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Symmetry in Anisotropic Rotating Turbulence

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In the turbulence that takes place in rotating fluids, the symmetry assumptions of the theory of fully developed turbulence may not hold, in particular, isotropy must be relaxed to axial symmetry. This symmetry reduction leads one to consider the most general axisymmetric *viscosity tensor* for a Newtonian fluid, which can be obtained by group theory methods. This tensor generates new turbulent effective forces on large scales in addition to the molecular viscous force.

1 Introduction

The turbulent state of a fluid is characterized by the presence of fluctuations of the velocity in a wide range of scales. These irregular velocity fluctuations enhance the dissipation of energy in the fluid well over the dissipation pertaining to molecular viscosity. Therefore, one speaks of an "effective" viscosity that operates in the same range of scales in which the turbulence takes place. For a general reference on modern ideas about turbulence, see the book by Uriel Frisch [1].

If we assume that the Navier–Stokes equations for viscous fluids are suitable to describe turbulence, this phenomenon appears when the *nonlinear* term becomes dominant, in particular, with respect to the dissipative term proportional to the molecular viscosity. More formally, as the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ grows with the driving motion, the laminar flow becomes unstable, giving rise to irregular velocity fluctuations, which in turn enhance the dissipative term $\nu\nabla^2\boldsymbol{u}$, because it is very sensitive to the variations of the velocity (due to the second derivative).

The most interesting turbulent state is precisely the one with the maximal amount of nonlinearity, in which the velocity of the driving motion U (associated with a scale L) is very large in comparison with the molecular viscosity. In technical terms, this state is the solution of the equations as the Reynolds number $Re = UL/\nu \to \infty$, and it is called the state of *fully developed* turbulence. As the range of scales with velocity fluctuations grows with Re, in the $Re \to \infty$ limit it becomes formally infinite, so the fully developed turbulence is independent of the large scale dynamics associated with the driving motion and of the small scale dynamics associated with molecular dissipation and, in this sense, is "universal".

Symmetry assumptions are essential in the theory of fully developed turbulence. In fact, the classical theory of fully developed turbulence applies to maximally symmetric states, namely, invariant under time translations, on the one hand, and invariant under the space translation and rotation groups, on the other hand. So fully developed turbulence is a *stationary, homogenous and isotropic* state. Of course, these symmetries must be understood in a statistical sense, as symmetries of suitables averages of the velocity field (*correlation functions*), since the instantaneous value of this field is highly irregular.

These symmetry assumptions can be questioned; in particular, the assumption of isotropy has been questioned [2, 3, 1, 4], arguing that anisotropy can arise from a mean flow or from anisotropic driving. However, if we preserve homogeneity (space translation symmetry), the dependence of any property of the flow on its mean velocity, contradicts Galilean invariance. In contrast, an anisotropic driving is likely to produce anisotropy but contradicts the fact that the fully developed turbulence state is independent of the driving motion.

In this symmetry breaking context, we consider here the case of *rotating turbulence*. Turbulence in uniformly rotating fluids, which is called rotating turbulence, is an example of anisotropic turbulence and an area of active research [5, 6]. To be precise, in rotating turbulence the full rotation symmetry must necessarily break down to axial symmetry (the symmetry axis being the fluid rotation axis).

To study the breakdown of rotation symmetry, we shall first review the fluid equations in a rotating frame and introduce the viscosity in the standard manner [7] but *without* recourse to isotropy. This leads to the viscosity being defined by a four-rank tensor, the *viscosity tensor*. Isotropy is replaced by only symmetry around the rotation axis, given by the angular velocity $\boldsymbol{\Omega}$. Then all the components of the viscosity tensor are determined in terms of $\boldsymbol{\Omega}$ by group theory arguments [8]. Finally, from this tensor we obtain the effective anisotropic forces.

2 Dynamical equations of rotating turbulence

The hydrodynamical equations for a fluid with density $\rho(\boldsymbol{x}, t)$, velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $P(\boldsymbol{x}, t)$ in a rotating frame are [5, 7]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P - 2\boldsymbol{\Omega} \times \boldsymbol{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) + \boldsymbol{f}. \tag{2}$$

The upper one, namely, the continuity equation, expresses conservation of mass while the lower one expresses conservation of momentum and includes the force from the gradient of pressure, the Coriolis and centrifugal forces, and an additional force that accounts for both friction and an external homogeneous and isotropic random force (in fact, all the forces are rather accelerations, since they have been divided by the density).

Assuming that the fluid is incompressible, with constant density, the continuity equation becomes $\nabla \cdot \boldsymbol{u} = 0$. If we further define $p = P/\rho$, every reference to the density disappears from the momentum equation. Moreover, we can also eliminate p by projecting the solenoidal (or transverse) components, obtaining

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{P}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] = -\mathcal{P}(2\boldsymbol{\Omega} \times \boldsymbol{u}) + \boldsymbol{f}, \tag{3}$$

where the transverse projector is

$$\mathcal{P}_{ij} = \delta_{ij} - \partial_i \frac{1}{\nabla^2} \partial_j. \tag{4}$$

We call equation (3) the transverse rotating fluid equation. If we substitute for $\boldsymbol{f}$ an *isotropic* viscous force, it becomes the transverse rotating Navier–Stokes equation, but we do not need to be so restrictive, as we shall see. Note that the transverse rotating fluid equation (3) is *translation invariant* (assuming that $\boldsymbol{f}$ is homogeneous), in contrast to equation (2), since the centrifugal force has disappeared. Therefore, its solutions are homogeneous velocity fields and, furthermore, one can make use of the Fourier transform.

3 The effective viscosity tensor

In a fluid with no global motion, we can express the forces acting on the fluid element of volume (or mass) in terms of the stress tensor, which has the general expression

$$T_{ij} = -P\delta_{ij} + \sigma_{ij}, \tag{5}$$

where the *deviatoric* part σ_{ij} corresponds to the internal relative motion of fluid elements, producing the viscous forces $f_i = \partial_j \sigma_{ij}$. Naturally, σ_{ij} vanishes if there is no velocity gradient, $\partial_i \boldsymbol{u} = 0$; so for moderate internal relative motion we can assume that σ_{ij} is proportional to the velocity gradient, that is,

$$\sigma_{ij} = \tau_{ijmn} \partial_m u_n. \tag{6}$$

This relation characterizes the Newtonian fluids. Furthermore, one can make two assumptions [7]:

1. The components of the velocity gradient that are antisymmetric in its two indices (associated to the vorticity) are excluded; in other words, one only includes the symmetric components $u_{mn} = \partial_{(m} u_{n)}$.
2. The stress tensor can be taken to be symmetric in its two indices because of angular momentum conservation.

Under these conditions, we can write

$$\sigma_{ij} = \eta_{ijmn} u_{mn}, \tag{7}$$

defining a four-rank tensor with symmetry in the first and second pairs of indices. This tensor has already appeared in studies of anisotropic turbulence (see, e.g., Ref. [3]). Of course, the assumption of *isotropy* leads to the existence of only two proportionality constants, namely, shear and bulk viscosities, the latter playing no role in incompressible fluids.

However, in a rotating frame, neither isotropy nor the assumptions leading to the symmetry in the indices hold, so we must content ourselves with the bare proportionality provided by equation (6). In any event, we can decompose the four-rank tensor τ_{ijmn} into components with definite symmetry in the first and second pairs of indices; namely, we can make $\tau = \eta + \chi + \xi + \zeta$, where η has symmetry in the first and second pairs of indices, χ has symmetry in the first pair and antisymmetry in the second pair, ξ has antisymmetry in the first pair and symmetry in the second pair, and ζ has antisymmetry in both pairs.

4 General form of axisymmetric viscosity tensor and forces

4.1 Irreducible four-rank tensors with definite symmetry by pairs of indices

To derive the mathematical form of the decomposition $\tau = \eta + \chi + \xi + \zeta$ according to the symmetry of pairs of indices (and further decomposition still allowed), let us first work out the resolution of the general four-rank tensor T_{ijmn} into a sum of tensors of definite symmetry type given by standard Young tableaux. Young tableaux indicate certain symmetry operations performed on the indices [8]. We can consider the general four-rank tensor as a tensorial product of four vectors and, therefore, write its resolution as the Clebsch–Gordan decomposition for the linear group $GL(3)$ of the corresponding direct product:

$$\boxed{i} \otimes \boxed{j} \otimes \boxed{m} \otimes \boxed{n} = \begin{array}{|c|c|c|c|}\hline i&j&m&n\\\hline\end{array} \oplus \begin{array}{|c|c|c|}\hline i&j&m\\\hline n\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline i&j&n\\\hline m\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline i&m&n\\\hline j\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline i&m\\\hline j&n\\\hline\end{array}$$

$$\oplus \begin{array}{|c|c|}\hline i&j\\\hline m\\\cline{1-1} n\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline i&m\\\hline j\\\cline{1-1} n\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline i&n\\\hline j\\\cline{1-1} m\\\cline{1-1}\end{array} \tag{8}$$

Now, we must project the above linear irreducible representations into irreducible representations with definite symmetry by pairs of indices. We can do it in two stages, the first one achieving symmetry in the first pair and the second one achieving symmetry in the second pair. Their diagrammatic representation is (the subindices denote either symmetrization or antisymmetrization with respect to the corresponding indices)

1. First stage:

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array} \otimes \begin{array}{|c|}\hline m\\\hline\end{array} \otimes \begin{array}{|c|}\hline n\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline i&j&m&n\\\hline\end{array} \oplus \left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline i&j\\\hline \multicolumn{1}{|c|}{m}&\multicolumn{1}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{n}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \right]_{(ij)} . \tag{9}$$

2. Second stage:

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline m&n\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline i&j&m&n\\\hline\end{array} \oplus \left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \right]_{(ij),(mn)} \oplus \left[\begin{array}{|c|c|}\hline i&j\\\hline m&n\\\hline\end{array} \right]_{(ij),(mn)} , \tag{10}$$

$$\begin{array}{|c|c|}\hline i&j\\\hline\end{array} \otimes \begin{array}{|c|}\hline m\\\hline n\\\hline\end{array} = \left[\begin{array}{|c|c|c|}\hline i&j&m\\\hline \multicolumn{1}{|c|}{n}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline i&j&n\\\hline \multicolumn{1}{|c|}{m}&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \right]_{(ij),[mn]} \oplus \left[\begin{array}{|c|c|}\hline i&j\\\hline \multicolumn{1}{|c|}{m}&\multicolumn{1}{c}{}\\\cline{1-1} \multicolumn{1}{|c|}{n}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \right]_{(ij),[mn]} . \tag{11}$$

The tensor with antisymmetry in ij and symmetry in mn is analogous to the one with the reversed symmetries, given by equation (11). The tensor with antisymmetry in both pairs can be obtained straightforwardly.

4.1.1 Computation of Young tableaux

The actual derivation of the irreducible representations with definite symmetry by pairs of indices requires to compute the Young tableaux above and to perform the symmetry operations given by the subindices. It is simple linear algebra but somewhat cumbersome (for more details, see Ref. [9]). The results are (left subscripts indicate dimensions of representations):

- Part with symmetry by pairs.
  Starting from the general four-rank tensor T_{ijmn}, define

$$S_{ijmn} = T_{ijmn} + T_{jimn} + T_{ijnm} + T_{jinm}. \tag{12}$$

  Then, the computation of the Young tableaux of equation (10) yields:

  1. Pair-symmetric part:
     i) Totally symmetric part (first Young tableau):

$${}_{15}S_{ijmn} = S_{ijmn} + S_{mnij} + S_{jmin} + S_{imjn} + S_{injm} + S_{jnim}. \tag{13}$$

     ii) Remaining part (last Young tableau):

$${}_{6}S_{ijmn} = S_{ijmn} + S_{mnij} - \frac{1}{2}\left[S_{imjn} + S_{inmj} + S_{mjin} + S_{njmi}\right]. \tag{14}$$

  2. Pair-antisymmetric part (middle Young tableaux):

$${}_{15}S'_{ijmn} = S_{ijmn} - S_{mnij}. \tag{15}$$

- Part with symmetry in the first pair and antisymmetry in the second pair.
  Now define

$$SA_{ijmn} = T_{ijmn} + T_{jimn} - T_{ijnm} - T_{jinm} \,. \tag{16}$$

The computation of the Young tableaux of equation (11) yields:

1. First irreducible representation:

$$_{15}SA_{ijmn} = SA_{ijmn} + \frac{1}{2}\left[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}\right]. \tag{17}$$

2. Second irreducible representation:

$$_{3}SA_{ijmn} = SA_{ijmn} - \frac{1}{2}\left[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}\right]. \tag{18}$$

- Part with antisymmetry in the first pair and symmetry in the second pair: analogous to the preceding one.
- Part with antisymmetry by pairs.
  Define

$$A_{ijmn} = T_{ijmn} - T_{jimn} - T_{ijnm} + T_{jinm}. \tag{19}$$

For this last part it is not really necessary to use Young tableaux. One obtains:

1. Pair-symmetric part:

$$_{6}A_{ijmn} = A_{ijmn} + A_{mnij}. \tag{20}$$

2. Pair-antisymmetric part:

$$_{3}A_{ijmn} = A_{ijmn} - A_{mnij}. \tag{21}$$

4.2 Rotation and axial symmetry

4.2.1 Reduction of the four-rank tensor under the rotation group

The preceding linearly irreducible four-rank tensors are, however, reducible under the rotation group $O(3)$ (which is a subgroup of the linear group $GL(3)$). The standard reduction of linear tensors under the rotation group $O(3)$ is performed by extracting and removing traces [8]. Indeed, the $O(3)$ irreducible representations are the symmetric traceless tensors, labelled by their rank J. Therefore, it is possible to reexpress each of the linear representations corresponding to the above four-rank tensors as a linear combination of symmetric traceless tensors of equal or lower rank. For example, from $_{15}S'$ we obtain the symmetric tensor $T_{ljn} = \epsilon_{lim}S'_{ijmn} + \epsilon_{jim}S'_{ilmn} + \epsilon_{nim}S'_{ijml}$, which after removing its trace (vector) leads to the $J = 3$ representation. From $_{15}S'$ we also obtain the symmetric rank-two tensor $T_{ij} = \delta_{mn}S'_{ijmn}$. Similarly, from $_{15}SA$ we obtain the symmetric tensor $T_{pij} = \epsilon_{pmn}\,SA_{ijmn} + \epsilon_{imn}\,SA_{jpmn} + \epsilon_{jmn}\,SA_{pimn}$, and we can also obtain a symmetric rank-two tensor.

The full result of the reduction under rotations of the linear representations corresponding to four-rank tensors is (in symbolic form) [8]:

- $_{15}S = 4 \oplus 2 \oplus 0$, while $_{6}S$ or $_{6}A = 2 \oplus 0$.
- $_{15}S'$, $_{15}SA$ or $_{15}AS$ yield $3 \oplus 2 \oplus 1$.

- Of course, ${}_3SA$ or ${}_3AS$, ${}_3A = 1$.

Now that we have the decomposition of the viscosity tensor in $O(3)$-irreducible components, it is easy to derive its general axisymmetric form. In order to do it, let us note that the axisymmetric form can only depend on the scalar tensors δ_{ij} and ϵ_{ijk} (the totally antisymmetric tensor), and the vector marking the symmetry axis, that is, the angular velocity Ω_i. In particular, the latter allows us to construct an infinite sequence of symmetric tensors for every rank J, namely:

$J = 1$: Ω_i

$J = 2$: $\Omega_i\Omega_j$

$J = 3$: $\Omega_i\Omega_j\Omega_k$

$\dots \quad \dots \quad \dots$

These tensors must be combined with the scalar tensors according to the $O(3)$ Clebsh–Gordan decomposition above.

4.2.2 Axisymmetric viscosity tensor

The resulting axisymmetric viscosity tensor parts are:

$$\begin{aligned}
\eta^S_{ijmn} &= a_1\,(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\
&\quad + a_2\,(\Omega_i\Omega_j\delta_{mn} + \Omega_m\Omega_n\delta_{ij} + \Omega_i\Omega_m\delta_{jn} + \Omega_j\Omega_m\delta_{in} + \Omega_i\Omega_n\delta_{jm} + \Omega_j\Omega_n\delta_{im}) \\
&\quad + a_3\,\Omega_i\Omega_j\Omega_m\Omega_n + a_4\,\delta_{ij}\delta_{mn} + a_5\,(\Omega_i\Omega_j\delta_{mn} + \Omega_m\Omega_n\delta_{ij})\,, \qquad (22)\\
\eta^A_{ijmn} &= b_1\,\Omega_q(\epsilon_{qim}\delta_{jn} + \epsilon_{qin}\delta_{jm} + \epsilon_{qjm}\delta_{in} + \epsilon_{qjn}\delta_{im}) \\
&\quad + b_2\,\Omega_q(\epsilon_{qim}\Omega_j\Omega_n + \epsilon_{qin}\Omega_j\Omega_m + \epsilon_{qjm}\Omega_i\Omega_n + \epsilon_{qjn}\Omega_i\Omega_m) \\
&\quad + b_3(\Omega_i\Omega_j\delta_{mn} - \Omega_m\Omega_n\delta_{ij}), \qquad (23)\\
\chi_{ijmn} &= (c_1\delta_{ij} + c_2\Omega_i\Omega_j)\epsilon_{lmn}\Omega_l + c_3(\Omega_i\Omega_m\delta_{jn} + \Omega_j\Omega_m\delta_{in} - \Omega_i\Omega_n\delta_{jm} - \Omega_j\Omega_n\delta_{im}) \\
&\quad + c_4(\epsilon_{imn}\Omega_j + \epsilon_{jmn}\Omega_i), \qquad (24)\\
\zeta_{ijmn} &= d_1(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) + d_2(\Omega_i\Omega_m\delta_{jn} - \Omega_j\Omega_m\delta_{in} - \Omega_i\Omega_n\delta_{jm} + \Omega_j\Omega_n\delta_{im}) \\
&\quad + d_3(\epsilon_{imn}\Omega_j - \epsilon_{jmn}\Omega_i). \qquad (25)
\end{aligned}$$

The tensor ξ with antisymmetry in the first pair and symmetry in the second pair has been suppressed, for its form is analogous to χ's, with different coefficients, say $c'_1, \dots, c'_4$. The total number of coefficients is 19. Of course, we can let depend these scalar coefficients on Ω^2.

4.3 Effective forces

Once we have the general form of the axisymmetric viscosity tensor, we can obtain the general form of the axisymmetric effective forces, according to equation (6), by the relation

$$f_i = \partial_j\sigma_{ij} = \tau_{ijmn}\partial_{jm}u_n. \qquad (26)$$

Note that this equation implies that only the components of τ_{ijmn} symmetric in jm contribute to the force. Therefore, the number of independent axisymmetric components of the force is smaller than 19, that is, some of the 19 coefficients of the axisymmetric viscosity tensor are redundant, as we shall see.

Taking into account that $\nabla \cdot \boldsymbol{u} = 0$ and suppressing gradient terms (which do not contribute to the transverse rotating fluid equation), we obtain:

$$\begin{aligned}\boldsymbol{f} = {} & (a_1 - d_1)\nabla^2\boldsymbol{u} - b_1(\boldsymbol{\Omega} \times \nabla^2\boldsymbol{u}) - b_2(\boldsymbol{\Omega} \cdot \nabla)^2(\boldsymbol{\Omega} \times \boldsymbol{u}) - (b_2 + c_2')(\boldsymbol{\Omega} \cdot \nabla)(\boldsymbol{\Omega} \times \nabla)(\boldsymbol{\Omega} \cdot \boldsymbol{u}) \\ & + (b_2 + c_2)\boldsymbol{\Omega}(\boldsymbol{\Omega} \cdot \nabla)(\boldsymbol{\Omega} \cdot \boldsymbol{\omega}) + (c_4 + c_4' + d_3)(\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{\omega} + (a_2 - c_3 + c_3' - d_2)\boldsymbol{\Omega}\nabla^2(\boldsymbol{\Omega} \cdot \boldsymbol{u}) \\ & + (a_2 + c_3 - c_3' - d_2)(\boldsymbol{\Omega} \cdot \nabla)^2\boldsymbol{u} + a_3\boldsymbol{\Omega}(\boldsymbol{\Omega} \cdot \nabla)^2(\boldsymbol{\Omega} \cdot \boldsymbol{u}). \end{aligned} \tag{27}$$

Several remarks about this effective force are in order. Note that the fifth and sixth terms of the force involve the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and are proportional to an odd power of $\boldsymbol{\Omega}$. The terms preceding them are also proportional to an odd power of $\boldsymbol{\Omega}$, except for the first one, which is isotropic. These terms proportional to an odd power of $\boldsymbol{\Omega}$ would not be allowed if isotropy were broken by a polar vector (instead of $\boldsymbol{\Omega}$). More importantly, 5 of the 19 coefficients are missing after making $\nabla \cdot \boldsymbol{u} = 0$ and suppressing gradient terms, and some of the remaining ones are redundant: inspecting equation (27), we see that there are two redundant coefficients among c_4, c_4', d_3, two redundant coefficients among a_2, c_3, c_3', d_2, and one redundant coefficient among a_1, d_1.

After taking into account that $\nabla \cdot \boldsymbol{u} = 0$ and suppressing gradient terms, there remain only the coefficients of the part of τ that is traceless in the first pair and in the second pair of indices. Gradient terms are longitudinal and the physical force must be transverse (solenoidal); but, after removing these terms, the force is still non-transverse and must be projected with the nonlocal operator $\mathcal{P}$ of equation (4). This operation brings back two supressed gradient terms, namely, $\nabla(\boldsymbol{\Omega} \cdot \boldsymbol{\omega}) = 0$ and $\nabla[(\boldsymbol{\Omega} \cdot \nabla)(\boldsymbol{\Omega} \cdot \boldsymbol{u})]$, in addition to producing nonlocal gradient terms.

5 Conclusions

We have studied the consequences of the axial symmetry of a uniformly rotating fluid for its homogeneous turbulent state, focusing on the four-rank tensor defining the linear relation between the stress tensor and the velocity derivatives (called the "viscosity tensor"). This study has been carried out with general methods of the theory of tensorial group representations.

The most general four-rank viscosity tensor comprises five parts:

- a tensor η^S symmetric by pairs of indices and pair symmetric, accounting for the usual proportionality relation between (anisotropic) stress and strain rate;
- a tensor η^A symmetric by pairs and pair antisymmetric embodying a new relation between stress and strain rate, typical of rotating fluids, since it does not lead to dissipation;
- a tensor χ symmetric in the first pair of indices and antisymmetric in the second, which accounts for a stress tensor coupling to vorticity;
- a tensor ξ antisymmetric in the first pair of indices and symmetric in the second, which accounts for the antisymmetric part of the stress tensor (angular momentum non-conserving) that couples to the strain rate.
- a tensor ζ antisymmetric in both pairs of indices, which accounts for the antisymmetric part of the stress tensor that couples to the vorticity. This tensor can be further decomposed into pair-symmetric and pair-antisymmetric parts, like η.

The theory of tensorial group representations allows us to find the $O(3)$ Clebsch–Gordan decomposition of these tensors and, hence, its axisymmetric form.

This variety of components of the "viscosity tensor" is reflected in the various effective forces that arise from them. However, given that the force is a vector, its number of axisymmetric independent components is smaller and, moreover, it is even smaller after suppressing gradient terms; namely, it boils down to just nine independent components.

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