Symmetry of Stochastic Equations

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We discuss symmetries of Ito equations, and the relations of these with symmetries of the associated Fokker–Planck equation.

Symmetry methods are by now recognized as one of the main tools to attack *deterministic* differential equations (both ODEs and PDEs); see e.g. [3, 8, 15, 20, 21, 24, 25]. The situation is quite different for what concerns *stochastic* differential equations [1,12,13,16,19]: here, symmetry considerations are of course quite widely used by theoretical physicists (see e.g. [4, 14, 17] in the context of KPZ theory), but a rigorous and general theory comparable to the one developed for deterministic equation is still lacking, and the attention of the community working on symmetry methods should maybe be called to this.

We would like to quote here early work by Misawa [18], mainly concerned with conservation laws for stochastic systems, and later on the work by Arnold and Imkeller [1,2] for normal forms of stochastic equations.

In the following I will report on some work I have done – to a large extent in collaboration with N. Rodríguez Quintero – on symmetries of stochastic (Ito) equations, and how these compare with the symmetries of the associated diffusion (Fokker–Planck) equations [5, 7, 22, 23]; further details can be found in [9, 11].

It should be stressed that in this note an Ito equation will be seen as describing a one-particle stochastic process only. The same equation does also describe N-particles processes, or a random dynamical system [1], and in this case there will be further symmetries beyond those admitted in the frame of the one-particle processes interpretation, as discussed in [9].

1 Projectable symmetries

We assume the reader to be fairly familiar with the concept of symmetry of a deterministic differential equation, and the methods to characterize these; so we will just recall some formula of use in the following, mainly to fix notation, identify the class of transformations we are going to consider, and compare with the situation for stochastic equations.

Let us first consider the case of systems of first order ODEs in $\mathbb{R}^n$,

$$\dot{x}^i - f^i(x,t) = 0. \tag{1}$$

The general form of Lie-point vector fields is in this case, writing $\partial_i = \partial/\partial x^i$, $X = \tau(x,t)\partial_t + \xi^i(x,t)\partial_i$. Such a vector field, however, is somehow too general: indeed, we would like the change on t not to depend on the value assumed by the solution $x(t)$; this is particularly relevant when we think of the ODE as describing not the motion of a single particle in $\mathbb{R}^n$, but that of an assembly of particles. We would thus like to keep the special role of time, i.e. require $\tau = \tau(t)$; in other words, we will consider

$$X_0 = \tau(t)\partial_t + \xi^i(x,t)\partial_i. \tag{2}$$

A similar discussion holds for symmetries of PDEs, first or higher order. Here we will deal only with scalar PDEs, i.e. equations for a single dependent variable $u = u(x,t)$. In this case

we would write in general $X = \tau(x, t, u)\partial_t + \xi^i(x, t, u)\partial_i + \phi(x, t, u)\partial_u$. On physical terms, it is preferable to consider only transformations such that the independent variables are transformed independently of the values assumed by the dependent ones, i.e. such that τ and the ξ^i do not depend on u; also, here we are primarily interested in evolution equations, and in order to guarantee that the time keeps its distinguished role we should also ask that τ does not depend on x. Thus, in the end we want to consider transformations of the form

$$X = \tau(t)\frac{\partial}{\partial t} + \xi^i(x, t)\frac{\partial}{\partial x^i} + \phi(x, t, u)\frac{\partial}{\partial u}. \tag{3}$$

Definition 1. The vector fields – and in particular symmetry generators – of the form (2), (3) will be called "fiber-preserving", or simply **projectable**.

Let us now consider in particular a system of ODEs, which we write in the form (1). The determining equations for the symmetry generators of this are easily determined (these should be compared to the determining equations for stochastic differential equations discussed in the next section). We will use, here and below, the notation

$$\{f, \xi\}^i := (f^j \cdot \partial_j)\xi^i - (\xi^j \cdot \partial_j)f^i. \tag{4}$$

Lemma 1. *The projectable symmetries of* (2) *are given by vector fields* X_0 *as above with coefficients satisfying*

$$\partial_t(\xi^i - \tau f^i) + \{f, \xi\}^i = 0. \tag{5}$$

For given functions $f^i(x, t)$ the partial differential equations (5) always [20, 24] have non-trivial solutions $\tau(t)$ and $\xi^i(x, t)$; for example we can fix $\tau(t)$ and compute $\xi^i(x, t)$. Although an infinite number of symmetries of (5) exist, in general there is no constructive way to find them.

For further detail on symmetries of (deterministic) ODEs and PDEs, and applications, the reader is referred to [3, 8, 15, 20, 21, 24, 25].

2   Stochastic differential equations

Let us turn to stochastic equations [1, 6, 12, 13, 16, 19], and let us consider an Ito equation[1]

$$dx^i = f^i(x, t)\, dt + \sigma_k^i(x, t)\, dw^k, \tag{6}$$

where f and σ are smooth functions, $\sigma(x, t)$ is a nonzero matrix and the w^k are independent homogeneous standard Wiener processes, so that

$$\langle |w^i(t) - w^j(s)|^2 \rangle = \delta^{ij}(t - s).$$

The equation (6) should be seen as a map from the vector Wiener process $\boldsymbol{w}(t) = \{w^1(t), \dots, w^n(t)\}$ to the stochastic process undergone by $\{x^1(t), \dots, x^n(t)\}$, and its meaning is precisely that of defining the vector stochastic process $\boldsymbol{x}(t)$.

If we consider a function $y = \Phi(x)$, its evolution under the Ito equation (6) is described (again in terms of a stochastic equation) by the *Ito formula*:

$$\begin{aligned}
dy^i &= \frac{\partial \Phi^i}{\partial x^j}\, dx^j + \frac{1}{2}\frac{\partial^2 \Phi^i}{\partial x^j \partial x^k}\, dx^j dx^k \\
&= \left[f^j \partial_j \Phi^i + (1/2)\partial_{jm}^2 \Phi^i (\sigma\sigma^T)^{jm} \right] dt + \left[(\partial_j \Phi^i)\sigma^{jk} \right] dw^k.
\end{aligned} \tag{7}$$

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[1]It is well known that any Ito equation is equivalent to a Stratonovich equation; for the Ito equation (6), the associated Stratonovich equation is $dx^i = b^i(x, t)dt + \sigma_k^i(x, t) \circ dw^k$, with $b^i = f^i - (1/2)[\sigma(\partial\sigma^T/\partial x)]$. We will stick to the Ito formulation in all of this note.

The Fokker–Planck equation

$$\partial_t u = -\partial_i \left(f^i u \right) + \frac{1}{2} \partial^2_{ij} \left[(\sigma \sigma^T)^{ij} u \right] \tag{8}$$

is associated to the Ito equation (6) and describes the evolution of the probability measure $\rho = u(x,t)$ under the stochastic process described by (6). Equations (6) and (8) contain the same *statistical* information [1, 12, 13, 16, 19] as far as one-particle processes are concerned (but not if we think of the Ito equation as describing a random dynamical system, see [1,9]), provided σ satisfies the **non-degeneracy condition**

$$A := \frac{1}{2} \sigma \, \sigma^T \neq 0$$

which we will assume throughout this note.

We stress that we are interested in equation (8) as far as it describes the time evolution of the probability measure $\rho = u(x,t)$ under the stochastic process (6). It is obvious that for this interpretation $u(x,t)$ should be non-negative and subject to the condition

$$\int_{-\infty}^{+\infty} u(x,t) \, dx^1 \cdots dx^n = 1. \tag{9}$$

This is relevant in connection with the allowed transformations in the $(x,t;u)$ space: only transformations preserving this normalization do represent symmetries of the Fokker–Planck equation compatible with its probabilistic interpretation[2], and one should expect a correspondence between symmetries of the Ito equation and this subclass – rather than all – of the symmetries of the Fokker–Planck equation.

Another simple but relevant point which should be stressed is the following: different Ito equations which have the same f but different matrices σ can provide the same term $\sigma \sigma^T$ and thus the same Fokker–Planck equation.

A simple example is provided e.g. by σ orthogonal: in this case we have by definition $\sigma \sigma^T = I$, so that all the Ito equations with the same f and any orthogonal matrix σ give the same Fokker–Planck equation[3]. Similarly, σ and $\widetilde{\sigma} = \sigma B$, with B any orthogonal matrix, will give the same Fokker–Planck equation (and conversely σ and $\widetilde{\sigma}$ give the same Fokker–Planck – with the same f – only if there is an orthogonal matrix B such that the above relation is satisfied.

Thus if we consider a continuous variation of σ, say $\sigma + \varepsilon \gamma$, in the Ito equation, the associated Fokker–Planck equation remains unchanged provided $(\sigma + \varepsilon \gamma)(\sigma + \varepsilon \gamma)^T = \sigma \sigma^T$, which at order ε is simply

$$\sigma \gamma^T + \gamma \sigma^T = 0. \tag{10}$$

3   Symmetries of stochastic ODEs

3.1   Spatial symmetries

We will at first consider symmetries not acting on the time variable t, i.e. $X_0 = \xi^i(x,t) \partial_i$. Thus, we consider a near-identity change of coordinates, passing from x to y via

$$x^i \to y^i = x^i + \varepsilon \xi^i(x,t). \tag{11}$$

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[2]In other words, we should set the FP equation in the function space $\mathcal{F}$ defined by (9), and thus accept only transformations mapping $\mathcal{F}$ to itself.

[3]The one-point stochastic processes described by two different Ito equations having the same associated Fokker–Planck equation have the same statistical properties (the probability measures evolve in the same way), but *are* different: the same realization of the Wiener process $\boldsymbol{w}(t)$ leads to different sample paths.

Using the Ito formula, we easily check that at first order in ε, this transformation maps the Ito equation (6) into a new Ito equation

$$dy^i = \widetilde{f}^i(y,t)\,dt + \widetilde{\sigma}^i_k(y,t)dw^k, \tag{12}$$

where we have explicitly

$$
\begin{aligned}
\widetilde{f}^i &= f^i + \varepsilon\,[\partial_t \xi^i + f^j \partial_j \xi^i - \xi^j \partial_j f^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i],\\
\widetilde{\sigma}^i_k &= \sigma^i_k + \varepsilon\,[\sigma^j_k \partial_j \xi^i - \xi^j \partial_j \sigma^i_k].
\end{aligned}
\tag{13}
$$

When (11) maps (6) into itself, i.e. when (12) coincides with (6) [up to terms $o(\varepsilon)$], we say that $X_0 = \xi^i(x,t)\partial_i$ is a (Lie-point) spatial **symmetry** of (6). Thus, Lie-point spatial symmetries of (6) are identified by the vanishing of terms $O(\varepsilon)$ in (13). In the following statement, we use the notations introduced above for A and $\{\cdot,\cdot\}$.

Proposition 1. *The (generators of) Lie-point spatial symmetries of the Ito equation (6) are given by vector fields $X_0 = \xi^i(x,t)\partial^i_x$ with coefficients ξ^i satisfying the* determining equations for spatial symmetries *of an Ito equation:*

$$
\begin{aligned}
&\partial_t \xi^i + \{f,\xi\}^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i = 0,\\
&\{\sigma_k,\xi\}^i = 0.
\end{aligned}
\tag{14}
$$

These are $n + n^2$ equations for the $n+1$ functions ξ^i, τ; thus for $n > 1$ they have some solution only in very exceptional cases. This should not be surprising, as symmetry is a non-generic property.

Note that if $\sigma = \sigma(t)$ does not depend on the spatial variables, the second equation of (14) reduces to $\sigma^j_k \partial_j \xi^i = 0$, which in turns implies vanishing of the term $(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i$, which in this case can be rewritten as $\sigma^k_p \partial_k(\sigma^j_p \partial_j \xi^i)$. Thus, (14) are in this case equivalent to the determining equations for symmetries of the deterministic part of the Ito equation [see (5) and recall now we are assuming $\tau = 0$] with the additional condition $\sigma^j_k \partial_j \xi^i = 0$.

3.2 Spatio-temporal symmetries

We could of course consider transformations acting also on t: in this case the t transformation would also affect the processes $w(t)$, and some extra care should be paid. We will not go over the details (which consist in studying how $w(t)$ is changed and include this in the change undergone by σ), and just report the result obtained in [11], with the same notation as above.

Proposition 2. *The projectable vector field $X_0 = \tau(t)\partial_t + \xi^i(x,t)\partial^i_x$ is a symmetry generator for the Ito equation (6) if and only if the coefficients $\{\tau,\xi\}^i$ satisfy the* full determining equations for projectable symmetries *of an Ito equation:*

$$
\begin{aligned}
&\partial_t\left(\xi^i - \tau f^i\right) + \{f,\xi\}^i - A^{jk}\partial^2_{jk}\xi^i = 0,\\
&\{\sigma_k,\xi\}^i - \tau\partial_t\sigma^i_k - (1/2)\sigma^i_k\partial_t\tau = 0.
\end{aligned}
\tag{15}
$$

Note that the symmetries which are linear in x, i.e. such that $\xi^i(t;\boldsymbol{x}) = M^i_j(t)x^j$, are given by the same equations as for symmetries of the deterministic part of the Ito equation (6), i.e. (5), plus the additional condition $M^i_j \sigma^j_k = M^j_p x^p \partial_j \sigma^i_k + \tau\partial_t\sigma^i_k + (1/2)\sigma^i_k\partial_t\tau$.

A relevant case in applications is the one where $f^i = f^i(\boldsymbol{x})$ and $\sigma^i_k = \sigma^i_k(t)$ or even $\sigma^i_k = \text{const} = S^i_k$; this correspond to an autonomous dynamical system subject to a noise which depends only on t or even a constant noise. In this case (14) can be discussed quite completely. Indeed, with σ independent of the spatial coordinates, the second of these reads $2\sigma^j_k \partial_j \xi^i =$

$2\tau \partial_t \sigma_k^i + \sigma_k^i \partial_t \tau$. As the r.h.s. depends only on t, by differentiating with respect to x^m we get the equation $\sigma_k^j (\partial^2 \xi^i / \partial x^j \partial x^m) = 0$: i.e., the (symmetric) matrix of second derivatives of ξ^i, H_{jm}^i must be such that $\sigma H^i = H^i \sigma = 0$. Notice that in particular, if $\sigma_k^i(t) = \lambda_{(i)}(t)\delta_k^i$ with all $\lambda_{(i)} \not\equiv 0$ (or however if σ^{-1} exists), this means that ξ can be at most linear in the x.

4   Symmetries of the Fokker–Planck equation

We will now discuss the determining equations for projectable symmetry generators of the Fokker–Planck equation (8) in arbitrary spatial dimensions; general symmetries of the Fokker–Planck equations (with some limitations on σ) in one and two space dimensions have been completely classified [5, 7, 23].

It will be convenient to rewrite the Fokker–Planck equation as

$$u_t + A^{ij} \partial_{ij}^2 u + B^i \partial_i u + Cu = 0; \tag{16}$$

the coefficients A, B, C depend on x and t only and are given explicitly by[4]

$$A = -(1/2)\sigma\sigma^T, \qquad B^i = f^i + 2\partial_j A^{ij}, \qquad C(x,t) = (\partial_i \cdot f^i) + \partial_{ij}^2 A^{ij}. \tag{17}$$

We consider a projectable vector field of the form (3). Computing the second prolongation of this and applying it to the equation, we obtain the determining equations for symmetries of the Fokker–Planck equation.

It follows by general results on Lie-point symmetries of linear equations [8, 20], or by an elementary explicit computation, that:

Lemma 2. *The projectable symmetries of the Fokker–Planck equation* (16) *are given by vector fields in the form* (8) *with* $\phi = \alpha(x,t) + \beta(x,t)u$.

Again by linearity, we will have "trivial" symmetries $X_\alpha = \alpha(x,t)\partial_u$, with $\alpha(x,t)$ an arbitrary solution to the FP equation itself; this is just expressing the linear superposition principle [20]. Note that in order to preserve the normalization (9), we should moreover require $\int \alpha(x,t)dx = 0$, which rules out nontrivial non-negative solutions.

We will thus from now on focus on other, i.e. "nontrivial", symmetries only, and take $\alpha(x,t) = 0$.

Proposition 3. *The nontrivial projectable symmetries of the Fokker–Planck equation* (16) *are given by vector fields in the form* (3) *with* $\phi = \beta(x,t)u$, *where* τ, ξ^i, β *satisfy the* determining equations

$$\begin{aligned}
&\partial_t(\tau A^{ik}) + (\xi^m \partial_m A^{ik} - A^{im} \partial_m \xi^k - A^{mk} \partial_m \xi^i) = 0, \\
&\partial_t(\tau B^i) - [\xi_t^i + B^m \partial_m \xi^i - \xi^m \partial_m B^i] + (A^{ik} \partial_k \beta + A^{mi} \partial_m \beta) - A^{mk} \partial_{mk}^2 \xi^i = 0, \\
&\partial_t(\tau C) + \beta_t + A^{ik} \partial_{ik}^2 \beta + B^i \partial_i \beta + \xi^m \partial_m C = 0.
\end{aligned} \tag{18}$$

It should be stressed that in the computations leading to (18) we have not required preservation of the normalization $\int u\, dx = 1$; when we require this, it turns out that

Lemma 3. *The nontrivial projectable symmetries of the Fokker–Planck equation* (16), *in the form* (3) *with* $\phi = \beta(x,t)u$, *preserve the normalization* (9) *if and only if*

$$\beta = -\operatorname{div}(\xi). \tag{19}$$

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[4]Recall that our non-degeneracy assumption guarantees $A \neq 0$.

In other words, this lemma guarantees that if we are looking for the (nontrivial) symmetries of a Fokker–Planck equation compatible with its probabilistic interpretation, we do not have to deal with the general form (3) of the symmetry vector field, but we can use instead the ansatz

$$X = \tau(t)\partial_t + \xi^i(x,t)\partial_i - [\text{div}(\xi)u]\partial_u; \tag{20}$$

this substantially simplifies the analysis of symmetries.

5 Symmetries of the Ito versus symmetries of the associated FP equation

We are specially interested in discussing how the symmetries of the partial differential equation (16) and those of the symmetries of the system of stochastic ODEs (6) are related.

In order to do this, we should express the coefficients A, B, C in terms of f and σ; indeed $A = -(1/2)\sigma\sigma^T$, $B^i = f^i + 2\partial_k A^{ik}$, and $C = \partial_i f^i + \partial_{ik}^2 A^{ik}$. With the latter two of these and with some manipulations, see [11], (18) reads

$$\partial_t(\tau A^{ik}) + \left(\xi^m \partial_m A^{ik} - A^{im}\partial_m\xi^k - A^{km}\partial_m\xi^i\right) = 0,$$
$$\left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial_{mk}^2\xi^i\right] - 2\left[A^{ik}\partial_k\beta + A^{im}\partial_{mk}^2\xi^k\right] = 0,$$
$$\left[\partial_t + f^i\partial_i - A^{ik}\partial_{ik}^2\right][\beta + \partial_m\xi^m] = 0. \tag{21}$$

It is convenient to introduce the shorthand notations

$$\Gamma_j^k = \sigma_j^m\partial_m\xi^k - \xi^m\partial_m\sigma_j^k - \tau\partial_t\sigma_j^k - (1/2)\sigma_j^k\partial_t\tau,$$
$$\Lambda^i = -\left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial_{mk}^2\xi^i\right]. \tag{22}$$

With these, (21) is rewritten as

$$\sigma_j^i\Gamma_s^k\delta^{js} + \sigma_j^k\Gamma_s^i\delta^{js} = 0,$$
$$\Lambda^i + 2\left[A^{ik}\partial_k\beta + A^{im}\partial_{mk}^2\xi^k\right] = 0,$$
$$\left[\partial_t + f^i\partial_i - A^{ik}\partial_{ik}^2\right][\beta + \partial_m\xi^m] = 0. \tag{23}$$

It should be stressed that with the notation just introduced, the determining equations (15) for symmetries of the Ito equation (6) read simply

$$\Lambda^i = 0, \qquad \Gamma_j^i = 0. \tag{24}$$

With this notation, the comparison of symmetries for the Ito and the associated Fokker–Planck equation is quite a simple matter.

5.1 From the Ito to the FP symmetries

We will first investigate if symmetries of an Ito equation result in symmetries of the associated Fokker–Planck equation.

We first note that for $\Gamma = 0$, the first of (23) is satisfied. As for the second of (23), when $\Lambda^i = 0$ this reduces to

$$A^{ik}\partial_k\beta = -A^{im}\partial_{mk}^2\xi^k. \tag{25}$$

Obviously β is not present in symmetries of the Ito equation, so we can choose it as to satisfy the third of (23) and (25): for this it suffices to choose

$$\beta = -\partial_m \xi^m + c_0 = -\mathrm{div}(\xi) + c_0. \tag{26}$$

We know from Lemma 2 that actually we have to take $c_0 = 0$, and that this is actually the only choice of β providing us with a normalization-preserving symmetry of the FP. Our findings are summarized as follows:

Proposition 4. *Let $X_0 = \tau \partial_t + \xi^i \partial_i$ be a symmetry of the Ito equation* (6). *Then X_0 extends to a unique normalization preserving symmetry $X := X_0 - [(\mathrm{div}\xi)u]\partial_u$ of the associated Fokker–Planck equation.*

5.2   From the FP to the Ito symmetries

Let us now consider the converse question, i.e. if and when a symmetry of the FP equation associated to an Ito equation can be projected to a symmetry of the Ito equation itself.

As the probabilistic interpretation of the FP requires the normalization condition (9), we expect only symmetries identified by Lemma 3 can provide symmetries of related Ito equations.

Moreover, as recalled above, there are transformations which map an Ito equation into a different one with the same statistical properties and thus the same associated FP equation: these would be symmetries of the FP but not of the Ito equation. Thus, not all the symmetries of the FP should be expected to produce symmetries of the Ito equation.

By Lemma 3, we assume $\beta = -\mathrm{div}(\xi)$; this guarantees that the third of (23) holds, and also that the second of (23) reduces to $\Lambda^i = 0$. Hence we only have to discuss the relation between the first of (23) and the second of (24). These two equations can be rewritten, respectively, as

$$\sigma\Gamma^T + \Gamma\sigma^T = 0 \qquad \text{and} \qquad \Gamma = 0. \tag{27}$$

The second of these implies the first, but the converse is not true: hence there is *not* a complete equivalence.

This just corresponds to a given FP equation corresponding to different Ito equations: the transformations with $\Gamma \neq 0$ but $\sigma\Gamma^T + \Gamma\sigma^T = 0$ will be precisely those which map an Ito equation E_0 into a different Ito equation E_1 which has the *same* Fokker–Planck associated equation, see (10).

Proposition 5. *Let $X = \tau(t)\partial_t + \xi^i(x,t)\partial_i - [(\mathrm{div}\xi)u]\partial_u$ be a symmetry of the Fokker–Planck equation associated to the Ito equation* (6). *Then X maps the Ito equation into a (generally, different) Ito equation with the same statistical properties; X is a symmetry of the Ito equation* (6) *if and only if Γ defined in* (22) *satisfies $\Gamma^i_k = 0$.*

We may note that from the above system (24), restricting to the case where $f(t,x) = f(x)$ and $\sigma(t,x) = \sigma(x)$, we recover the results of Cicogna and Vitali [5] for the one-dimensional setting. From (22) it is also possible to recover the results of Shtelen and Stogny [23] for the two-dimensional Kramers equation, as well as the results of Finkel [7] for certain two-dimensional FP equations.

5.3   Examples

Example 1. As the first example, we consider the case $f(t,x) = 0$, $\sigma(t,x) = \sigma_0 = \mathrm{const} \neq 0$, i.e. the equation

$$dx = \sigma_0 \, dw(t) \tag{28}$$

which represents a free particle subject to constant noise. The corresponding Fokker–Planck equation is simply the heat equation $u_t = (\sigma_0^2/2)u_{xx}$. The symmetries of the heat equation (other than the trivial ones $v_\alpha = \alpha(x,t)\partial_u$ with $\alpha_t = \alpha_{xx}$) are well-known to be [20, 24]

$$v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = u\partial_u, \qquad v_4 = \sigma_0^2 t\partial_x - \sigma_0 xu\partial_u,$$
$$v_5 = 2t\partial_t + x\partial_x, \qquad v_6 = t^2\partial_t + xt\partial_x - (1/2)(t + x^2/\sigma_0^2)u\partial_u.$$

Of these, v_1, v_2 and v_5 (which do not act on u) are also symmetries of the Ito equation (28), as is easily checked using (14). Notice that (19) is satisfied for these, and is not satisfied for v_3, v_4 and v_6. The vector fields v_1, v_2 and v_5 do actually span the symmetry algebra of (28).

Example 2. As an example in two space dimensions [with coordinates $(x_1, x_2) = (x, y)$], we choose

$$dx = y\,dt,$$
$$dy = -k^2 y\,dt + \sqrt{2k^2}\,dw(t) \tag{29}$$

with k^2 a positive constant.

The corresponding Fokker–Planck equation is the Kramers equation

$$u_t = k^2 u_{yy} - yu_x + k^2 yu_y + k^2 u; \tag{30}$$

the symmetries of this were studied in [23] and, apart from the trivial ones v_α, are

$$v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = e^{-k^2 t}[k^{-2}\partial_x - \partial_y], \qquad v_4 = u\partial_u,$$
$$v_5 = t\partial_x + \partial_y - (1/2)(y + k^2 x)u\partial_u, \qquad v_6 = e^{k^2 t}[k^{-2}\partial_x + \partial_y - yu\partial_u].$$

Here v_1, v_2 and v_3 satisfy (19), while for v_4, v_5 and v_6 this is violated. According to our definition, the symmetries of the equations (29) are again v_1, v_2 and v_3, as easily checked.

Example 3. We consider now an example where the correspondence between normalization-preserving symmetries of the FP equation and symmetries of the Ito equation is not complete, i.e. the two-dimensional Ito system (with zero drift)

$$dx^1 = \cos(t)dw^1 - \sin(t)dw^2,$$
$$dx^2 = \sin(t)dw^1 + \cos(t)dw^2; \tag{31}$$

the corresponding Fokker–Planck equation is now just the two dimensional heat equation $u_t = (1/2)\triangle u$.

It is now immediate to check that the vector field $X_0 = \partial_t$ is a symmetry of the FP equation, but *not* a symmetry of (31). Obviously, the case of any orthogonal σ with $\partial_t \sigma_k^i \not\equiv 0$ will be exactly the same.

Example 4. Consider n uncoupled equations for equal "Langevin harmonic oscillators" subject to independent stochastic noises [16, 22]; this system is described by the Ito system

$$dx^i = -x^i\,dt + \sqrt{2s_i}\,dw^i, \qquad i = 1, 2, \ldots, n \tag{32}$$

(no sum on i) where we assume all the s_i are strictly positive; the corresponding Fokker–Planck equation is

$$\partial_t u = \sum_{i=1}^n [s_i \partial_{ii}^2 u + x^i \partial_i u + u].$$

In this case we get the symmetries ($i = 1, 2, \ldots, n$)

$$v_1 = \partial_t, \qquad v_2 = e^{-2t} \left[\partial_t - \sum_{i=1}^{n} x^i \partial_i + nu \partial_u \right], \qquad v_{q_i} = e^{-t} \partial_i.$$

Example 5. Finally, we consider now the n-dimensional nonlinear case

$$dx^i = -(1 - \lambda \|x\|^2) x^i \, dt + dw^i, \tag{33}$$

where $\|x\|$ is the norm of the vector $\boldsymbol{x}$ and $\lambda \neq 0$. By inserting $f^i = -x^i(1 - \lambda\|x\|^2)$ and $\sigma_j^i = \delta_{ij}$ in (24) we obtain, from the second of these, $\xi^i = h^i(t) + (\tau_t/2)x^i$. Inserting this into the first of (24) and isolating the coefficients of different powers of x, we get that the only symmetry is given by $v = \partial_t$.

Note that we got no rotation symmetry; this is because within the class of transformations we are considering we can rotate the vector x, but *not* the vector Wiener process $w(t)$; transformations allowing to rotate $w(t)$ as well will be considered in the next section.

6   W-symmetries

We will now consider symmetries involving not only the spatial and time variables (x, t), but also the vector Wiener processes $w(t)$ entering in the n-dimensional Ito equation (6). We will specifically consider infinitesimal transformations of the form

$$\begin{aligned}
&x^i \rightarrow y^i = x^i + \varepsilon \xi^i(x, t), \\
&t \rightarrow s = t + \varepsilon \tau(t), \\
&w^i \rightarrow z^i = w^i + \varepsilon \mu^i(w, t).
\end{aligned} \tag{34}$$

We also call symmetry generators of this form, "*W-symmetries*".

Note that this is not the most general possible form of a transformation for the variables involved in (1); some words on this restriction are in order here.

The restriction on τ and ξ is the same as considered above. Moreover, we only allow "internal" transformations of w, i.e. they cannot depend on $x(t)$. As we think of the stochastic process $w(t)$ to be independent of the evolution of the $x(t)$, we like its transformation not to depend on the latter. Finally, we have allowed the transformation on the spatial coordinates $x(t)$ to depend on x and t, but not on the w; this means that we do not want to consider transformations of the spatial coordinates which depend on the realization of the stochastic process $w(t)$. This again is somewhat a natural requirement in physical terms[5].

In order to compute how (34) acts on the Ito equation (6), we use the Ito formula (7). Moreover, due to the form of (34), we can adopt a "two-steps procedure", see [9]: the transformations of the (x, t) variables and of the stochastic process $w(t)$ do not interfere with each other. It turns out that some strong limitation on the functions μ^i arise.

Lemma 4. *The infinitesimal transformation $w \rightarrow z = w + \varepsilon\mu(w, t)$ maps the vector Wiener process $w(t)$ into a vector Wiener process $z(t)$ if and only if $\mu = Bw$, with B a real antisymmetric matrix. Equivalently, if and only if $z = Mw$ with M orthogonal.*

Note that as B is antisymmetric matrix, it vanishes in dimension one: for one-dimensional Ito equations we have no new symmetries with respect to those discussed above.

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[5]It should be however stressed that in this way we are also discarding the transformations needed to obtain *normal forms* of stochastic differential equations, see [1, 2] and references therein.

Proposition 6. *Under the infinitesimal transformation* (34), *with B a real antisymmetric matrix, the Ito equation* (6) *is changed into a (generally, different) Ito equation. The Ito equation* (6) *is invariant under* (34) *if and only if τ, ξ, μ satisfy the determining equations for W-symmetries:*

$$\partial_t \xi^i + \{f, \xi\}^i - \partial_t(\tau f^i) + A^{jk}\partial^2_{jk}\xi^i = 0,$$
$$\{\sigma_k, \xi\}^i - \tau\partial_t\sigma^i_k - (1/2)(\partial_t\tau)\sigma^i_k - \sigma^i_p B^p_k = 0. \tag{35}$$

If we just consider a transformation $w \to z = Mw$, with M orthogonal as required by Lemma 4, the Fokker–Planck equations associated to the Ito equation and to the transformed one will be the same. Thus all the discussion conducted in Section 5 about the relations betwen symmetries of an Ito equation and of the associated FP equation also applies to transformations of the form (34)[6].

Example 6. Let us consider again the two-dimensional Ito equation (29). If we look at the second set of determining equations and single out the one for $i = 2$ and $k = 1$, we have immediately that $B = 0$, i.e. in this case we have no new symmetry by allowing transformation of w.

Example 7. Let us consider again the two-dimensional Ito system (31) We will write $\xi^1 = \alpha$, $\xi^2 = \beta$, and $B_{12} = -B_{21} = b$. The most general solution to the determining equations turns out to be

$$\alpha = [\tau_t/2 - \sin(2t)\tau]x + [b + \cos(2t)\tau]y + \alpha_0(t),$$
$$\beta = -[b + \tau]x + [\tau_t/2]y + \beta_0(t).$$

By setting $\tau = \alpha_0 = \beta_0 = 0$, we obtain a new symmetry, corresponding to $\alpha = by$ and $\beta = -bx$ with b arbitrary, i.e. given by

$$X = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left(w^2\frac{\partial}{\partial w^1} - w^1\frac{\partial}{\partial w^2}\right).$$

Note this is nothing else than a simultaneous (and identical) rotation in the (x, y) and in the (w_1, w_2) planes.

Example 8. We will now consider again the case of n uncoupled equal "Langevin harmonic oscillators", see (32) above.

If we discard symmetries with $B = 0$ (i.e. those obtained above), we are still left with a nontrivial possibility, i.e. B an arbitrary (real, antisymmetric) constant matrix and $\xi^i = C^i{}_k x^k$, where $C^i{}_k = (s_i/s_k)B^i_k$ (no sum on i, k here).

The meaning of this result is obvious: we can act on this system by an arbitrary $SO(n)$ rotation in the w space and a related rotation in the x space.

If we assume $s_1 = s_2 = \cdots = s_n = s$, we have indeed $C = B$ and the rotations in w and x spaces do just coincide. For general $s_1, \ldots, s_n$, the (s_i/s_k) factors relating the C and B matrices are also easily understood: we could rescale each of the x^i by a factor $\sqrt{2s_i}$, $x^i = \sqrt{2s_i}y^i$ (no sum on i), arriving at the manifestly rotationally invariant n-dimensional Ito equation $dy^i = -y^i dt + dw^i$.

Example 9. Let us consider again (33). We easily check that it is rotationally invariant with a W-symmetry: take $\tau = 0$ and $\xi^i = B^i_k x^k$, so that

$$\{\xi, f\}^i = \{Bx, (1 - \lambda||x||^2)x\} = -2\lambda x^i \left(x^j B^j_k x^k\right) = 0,$$

the last equality following from $B = -B^T$. That is, (33) is indeed symmetric under simultaneous identical rotations of the x and w vectors.

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[6]Needless to say, for more general transformations, for which the transformations of the (x, t) variables and of the vector Wiener process $w(t)$ do interfere with each other, this would not be the case.

7   Discrete symmetries

We will now briefly consider, for the sake of completeness, *discrete* symmetries of stochastic differential equations. Similarly to what happens in the deterministic case (see e.g. [10]), the resulting determining equations are in general too difficult to be attacked except for very simple classes of transformations (e.g. reflections, or however linear ones). On the other side, they can be used to check if a given discrete transformation is a symmetry of a given stochastic differential equation; and/or to determine the stochastic differential equations which admit a given discrete transformation as a symmetry.

We consider again an Ito equation of the form (6); here we will not consider transformations acting on the t coordinate, and will thus limit to consider the change of coordinates in the x and w spaces given by

$$y^i = \phi^i(x,t), \qquad w^k(t) = R^k_{\ p}(t) z^p(t). \tag{36}$$

As required by Lemma 4, we take $R \in O(n)$. We write as usual $A = -(1/2)\sigma\sigma^T$.

With the Ito formula we have at once that y obeys the Ito equation

$$dy^i = \left[\frac{\partial \phi^i}{\partial x^j} f^j - A^{jk} \frac{\partial^2 \phi^i}{\partial x^j \partial x^k} + \frac{\partial \phi^i}{\partial t} \right] dt + \left[\frac{\partial \phi^i}{\partial x^j} \sigma^j_{\ p} R^p_{\ k} \right] dz^k. \tag{37}$$

Proposition 7. *The transformation* (34) *is a symmetry of the Ito equation* (6) *if and only if* ϕ, R *satisfy the* determining equations for discrete symmetries *of an Ito equation:*

$$\frac{\partial \phi^i(x,t)}{\partial x^j} f^j(x,t) + A^{jk}(x,t) \frac{\partial^2 \phi^i(x,t)}{\partial x^j \partial x^k} + \frac{\partial \phi^i(x,t)}{\partial t} = f^i(\phi(x,t),t),$$
$$\frac{\partial \phi^i(x,t)}{\partial x^j} \sigma^j_{\ p}(x,t) R^p_{\ k} = \sigma^i_{\ k}(\phi(x,t),t). \tag{38}$$

Example 10. The simplest case of discrete transformation is provided by $\phi^i(x,t) = -x^i$, $R = \pm I$. In these cases, (38) reduce to

$$f^i(x,t) = -f^i(-x,t), \qquad \sigma(x,t) = \mp \sigma(-x,t). \tag{39}$$

Example 11. For the n independent "Langevin oscillators", see (32), it is immediate to check that (39) is satisfied (for $R = -I$). More generally, (38) read now (no sum over i, sum over j when repeated)

$$a_{(j)} \frac{\partial \phi^i}{\partial x_j} - \frac{s^2_{(j)}}{2} \frac{\partial^2 \phi^i}{\partial x^j \partial x^j} = a_{(i)} \phi^i, \qquad s_{(j)} \frac{\partial \phi^i}{\partial x_j} R^j_{\ k} = s_{(i)} \delta^i_{\ k}. \tag{40}$$

Example 12. For equations with $\sigma^i_{\ k}(x,t) = s_0 \delta^i_{\ k}$ (with s_0 a real constant), the second of (38) always reduces to $(\partial \phi^i/\partial x^j) = (R^{-1})^i_{\ j}$. But this implies ϕ must be a linear function of the x, $\phi^i(x,t) = L^i_{\ j}(t) x^j$ with $L \in O(n)$ an orthogonal matrix ($L = R^{-1}$). Thus in this case we are always reduced to the simple equation $L^i_{\ j} f^j(x,t) = f^i(Lx,t)$.

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