}_1 = \{x^a, H_1\}_{\pm 2}.$$

is the left(right) chiral string

$$\dot{x}^a = \pm x^{'a}, \qquad \dot{p}_a = \pm p_a^{'}.$$

In the terms of the Virasoro operators

$$L_k = \frac{1}{4\pi}\int_0^\pi (h_1 + h_2)e^{ik\sigma}d\sigma, \qquad \bar{L}_k = \frac{1}{4\pi}\int_0^\pi (h_1 - h_2)e^{ik\sigma}d\sigma,$$

the first kind constraints form the Vir $\oplus$ Vir algebra under the PB $\{\,,\,\}_1$.

$$\{L_n, L_m\}_1 = -i(n-m)L_{n+m}, \qquad \{\bar{L}_n, \bar{L}_m\}_1 = -i(n-m)\bar{L}_{n+m}, \qquad \{L_n, \bar{L}_m\}_1 = 0.$$

The dual set of the integrals of motion is obtained from initial it by the mirror transformations

$$H_1 \to \pm H_2, \qquad L_0 \to \pm L_o, \qquad \bar{L}_0 \to \mp\bar{L}_0, \qquad \tau \to \sigma.$$

and by the contraction of the first kind constraints algebra $L_n = 0$, or $\bar{L}_n = 0$, $n \neq 0$. Another way to obtain the dual brackets is the imposition of the second kind constraints on the initial dynamical system, by such manner, that $F_i = F_k$ for $i \neq k$, $i,k = 1,2,\ldots$ on the constraints surface $f(x^a, p_a) = 0$. The constraints $f_a^{(-)}(x,p) = p_a - g_{ab}x'^b \approx 0$ or $f_a^{(+)} = p_a + g_{ab}x'^b \approx 0$ (do not simultaneously) are the second kind constraints.

$$\{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 = C_{ab}^{(\pm)}(\sigma - \sigma') = \pm 2g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma' - \sigma).$$

The inverse matrix $(C^{(\pm)})^{-1}$ has following form $C^{(\pm)ab}(\sigma - \sigma') = \pm\frac{1}{2}g^{ab}\nu(\sigma' - \sigma)$. There is only one set of the constraints, because consistency condition

$$\{f^{(\pm)}(\sigma), H_1\}_1 = f^{'(\pm)}(\sigma) \approx 0, \quad \ldots, \quad \{f^{(\pm)(n)}(\sigma), H_1\}_1 = f^{(\pm)(n+1)}(\sigma) \approx 0.$$

is not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates

$$\{x^a(\sigma), x^b(\sigma')\}_D = \pm\frac{1}{2}g^{ab}\nu(\sigma'-\sigma), \qquad \{p_a(\sigma), p_b(\sigma')\}_D = \mp\frac{1}{2}g_{ab}\frac{\partial}{\sigma'}\delta(\sigma'-\sigma),$$
$$\{x^a(\sigma), p_b(\sigma')\}_D = \frac{1}{2}\delta^a_b\delta(\sigma'-\sigma).$$

The equations of motion under the Hamiltonians $H_1 = h_1$, $H_2 = h_2$ and Dirac bracket

$$\dot{x}^a = \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab}p_b = \pm x'^a,$$
$$\dot{p}_a = \{p_a, H_1\}_D = \{p_a, H_2\}_D = g_{ab}x'^b = \pm p'_a.$$

are coincide on the constraints surface. The dual brackets $\{\,,\,\}_{\pm 2}$ are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion $H_1 = H_2$ are coincide on the constraints surface too.

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