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Associative Algebras, Punctured Disks and the Quantization of Poisson Manifolds

Domenico FIORENZA and Riccardo LONGONI

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", P.le A. Moro 2, I-00185 Roma, Italy
E-mail: *fiorenza@mat.uniroma1.it, longoni@mat.uniroma1.it*

The aim of the note is to provide an introduction to the algebraic, geometric and quantum field theoretic ideas that lie behind the Kontsevich–Cattaneo–Felder formula for the quantization of Poisson structures. We show how the quantization formula itself naturally arises when one imposes the following two requirements to a Feynman integral: on the one side it has to reproduce the given Poisson structure as the first order term of its perturbative expansion; on the other side its three-point functions should describe an associative algebra. It is further shown how the Magri–Koszul brackets on 1-forms naturally fits into the theory of the Poisson sigma-model.

1 Deformation quantization as a Feynman diagrams expansion

A Poisson manifold is a differentiable manifold M endowed with a bi-vector $\alpha \in \Gamma(M; TM \wedge TM)$ such that $[\alpha, \alpha] = 0$, where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket (see e.g. [7]). The bi-vector α defines a Poisson algebra structure on the space of smooth functions on M by

$$\{f, g\} := \langle \alpha \,|\, \mathrm{d}f \wedge \mathrm{d}g \rangle.$$

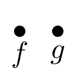
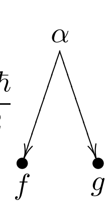
The problem of deformation quantization of the given Poisson structure is that of finding an associative $\star$-product on $C^\infty(M)[[\hbar]]$ deforming the usual pointwise product on $C^\infty(M)$ and having the Poisson bracket as the first order term in $\hbar$:

$$(f \star g)(x) = f(x)g(x) + \frac{\mathtt{i}\hbar}{2}\{f, g\}(x) + O(\hbar^2), \tag{1}$$

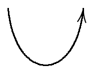
or, more generally,

$$(f \star g)(x) = f(x)g(x) + \frac{\mathtt{i}\hbar}{2}\big(\{f, g\} + B(f, g)\big)(x) + O(\hbar^2), \tag{2}$$

where B is a symmetric bi-differential operator. This problem has been solved by M. Kontsevich [3], and his solution was then interpreted in the language of quantum field theories by A. Cattaneo and G. Felder [2]. These notes are an attempt to explain why the Cattaneo–Felder model naturally arises when one tries to look at (1) as the perturbative expansion of a Feynman integral:

$f \star g =$  $+ \quad \dfrac{\mathtt{i}\hbar}{2}$  $+ O(\hbar^2).$

We see from this formula that there are two types of vertices, namely the ones labelled by the functions f, g and the ones labelled by the bi-vector α, and that the propagator is



$$= \mathrm{d}x^i \otimes \partial_i,$$

where ∂_i is a shorthand notation for $\partial/\partial x^i$. By the above description, we see that our fields are tangent and cotangent vectors at x; moreover, in order to look at α as to a function of the fields, we have to consider the cotangent vectors as odd fields, i.e., the coordinates η_i of a cotangent vector η are anticommuting variables. Therefore, the natural choice for the space of fields is $T_xM \oplus \Pi T_x^*M$, endowed with the natural pairing $\langle \partial_i | \mathrm{d}x^j \rangle = \delta_i^j$.

The functions f and g and the Poisson bi-vector α can be seen as functions on the space of fields, by using the Taylor expansions:

$$\begin{aligned}
f(\xi,\eta) &:= f(x+\xi) = f(x) + \partial_i f(x)\xi^i + \frac{1}{2}\partial_i\partial_j f(x)\xi^i\xi^j + \cdots\,, \\
g(\xi,\eta) &:= g(x+\xi) = g(x) + \partial_i g(x)\xi^i + \frac{1}{2}\partial_i\partial_j g(x)\xi^i\xi^j + \cdots\,, \\
\alpha(\xi,\eta) &:= \langle \alpha(x+\xi) | \eta\wedge\eta\rangle = \alpha^{ij}(x)\eta_i\eta_j + \partial_k\alpha^{ij}(x)\eta_i\eta_j\xi^k + \cdots\,,
\end{aligned}$$

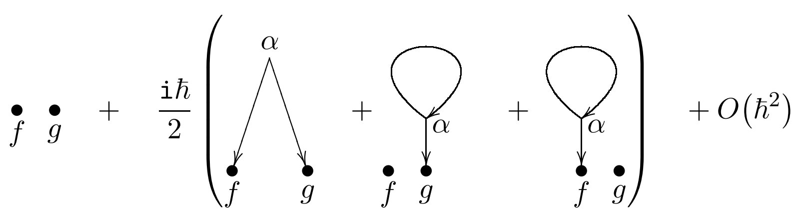
where $\xi \in T_xM$ and $\eta \in \Pi T_x^*M$. Now consider

$$\int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, f(x+\xi)\, g(x+\xi)\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)} \Bigg/ \int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\langle\xi|\eta\rangle}, \tag{3}$$

where the action is

$$S(\xi,\eta) = S_{\text{free}}(\xi,\eta) + S_{\text{int}}(\xi,\eta) := \langle \xi|\eta\rangle + \langle \alpha(x+\xi)|\eta\wedge\eta\rangle.$$

By the usual Feynman rules, the perturbative expansion of (3) is



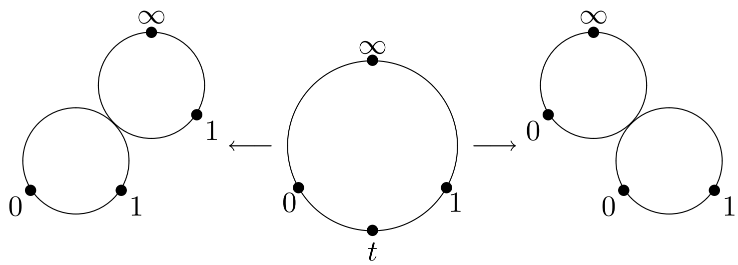
which is of the form (2). Note that, if α is constant as a function of $x \in M$, then the perturbative expansion of (3) is

$$(f \star g)(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n \alpha^{i_1j_1}\cdots\alpha^{i_nj_n}\partial_{i_1}\cdots\partial_{i_n} f(x)\,\partial_{j_1}\cdots\partial_{j_n} g(x)$$

which is precisely the Moyal $\star$-product formula. However, for general α, formula (3) does not yield an associative $\star$-product. A way to remedy this is to consider a topological space whose geometry describes the structure of associative algebras, and pull back our integral onto this space.

2 Punctured disks and associative algebras

Let D be the unit complex disk, and let B_n be the moduli space of $(n+1)$ points on the boundary of D, for $n \geq 2$. The disk D is identified with the complex upper half plane and its boundary with $\mathbb{R} \cup \{\infty\}$. Since the group of the biholomorphisms acts 3-transitively on the set of boundary points on D, we can fix three of them to be 0, 1 and ∞, and make all the others lie in the interval $(0, 1)$. Therefore B_n is just the open $(n-2)$-dimensional simplex $0 < t_1 < \cdots < t_{n-2} < 1$. One can define a compactification $\overline{B}_n$ of B_n by adding products of $B_{n'}$, $n' < n$; these new boundary components correspond to the collapsing of two or more points in the boundary. For instance, there are two boundary components in $\overline{B}_3$ corresponding to the degenerations as $t = t_1$ goes to 0 or to 1.



Now, we look at B_2 as to an operation m_2 with two inputs (the points 0 and 1) and one output (the point ∞). Note that the two boundary components of $\overline{B}_3$ correspond to the two ways of composing m_2 with itself, namely $m_2(m_2 \otimes \mathrm{id})$ and $m_2(\mathrm{id} \otimes m_2)$. So, if we find a continuous family of operations $m_3(t)$, $t \in (0, 1)$, with three inputs and one output, which extends to the compactification $\overline{B}_3$ (in a way compatible with the product structure of the boundary), then the associativity of m_2 is equivalent to $m_3(0) = m_3(1)$. If moreover $m_3(t)$ is differentiable, this is equivalent to

$$m_2 \text{ associative} \quad \Leftrightarrow \quad \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}m_3(t)}{\mathrm{d}t} = 0.$$

Remark 1. In the language of operads, the above discussion corresponds to the well-known fact that the chain complex $C_*(\overline{B}_n)$ is the operad governing A_∞ algebras. In particular one says that m_2 is associative only up to the homotopy m_3.

Now, we want to define m_2 and m_3 on the space of smooth functions on the Poisson manifold M, in such a way that m_2 is related to equation (3). The most natural choice is to consider the "expectation value" over the maps $X \colon D \to M$ of the product $f(X(0))\, g(X(1))\, h(X(\infty))$ w.r.t. some measure to be defined, and "raise" the indices, i.e., set h to be the Dirac delta function δ_x. In other words we are looking for an operation m_2 of the form

$$m_2(f, g)(x) = \int \mathrm{d}\mu(X)\, f(X(0))\, g(X(1))\, \delta_x(X(\infty)). \tag{4}$$

As for $m_3 = m_3(t)$, we set

$$m_3(f, g, h)(x) = \int \mathrm{d}\mu(X)\, f(X(0))\, g(X(t))\, h(X(1))\, \delta_x(X(\infty))$$

so that the associativity of m_2 becomes

$$\int \mathrm{d}\mu(X) \int_0^1 \mathrm{d}t \left(f(X(0))\, \frac{\mathrm{d}g(X(t))}{\mathrm{d}t}\, h(X(1))\, \delta_x(X(\infty)) \right) = 0. \tag{5}$$

3 The Poisson sigma-model

In this Section we want to combine equation (4), which defines an associative product, with equation (3), which has the correct first term in its perturbative expansion. First, the measure

$\mathrm{d}\mu(X)$ in equation (4) should be of the form $(1/C)\,\mathrm{d}\xi\mathrm{d}e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}$ as in equation (3), where C is a suitable normalization constant. In order to accomplish this, a new field, denoted by η, has to be introduced: it has to be defined on the disk and take values in ΠT^*_xM. Moreover, since the new action S will be an integral over D, it is natural to take $\eta \in \Omega^1(D; X^*(\Pi T^*M))$. We are therefore led to consider the following object

$$\int \mathrm{d}X\mathrm{d}\eta\, f(X(0))\, g(X(1))\delta_x(X(\infty))\, e^{\frac{\mathrm{i}}{\hbar}S(X,\eta)} \Big/ \int \mathrm{d}X\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\int_D\langle \mathrm{d}X|\eta\rangle}, \tag{6}$$

where $S(X,\eta) = \int_D\langle \mathrm{d}X|\eta\rangle + \frac{1}{2}\int_D\langle\alpha(X)|\eta\wedge\eta\rangle$.

Notice however that in equation (3), we have a tangent vector $\xi \in T_xM$, where x is some point in M. Hence, what we should consider are infinitesimal variations of the map X around the constant map $X \equiv x$. In other terms, in equation (6) we have to replace X with $x+\xi$ where $\xi \in \Omega^0(D; X^*(TM))$.

Since the map X at the point ∞ is fixed to be equal to x by the term $\delta_x(X(\infty))$, we have to impose the boundary condition $\xi(\infty) = 0$; finally the 1-form η is required to vanish on tangent vectors to the boundary of the disk D. The action now reads

$$S(\xi,\eta) = \int_D\langle \mathrm{d}\xi|\eta\rangle + \frac{1}{2}\int_D\langle\alpha(x+\xi)|\eta\wedge\eta\rangle,$$

and we define

$$(f\star g)(x) := \frac{\displaystyle\int \mathrm{d}\xi\mathrm{d}\eta\, f(x+\xi(0))\, g(x+\xi(1))\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}}{\displaystyle\int \mathrm{d}\xi\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\int_D\langle \mathrm{d}\xi|\eta\rangle}}. \tag{7}$$

In order to perform the perturbative expansion of (7), symmetries of the action have to be taken into account. A systematic way of doing this is via the superfield formalism, namely we consider the superdisk $D^{2|2}$ with even coordinates u^1, u^2 and Grassmann coordinates θ^1, θ^2 and set

$$\tilde{\xi}^i = \xi^i + \eta^{+\,i}_{\mu}\theta^\mu + \frac{1}{2}\beta^{+\,i}_{\mu\nu}\theta^\mu\theta^\nu, \qquad \tilde{\eta}_i = \beta_i + \eta_{i\,\mu}\theta^\mu + \frac{1}{2}\xi^{+}_{i\,\mu\nu}\theta^\mu\theta^\nu.$$

The de Rham differential now reads $\mathrm{D} = \theta_\mu\frac{\partial}{\partial u^\mu}$ and the $\star$-product becomes

$$(f\star g)(x) = \frac{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, f(x+\tilde{\xi}(0))\, g(x+\tilde{\xi}(1))\, e^{\frac{\mathrm{i}}{\hbar}S(\tilde{\xi},\tilde{\eta})}}{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, e^{\frac{\mathrm{i}}{\hbar}\int_{D^{2|2}}\langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle}}, \tag{8}$$

where the superaction is

$$S(\tilde{\xi},\tilde{\eta}) := \int_{D^{2|2}}\langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle + \frac{1}{2}\int_{D^{2|2}}\langle\alpha(x+\tilde{\xi})|\tilde{\eta}\wedge\tilde{\eta}\rangle. \tag{9}$$

Notice that besides of the original fields ξ, η (and their "antifields" ξ^+, η^+), a new field β has appeared, which can be interpreted as an infinitesimal symmetry of the original action (see Remark 4 below).

The advantage of this reformulation of the Poisson sigma-model is that we can now apply the Batalin–Vilkovisky formalism and deform the subspace $\xi^+ = \eta^+ = \beta^+ = 0$ over which the integration is performed, in such a way that the perturbative expansion is well defined.

4 Batalin–Vilkovisky formalism

We recall that for any vector space V, the space of functions on $V \oplus \Pi V^*$ is naturally endowed with a BV algebra structure [1, 6]. Using the standard terminology, we call *fields* the coordinates v^i on V and *antifields* the coordinates v_i^+ on ΠV^*. The BV bracket between two functionals $f, g\colon V \oplus \Pi V^* \to \mathbb{R}$ is given by

$$(f, g) := \frac{\overleftarrow{\partial} f}{\partial v^i} \frac{\overrightarrow{\partial} g}{\partial v_i^+} - \frac{\overleftarrow{\partial} f}{\partial v_i^+} \frac{\overrightarrow{\partial} g}{\partial v^i}$$

while the BV Laplacian is

$$\Delta f = \frac{\overrightarrow{\partial}}{\partial v_i^+} \frac{\overleftarrow{\partial}}{\partial v^i} f.$$

The BV bracket and the BV Laplacian satisfy, together with the pointwise product, the axioms of a BV algebra, namely

$$\begin{aligned}
&(f, g) = -(-1)^{(|f|-1)\,(|g|-1)}(g, f),\\
&(f, (g, h)) = ((f, g), h) + (-1)^{(|f|-1)(|g|-1)}(g, (f, h)) = 0,\\
&(f, gh) = (f, g)h + (-1)^{(|f|-1)|g|} g(f, h),\\
&(f, g) = \Delta(fg) - \Delta(f)g + (-1)^{|f|} f \Delta(g),\\
&\Delta^2 = 0.
\end{aligned}$$

In particular a Δ-cohomology is defined on the space of functional on the fields-antifields.

In our case $(\xi, \eta, \beta) \in V = \Omega^0(D, X^*(TM)) \oplus \Omega^1(D, X^*(\Pi T^*M)) \oplus \Omega^0(D, X^*(\Pi T^*M))$ and $(\xi^+, \eta^+, \beta^+) \in \Pi V^* = \Omega^2(D, X^*(\Pi T^*M)) \oplus \Omega^1(D, X^*(TM)) \oplus \Omega^2(D, X^*(TM))$. A "total degree" is then introduced by saying that a form on D with values in $X^*(TM)$ has total degree zero, while a form with values in $X^*(\Pi T^*M)$ has total degree 1. Next, we define the "ghost number" *gh* as the difference between the total degree and the degree *deg* as a differential form on D. We summarize the degrees and ghost numbers of our fields and antifields in the following table:

$gh \backslash deg$	0	1	2
-2			β^+
-1		η^+	ξ^+
0	ξ	η	
1	β		

A main feature of the BV formalism is that the integral of a Δ-closed functional $\mathcal{H}$ performed over a Lagrangian submanifold L in the space of fields-antifields, depends only on the homology class of L and that the integral of a Δ-exact functional is zero. Hence, integration defines a pairing between homology classes of Lagrangian submanifolds and Δ-cohomology classes. An easy computation shows that a functional of the form $e^{\frac{i}{\hbar}S}$ is Δ-closed if and only if S satisfies the "quantum master equation"

$$(S, S) - 2i\hbar\Delta(S) = 0 \tag{10}$$

as indeed happens for the superaction (9) of the Poisson sigma-model [2] (see also Remark 3 below). More generally, if the functional $\mathcal{H}$ is of the form $\mathcal{O}\, e^{\frac{i}{\hbar}S}$ for some functional $\mathcal{O}$ and some S satisfying equation (10), we have that $\Delta(\mathcal{O}\, e^{\frac{i}{\hbar}S}) = 0$ if and only if $\Omega(\mathcal{O}) = 0$, where $\Omega(\mathcal{O}) :=$

$(S, \mathcal{O}) - i\hbar\Delta(\mathcal{O})$. Equation (10) immediately implies $\Omega^2 = 0$ and the relevant cohomology classes are called "observables" of the theory. Since the "expectation value" $\langle\mathcal{O}\rangle := \int_L \mathcal{O}\, e^{\frac{i}{\hbar}S}$ of an observable $\mathcal{O}$ depends only on the homology class of L, the perturbative expansion of the original path integral (7), which corresponds to integrating over the Lagrangian submanifold $\xi^+ = \eta^+ = \beta^+ = 0$ (and which is actually ill-defined due to the symmetries), can be effectively computed by choosing an appropriate submanifold where the quadratic part of the action is non-degenerate (see [2] for details).

Remark 2. For any point u in the boundary of D, one has

$$\Omega(\tilde{\xi}^i(u)) = \Omega(\tilde{\eta}_j(u)) = 0. \tag{11}$$

This gives a way to construct observables for the Poisson sigma-model from a point $u \in \partial D$ and a smooth function φ of $\tilde{\xi}$ and $\tilde{\eta}$. Indeed, the functional $\mathcal{O}_{\varphi,u}(\tilde{\xi}, \tilde{\eta}) := \varphi(\tilde{\xi}(u), \tilde{\eta}(u))$ is clearly Ω-closed. In particular, $f(x + \tilde{\xi}(0))$ and $g(x + \tilde{\xi}(1))$ from equation (8) are observables.

Remark 3. Given a p-multivector field ψ, written in coordinates as $\psi(x)^{i_1,\ldots,i_p}\partial_{i_1} \wedge \cdots \wedge \partial_{i_p}$, we can consider

$$S_\psi(\tilde{\xi}, \tilde{\eta}) := \int_{D^{2|2}} \psi(x + \tilde{\xi})^{i_1,\ldots,i_p}\tilde{\eta}_{i_1}\cdots\tilde{\eta}_{i_p}.$$

Notice that with this notation the superaction (9) becomes $S(\tilde{\xi}, \tilde{\eta}) = S_{\text{free}}(\tilde{\xi}, \tilde{\eta}) + S_\alpha(\tilde{\xi}, \tilde{\eta})$. An explicit calculation shows that the map $\psi \mapsto S_\psi$ is a Lie algebra morphism

$$(S_{\psi_1}, S_{\psi_2}) = S_{[\psi_1,\psi_2]}$$

where we have the BV bracket on the l.h.s. and the Schouten–Nijenhuis bracket on the r.h.s. In particular, since the bi-vector α is Poisson, we have $(S_\alpha, S_\alpha) = 0$. When the "free" part of the superaction is taken into account, it is not difficult to show that $(S_{\text{free}}, S_{\text{free}}) = 0$ and $(S_{\text{free}}, S_\psi) = 0$, which in turn imply the so-called "master equation" for (9)

$$(S, S) = (S_{\text{free}} + S_\alpha, S_{\text{free}} + S_\alpha) = 0. \tag{12}$$

A consequence of this equality is that $\delta f := (S, f)$ is a coboundary operator. Finally, notice that the quantum master equation (10) descends immediately from the relations $\Delta(S_{\text{free}}) = \Delta(S_\alpha) = 0$.

Remark 4. Using the operator δ defined above, we can rewrite equation (12) as

$$\delta S = 0. \tag{13}$$

On the other hand one explicitly computes

$$\delta\tilde{\xi}^i = \mathrm{D}\tilde{\xi}^i + \alpha^{ij}(x + \tilde{\xi})\tilde{\eta}_j, \tag{14}$$

$$\delta\tilde{\eta}_i = \mathrm{D}\tilde{\eta}_i + \frac{1}{2}\partial_i\alpha^{jk}(x + \tilde{\xi})\tilde{\eta}_j\tilde{\eta}_k. \tag{15}$$

The operator $\delta|_{\xi^+=\eta^+=\beta^+=0}$ can be seen as a vector field on the space of functionals of (ξ, η) depending on the choice of β. We denote by δ_β this vector field. Now, equations (13)–(15) together imply that δ_β is an infinitesimal symmetry of the original action $S(\xi, \eta)$. Explicitly this symmetry reads

$$\delta_\beta\xi^i = \alpha^{ij}(x + \xi)\beta_j,$$

$$\delta_\beta\eta_i = -\mathrm{d}\beta_i - \partial_i\alpha^{jk}(x + \xi)\eta_j\beta_k.$$

5 Ward identities

The equation $\int_L \Delta(\mathcal{H}) = 0$ produces non-trivial identities (called "Ward identities") among the expectation values. For instance if $\phi(\tilde{\xi}, \tilde{\eta})$ is a Δ-closed functional, the following equality easily descends from the axioms of a BV algebra

$$0 = \int_L \Delta\big(e^{\frac{\mathrm{i}}{\hbar}S}\phi\big) = \int_L e^{\frac{\mathrm{i}}{\hbar}S}\delta\phi. \tag{16}$$

Now consider

$$\phi = \int_0^1 \mathrm{d}t \int \mathrm{d}\theta\, f(x+\tilde{\xi}(0))\, g(x+\tilde{\xi}(t,\theta))\, h(x+\tilde{\xi}(1)).$$

An explicit computation using equation (14) shows that

$$\delta\phi = \int_0^1 \mathrm{d}t \left(f(x+\tilde{\xi}(0))\, \frac{\mathrm{d}g(x+\tilde{\xi}(t))}{\mathrm{d}t}\, h(x+\tilde{\xi}(1)) \right).$$

Therefore equation (16) has precisely the form of equation (5) and the Ward identity for this choice of ϕ is the associativity equation

$$0 = \int_L \Delta\left(e^{\frac{\mathrm{i}}{\hbar}S}\phi\right) = ((f \star g) \star h) - (f \star (g \star h)).$$

6 The Magri–Koszul bracket

If ω is a 1-form on M we can associate to it a function on $T_xM \oplus \Pi T_x^*M$ by

$$\omega(\xi,\eta) = \omega(\xi) := \langle \omega(x+\xi)|\xi\rangle = \omega_i(x)\xi^i + \partial_j\omega_i(x)\xi^i\xi^j + \cdots .$$

Similarly, to a vector field χ we can associate the function

$$\chi(\xi,\eta) := \langle \chi(x+\xi)|\eta\rangle = \chi^i(x)\eta_i + \partial_j\chi^i(x)\eta_i\xi^j + \cdots .$$

The perturbative expansion of the integral

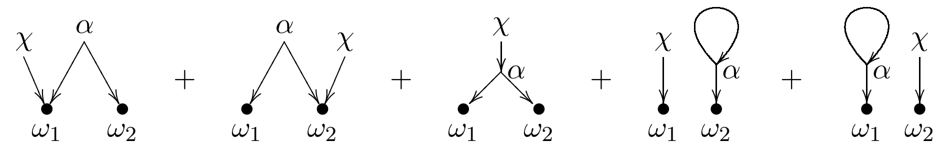
$$\frac{\displaystyle\int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, \omega_1\big(\xi\big)\, \omega_2\big(\xi\big)\, \chi\big(\xi,\eta\big)\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}}{\displaystyle\int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\langle\xi|\eta\rangle}}$$

is closely related to the Magri–Koszul bracket on 1-forms [4, 5]. More precisely, if we apply the Poisson sigma-model techniques to this situation, the function $\omega_1\big(\xi\big)\,\omega_2\big(\xi\big)\,\chi\big(\xi,\eta\big)$ is changed into $\omega_1\big(\tilde{\xi}(0)\big)\,\omega_2\big(\tilde{\xi}(1)\big)\,\chi\big(\tilde{\xi}(\infty),\tilde{\eta}(\infty)\big)$. Since $\xi(\infty) = 0$, we have $\chi(\tilde{\xi}(\infty),\tilde{\eta}(\infty)) = \chi^i(x)\tilde{\eta}_i(\infty)$. Therefore the perturbative expansion of the path integral:

$$\frac{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, \omega_1(\tilde{\xi}(0))\, \omega_2(\tilde{\xi}(1))\, \chi(\tilde{\xi}(\infty),\tilde{\eta}(\infty))\, e^{\frac{\mathrm{i}}{\hbar}S(\tilde{\xi},\tilde{\eta})}}{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, e^{\frac{\mathrm{i}}{\hbar}\int_{D^{2|2}}\langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle}}, \tag{17}$$

will depend on $\chi(x)$ but not on its derivatives. The first order expansion of the integral (17) is $\frac{\mathrm{i}\hbar}{2}\left\langle\omega_1\bullet\omega_2\middle|\chi\right\rangle+O\left(\hbar^2\right)$ where

$$\left\langle\omega_1\bullet\omega_2\middle|\chi\right\rangle :=$$



If we define

$$[\omega_1,\omega_2] := \frac{\omega_1\bullet\omega_2-\omega_2\bullet\omega_1}{2}$$

then

$$\begin{aligned}
[\omega_1,\omega_2] &= \alpha^{ij}(\partial_i\omega_{1\,k}+\partial_k\omega_{1,i})\omega_{2,j}\mathrm{d}x^k+\alpha^{ij}\omega_{1,i}(\partial_j\omega_{2\,k}+\partial_k\omega_{2,j})\mathrm{d}x^k+\partial_k\alpha^{ij}\omega_{1,i}\omega_{2,j}\mathrm{d}x^k\\
&= (\partial_k\alpha^{ij}\omega_{1,i}\omega_{2,j}+\alpha^{ij}\partial_k\omega_{1,i}\omega_{2,j}+\alpha^{ij}\omega_{1,i}\partial_k\omega_{2,j})\mathrm{d}x^k\\
&\quad -\alpha^{ij}\partial_j\omega_{1\,k}\omega_{2,i}\mathrm{d}x^k+\alpha^{ij}\omega_{1,i}\partial_j\omega_{2\,k}\mathrm{d}x^k\\
&= \mathrm{d}\langle\alpha|\omega_1\wedge\omega_2\rangle+\mathcal{L}_{\alpha\lrcorner\omega_1}\omega_2-\mathcal{L}_{\alpha\lrcorner\omega_2}\omega_1,
\end{aligned}$$

i.e., the bracket $[\omega_1,\omega_2]$ is precisely the Magri–Koszul bracket on 1-forms.

In particular one can recover the Jacobi identity for the Magri–Koszul bracket as a Ward identity (see Section 5) by choosing

$$\phi=\int_0^1\mathrm{d}t\int\mathrm{d}\theta\,\omega_1(x+\tilde\xi(0))\,\omega_2(x+\tilde\xi(t,\theta))\,\omega_3(x+\tilde\xi(1))\,\chi(\tilde\xi(\infty),\tilde\eta(\infty)).$$

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