Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 92–97

On Symmetry Reduction of Some Classes of the First-Order Differential Equations in the Space $M(1,4)\times R(u)$

Volodymyr I. FEDORCHUK

Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine
E-mail: *fedorchukv@ukr.net*

The symmetry reduction of some classes of the first-order differential equations (in the space $M(1,4)\times R(u)$), invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$, to differential equations with fewer independent variables are done.

1 Introduction

It is well known that among equations, important for theoretical and mathematical physics, there are also ones which have nontrivial symmetry groups. In the space $M(1,4)\times R(u)$ we have the linear and nonlinear wave equations and the Dirac equation. Here, and in what follows, $R(u)$ is the number axis of the dependent variable u. These equations are invariant under the generalized Poincaré group $P(1,4)$ (see, for example, [1–3]). The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in the theoretical and mathematical physics [1–3]. The group $P(1,4)$ has many subgroups used in the theoretical physics [4–6]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\widetilde{G}(1,3)$ (see also [7]). Thus, the results obtained with the use of the subgroup structure of the group $P(1,4)$ will be useful in the relativistic and non-relativistic physics. Therefore, it is important from the physical and mathematical points of view, that we are able to construct, in the space $M(1,4)\times R(u)$, new differential equations invariant under continuous subgroups of the generalized Poincaré group $P(1,4)$. The paper [8] is devoted to the construction of the first-order differential equations invariant under splitting subgroups of the group $P(1,4)$, defined in the space $M(1,4)\times R(u)$.

In the present paper we continue to study this type of equations. We concentrate our attention on the symmetry reduction of the first-order differential equations, invariant under splitting subgroups of the group $P(1,4)$, to differential equations with fewer independent variables.

Our paper is organized as follows: in Section 2 we introduce some notations and results concerning the Lie algebra of the group $P(1,4)$, which are used in the next chapter. Section 3 presents our main results.

2 The Lie algebra of the group $P(1,4)$ and its subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu}=-M_{\nu\mu}$ $(\mu,\nu=0,1,2,3,4)$ and P'_μ $(\mu=0,1,2,3,4)$, which satisfy the commutation relations

$$\left[P'_\mu,P'_\nu\right]=0,\qquad \left[M'_{\mu\nu},P'_\sigma\right]=g_{\mu\sigma}P'_\nu-g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu},M'_{\rho\sigma}\right]=g_{\mu\rho}M'_{\nu\sigma}+g_{\nu\sigma}M'_{\mu\rho}-g_{\nu\rho}M'_{\mu\sigma}-g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

We consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Below, we will use the following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1,2,3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

For the study of the subgroup structure of the group $P(1,4)$ we used the method proposed in [9]. Continuous subgroups of the group $P(1,4)$ have been described in [4–6].

3 The first-order differential equations in the space $M(1,4) \times R(u)$

In this section we present some of the new results concerning the first-order differential equations in the space $M(1,4) \times R(u)$.

The equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$, can be written in the following form (see, for example [10–12]):

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of splitting subgroups of the group $P(1,4)$.

The classes of the first-order differential equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$, are constructed. Some of the results obtained have been presented in [8]. To study these classes of the differential equations, we used only their symmetry properties.

Taking into account the functional bases of the invariants of some splitting subgroups of the group $P(1,4)$, we have constructed the ansatzes, which reduce majority of the equations obtained to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Among the reduced equations there are one-, two-, three- and four-dimensional ones.

It is impossible to present here all the results obtained. Therefore, we will give a short review of the results concerning the symmetry reduction of the considered classes of differential equations to classes of ODEs.

Let us consider the classes of the first-order differential equations, which are constructed with the use of the functional bases of the first-order differential invariants containing two usual invariants. One of them always is u, since it is one of the invariants of the group $P(1,4)$. On the base of these two invariants we have constructed the ansatzes, which reduce the classes considered to classes of ODEs. The corresponding symmetry reduction is done. Among the considered classes there are ones which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements: L_1, L_2, L_3, P_1, P_2, P_3, X_0, X_1, X_2, X_3, X_4. We have 32 such classes of differential equations. From among them, 17 classes have been reduced to the classes of functional equations. From these functional equations we can find solutions for the corresponding classes of differential equations.

Below, for some splitting subgroups of the group $P(1,4)$, we write the basis elements of their respective Lie algebras, the corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F, the ansatzes obtained, as well as the reduced equation for each case.

1. $\langle P_1, P_2, P_3, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_1 + x_1 \frac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = u_2 + x_2 \frac{u_0 - u_4}{x_0 + x_4},$$
$$J_5 = \frac{u_0 - u_4}{x_0 + x_4} x_3 + u_3, \quad J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F(\omega, \varphi, 0, 0, 0, 0, 0) = 0; \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$

2. $\langle P_1, P_2, X_3, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_1(x_0 + x_4) + x_1(u_0 - u_4),$$
$$J_4 = u_2(x_0 + x_4) + x_2(u_0 - u_4), \quad J_5 = u_3, \quad J_6 = u_0 - u_4,$$
$$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F(\omega, \varphi, 0, 0, 0, 0, 0) = 0;$$

3. $\langle P_3, X_1, X_2, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_4 = u_1,$$
$$J_5 = u_2, \quad J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F(\omega, \varphi, 0, 0, 0, 0, 0) = 0;$$

4. $\langle L_3, P_1, P_2, P_3, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4} x_3 + u_3, \quad J_4 = \left(u_1 + x_1 \frac{u_0 - u_4}{x_0 + x_4} \right)^2 +$$
$$+ \left(u_2 + x_2 \frac{u_0 - u_4}{x_0 + x_4} \right)^2, \quad J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F(\omega, \varphi, 0, 0, 0, 0) = 0;$$

5. $\langle L_3, P_1, P_2, X_3, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \left(u_1 + x_1 \frac{u_0 - u_4}{x_0 + x_4} \right)^2 + \left(u_2 + x_2 \frac{u_0 - u_4}{x_0 + x_4} \right)^2,$$
$$J_4 = u_3, \quad J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F(\omega, \varphi, 0, 0, 0, 0) = 0;$$

6. $\langle L_3, P_3, X_1, X_2, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3,$$
$$J_4 = u_0 - u_4, \quad J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F(\omega, \varphi, 0, 0, 0, 0) = 0;$$

7. $\langle L_3 - P_3, P_1, P_2, X_1, X_2, X_4 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_4 = u_0 - u_4,$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F(\omega, \varphi, 0, 0, 0) = 0.$$

However, among the classes of the first-order differential equations, which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$, there exist classes, which are reduced to classes of ODEs. We obtained 15 such classes. Let us present some of them.

1. $\langle L_3, X_0, X_3, X_4\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1, \quad J_4 = u_0, \quad J_5 = u_3, \quad J_6 = u_4,$$
$$J_7 = u_1^2 + u_2^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2\right)^{1/2}; \quad F\left(\omega, \varphi, 0, 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$$

2. $\langle X_1, X_2, X_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_0, \quad J_4 = u_1, \quad J_5 = u_2, \quad J_6 = u_3, \quad J_7 = u_4;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, \varphi', 0, 0, 0, \varphi'\right) = 0;$$

3. $\langle L_1, L_2, L_3, X_0, X_4\rangle$,

$$J_1 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_1 + x_2u_2 + x_3u_3, \quad J_4 = u_0, \quad J_5 = u_4,$$
$$J_6 = u_1^2 + u_2^2 + u_3^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}; \quad F\left(\omega, \varphi, \omega\varphi', 0, 0, \left(\varphi'\right)^2\right) = 0;$$

4. $\langle L_1, L_2, L_3, P_1, P_2, P_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (u_0 - u_4)(x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2) - 2(x_0u_0 + x_1u_1$$
$$+ x_2u_2 + x_3u_3 + x_4u_4)(x_0 + x_4), \quad J_4 = u_0 - u_4, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, -2\omega^2\varphi', 0, 0\right) = 0.$$

The remainder of the considered classes of the first-order differential equations are invariant under the splitting subgroups of the group $P(1,4)$, which do not belong to the splitting subgroups of the group $\widetilde{G}(1,3)$. We have 43 such classes. All these classes are reduced to ODEs. Now, we give some examples of this type of reduction.

1. $\langle G, P_1, P_2, P_3\rangle$,

$$J_1 = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = x_1 + u_1\frac{x_0 + x_4}{u_0 - u_4},$$
$$J_5 = x_2 + u_2\frac{x_0 + x_4}{u_0 - u_4}, \quad J_6 = x_3 + \frac{x_0 + x_4}{u_0 - u_4}u_3, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}; \quad F\left(\omega, \varphi, \frac{\varphi'}{\omega}, 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$$

2. $\langle G, P_3, L_3, X_4\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1, \quad J_4 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_5 = \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3,$$
$$J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2\right)^{1/2};$$
$$F\left(\omega, \varphi, 0, 0, 0, \left(\varphi'\right)^2, 0\right) = 0;$$

3. $\langle G, P_1, P_2, X_3\rangle$,

$$J_1 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = x_1 + u_1\frac{x_0 + x_4}{u_0 - u_4},$$
$$J_5 = x_2 + u_2\frac{x_0 + x_4}{u_0 - u_4}, \quad J_6 = u_3, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2; \quad u = \varphi(\omega),$$

$$\omega = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \frac{\varphi'}{\omega}, 0, 0, 0, (\varphi')^2\right) = 0;$$

4. $\langle G, L_3, X_3, X_4 \rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1 u_2 - x_2 u_1, \quad J_4 = (x_0 + x_4)(u_0 + u_4),$$
$$J_5 = u_3, \quad J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2\right)^{1/2};$$
$$F\left(\omega, \varphi, 0, 0, 0, (\varphi')^2, 0\right) = 0;$$

5. $\langle G, P_3, X_1, X_2 \rangle$,

$$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = \frac{u_0 - u_4}{x_0 + x_4} x_3 + u_3,$$
$$J_5 = u_1, \quad J_6 = u_2, \quad J_7 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, \frac{\varphi'}{\omega}, 0, 0, 0, (\varphi')^2\right) = 0;$$

6. $\langle G, X_1, X_2, X_3 \rangle$,

$$J_1 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = (x_0 + x_4)(u_0 + u_4), \quad J_4 = u_1, \quad J_5 = u_2,$$
$$J_6 = u_3, \quad J_7 = u_0^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, \omega\varphi', 0, 0, 0, (\varphi')^2\right) = 0;$$

7. $\langle P_3 + C_3 + 2L_3, X_1, X_2, X_0 + X_4 \rangle$,

$$J_1 = \left(x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_3 u_4 - x_4 u_3, \quad J_4 = x_3 u_2 - x_4 u_1, \quad J_5 = u_0,$$
$$J_6 = u_1^2 + u_2^2, \quad J_7 = u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_3^2 + x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, 0, 0, 0, 0, (\varphi')^2\right) = 0;$$

8. $\langle L_1 + \frac{1}{2}\left(P_1 + C_1\right), L_2 + \frac{1}{2}\left(P_2 + C_2\right), L_3 + \frac{1}{2}\left(P_3 + C_3\right), X_0 + X_4 \rangle$,

$$J_1 = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4,$$
$$J_4 = x_1 u_2 - x_2 u_1 + x_4 u_3 - x_3 u_4, \quad J_5 = x_2 u_3 + x_4 u_1 - x_1 u_4 - x_3 u_2, \quad J_6 = u_0,$$
$$J_7 = u_1^2 + u_2^2 + u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, \omega\varphi', 0, 0, 0, (\varphi')^2\right) = 0;$$

9. $\langle G, L_1, L_2, L_3, X_4 \rangle$,

$$J_1 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = (x_0 + x_4)(u_0 + u_4), \quad J_4 = x_1 u_1 + x_2 u_2 + x_3 u_3,$$
$$J_5 = u_0^2 - u_4^2, \quad J_6 = u_1^2 + u_2^2 + u_3^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2};$$
$$F\left(\omega, \varphi, 0, \omega\varphi', 0, (\varphi')^2\right) = 0;$$

10. $\langle G, P_3, C_3, X_1, X_2 \rangle$,

$$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_0 u_0 + x_3 u_3 + x_4 u_4, \quad J_4 = u_1, \quad J_5 = u_2,$$

$$J_6 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \omega\varphi', 0, 0, (\varphi')^2\right) = 0;$$

11. $\langle L_1, L_2, L_3, P_1 + C_1, P_2 + C_2, P_3 + C_3, X_0 + X_4 \rangle$,

$$J_1 = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4, \quad J_4 = u_0,$$
$$J_5 = u_1^2 + u_2^2 + u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, \omega\varphi', 0, (\varphi')^2\right) = 0;$$

12. $\langle G, P_3, C_3, L_3, X_1, X_2 \rangle$,

$$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_0u_0 + x_3u_3 + x_4u_4, \quad J_4 = u_1^2 + u_2^2,$$
$$J_5 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \omega\varphi', 0, (\varphi')^2\right) = 0;$$

13. $\langle G, L_1, L_2, L_3, P_1, P_2, P_3 \rangle$,

$$J_1 = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4},$$
$$J_4 = x_0u_0 + x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}; \quad F\left(\omega, \varphi, \frac{\varphi'}{\omega}, \omega\varphi', (\varphi')^2\right) = 0.$$

[1] Fushchych W.I., Representations of full inhomogeneous de Sitter group and equations in five-dimensional approach. I, *Teoret. i Mat. Fiz.*, 1970, V.4, N 3, 360–367.

[2] Fushchych W.I. and Nikitin A.G., Symmetries of equations of quantum mechanics, New York, Allerton Press Inc., 1994.

[3] Kadyshevsky V.G., New approach to theory electromagnetic interactions, *Fizika Elementar. Chastitz i Atomn. Yadra*, 1980, V.11, N 1, 5–39.

[4] Fedorchuk V.M., Nonsplitting subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$, *Ukr. Math. J.*, 1981, V.33, N 5, 696–700.

[5] Fedorchuk V.M., Splitting subalgebras of the Lie algebra of the generalized Poincaré group $P(1,4)$, *Ukr. Math. J.*, 1979, V.31, N 6, 717–722.

[6] Fushchich W.I., Barannik A.F., Barannik L.F. and Fedorchuk V.M., Continuous subgroups of the Poincaré group $P(1,4)$, *J. Phys. A: Math. Gen.*, 1985, V.18, N 14, 2893–2899.

[7] Fushchich W.I. and Nikitin A.G., Reduction of the representations of the generalized Poincaré algebra by the Galilei algebra, *J. Phys. A: Math. Gen.*, 1980, V.13, N 7, 2319–2330.

[8] Fedorchuk V.I., Differential equations of the first-order in the space $M(1,4) \times R(u)$ with nontrivial symmetry groups, in Group and Analytic Methods in Mathematical Physics, Kyiv, Institute of Mathematics, 2001, V.36, 283–292.

[9] Patera J., Winternitz P. and Zassenhaus H., Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group, *J. Math. Phys.*, 1975, V.16, N 8, 1597–1614.

[10] Lie S., Über Differentialinvarianten, *Math. Ann.*, 1884, V.24, N 1, 52–89.

[11] Olver P.J., Applications of Lie groups to differential equations, New York, Springer-Verlag, 1986.

[12] Ovsiannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.