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On the Differential First-Order Invariants of the Non-Splitting Subgroups of the Poincaré Group $P(1,4)$

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The differential first-order invariants for all non-splitting subgroups of the generalized Poincaré group $P(1,4)$ are constructed.

1 Introduction

The differential invariants play an important role in geometry, group analysis of differential equations, theoretical and mathematical physics, mechanics, gas dynamics, etc. They have been studied in many works (see, for example [1–14]). In particular, the papers [12–14] are devoted to the construction of the first-order differential invariants for the splitting subgroups [15–17] of the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group is very useful in the theoretical and mathematical physics [18–20].

The objective of this paper is to present some of the new results concerning the differential first-order invariants for the non-splitting subgroups [17, 21, 22] of the group $P(1,4)$.

For this purpose, we have to consider the Lie algebra of the group $P(1,4)$.

2 The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is generated by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ ($\mu,\nu = 0,1,2,3,4$) and P'_μ ($\mu = 0,1,2,3,4$), satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in the following: $M'_{\mu\nu} = iM_{\mu\nu}$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Next, we will use the following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1, 2, 3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1, 2, 3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

We used the method proposed in [23] to study the subgroup structure of the group $P(1,4)$.

Non-conjugate subgroups of the group $P(1,4)$ have been described in [15–17, 21, 22]. The conjugation was considered under the group $P(1,4)$.

From the results obtained it follows that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [20]. The Lie algebra of the group $\widetilde{G}(1,3)$ has the following basis elements: $L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4$.

3 The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The functional bases of the differential first-order invariants have been constructed for all non-splitting subgroups of the group $P(1,4)$. Since it is impossible to present here all the results obtained, only some of them are given in this section.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ we write their respective basis elements and corresponding functional basis of the differential invariants.

One-dimensional subalgebras

1. $\langle P_3 + X_1\rangle$,

$$J_1 = x_2, \quad J_2 = x_0 + x_4, \quad J_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_4 = x_1 + \frac{x_3}{x_0 + x_4}, \quad J_5 = u,$$
$$J_6 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_0 - u_4,$$
$$J_{10} = u_0^2 - u_3^2 - u_4^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$

2. $\langle G + cX_1,\ c < 0\rangle$,

$$J_1 = x_1 - c\ln(x_0 + x_4), \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = (x_0^2 - x_4^2)^{1/2}, \quad J_5 = u,$$
$$J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_3, \quad J_{10} = u_0^2 - u_4^2;$$

3. $\langle L_3 - P_3 + \alpha_0 X_0,\ \alpha_0 < 0\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_0 + x_4)^2 + 2\alpha_0 x_3, \quad J_3 = \alpha_0 \arctan\frac{x_1}{x_2} - x_0 - x_4,$$
$$J_4 = 2(x_0 + x_4)^3 + 6\alpha_0 x_3(x_0 + x_4) + 3\alpha_0^2(x_0 - x_4), \quad J_5 = u, \quad J_6 = x_1u_2 - x_2u_1,$$
$$J_7 = x_0 + x_4 - \alpha_0\frac{u_3}{u_0 - u_4}, \quad J_8 = u_0 - u_4, \quad J_9 = u_1^2 + u_2^2, \quad J_{10} = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle L_3 + \tilde{d}(X_0 + X_4),\ \tilde{d} < 0\rangle$,

$$J_1 = x_3, \quad J_2 = x_4, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = \tilde{d}\arctan\frac{x_1}{x_2} - x_0, \quad J_5 = u,$$
$$J_6 = x_1u_2 - x_2u_1, \quad J_7 = u_0, \quad J_8 = u_3, \quad J_9 = u_4, \quad J_{10} = u_1^2 + u_2^2.$$

Two-dimensional subalgebras

1. $\langle L_3 - X_4,\ P_3 + hX_0,\ h > 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = (x_0 + x_4)^2 - 2hx_3,$$
$$J_3 = 2(x_0 + x_4)^3 - 3h(2x_3 - h)(x_0 + x_4) + 3h^2 \arctan\frac{x_1}{x_2} - 6h^2 x_4, \quad J_4 = u,$$
$$J_5 = x_1 u_2 - x_2 u_1, \quad J_6 = (x_0 + x_4) + \frac{hu_3}{u_0 - u_4}, \quad J_7 = u_0 - u_4, \quad J_8 = u_1^2 + u_2^2,$$
$$J_9 = u_0^2 - u_3^2 - u_4^2;$$

2. $\langle G + aX_3,\ L_3,\ a < 0\rangle$,

$$J_1 = x_3 - a\ln(x_0 + x_4), \quad J_2 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u,$$
$$J_5 = x_1 u_2 - x_2 u_1, \quad J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_3, \quad J_8 = u_0^2 - u_4^2, \quad J_9 = u_1^2 + u_2^2;$$

3. $\langle G + aX_1,\ P_3,\ a < 0\rangle$,

$$J_1 = x_1 - a\ln(x_0 + x_4), \quad J_2 = x_2, \quad J_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_4 = u, \quad J_5 = \frac{x_0 + x_4}{u_0 - u_4},$$
$$J_6 = \frac{u_0 - u_4}{x_0 + x_4} x_3 + u_3, \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle G,\ L_3 + dX_3,\ d < 0\rangle$,

$$J_1 = x_3 + d\arctan\frac{x_1}{x_2}, \quad J_2 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u,$$
$$J_5 = x_1 u_2 - x_2 u_1, \quad J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_3, \quad J_8 = u_0^2 - u_4^2, \quad J_9 = u_1^2 + u_2^2.$$

Three-dimensional subalgebras

1. $\langle L_3 + cG + bX_3,\ P_1,\ P_2,\ c > 0,\ b < 0\rangle$,

$$J_1 = cx_3 - b\ln(x_0 + x_4), \quad J_2 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}, \quad J_3 = u,$$
$$J_4 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = \left(x_1 + \frac{x_0 + x_4}{u_0 - u_4} u_1\right)^2 + \left(x_2 + \frac{x_0 + x_4}{u_0 - u_4} u_2\right)^2,$$
$$J_6 = c\arctan\left(\frac{u_1(x_0 + x_4) + x_1(u_0 - u_4)}{u_2(x_0 + x_4) + x_2(u_0 - u_4)}\right) + \ln(x_0 + x_4), \quad J_7 = u_3,$$
$$J_8 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

2. $\langle L_3 + eG + \kappa_3 X_3,\ X_0,\ X_4,\ e > 0,\ \kappa_3 < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1 u_2 - x_2 u_1, \quad J_4 = \ln(u_0 + u_4) - e\arctan\frac{x_1}{x_2},$$
$$J_5 = \kappa_3 \ln(u_0 + u_4) + ex_3, \quad J_6 = u_3, \quad J_7 = u_0^2 - u_4^2, \quad J_8 = u_1^2 + u_2^2;$$

3. $\langle L_3 + d_3 X_3,\ X_0,\ X_4,\ d_3 < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_2 u_1 - x_1 u_2, \quad J_4 = d_3 \arctan\frac{x_1}{x_2} + x_3,$$
$$J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_4, \quad J_8 = u_1^2 + u_2^2;$$

4. $\langle L_3 + dX_3,\ P_3 + X_0,\ X_4,\ d < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = 2d\arctan\frac{x_1}{x_2} - (x_0 + x_4)^2 + 2x_3, \quad J_3 = u, \quad J_4 = x_1 u_2 - x_2 u_1,$$

$$J_5 = (x_0 + x_4)(u_0 - u_4) + u_3, \quad J_6 = u_0 - u_4, \quad J_7 = u_1^2 + u_2^2, \quad J_8 = u_0^2 - u_3^2 - u_4^2.$$

Four-dimensional subalgebras

1. $\langle G + a_3X_3,\ L_3 + d_3X_3,\ P_1,\ P_2,\ a_3 < 0,\ d_3 < 0\rangle$,

$$J_1 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{x_0 + x_4}{u_0 - u_4},$$
$$J_4 = \left(x_1 + u_1\frac{x_0 + x_4}{u_0 - u_4}\right)^2 + \left(x_2 + u_2\frac{x_0 + x_4}{u_0 - u_4}\right)^2,$$
$$J_5 = x_3 - a_3\ln(x_0 + x_4) + d_3\arctan\left(\frac{(x_0 + x_4)u_1 + (u_0 - u_4)x_1}{(x_0 + x_4)u_2 + (u_0 - u_4)x_2}\right),$$
$$J_6 = u_3, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

2. $\langle G + a_3X_3,\ L_3 + d_3X_3,\ P_3,\ X_4,\ a_3 < 0,\ d_3 < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_4 = x_1u_2 - x_2u_1,$$
$$J_5 = d_3\arctan\frac{x_1}{x_2} + x_3 - a_3\ln(u_0 - u_4) + u_3\frac{x_0 + x_4}{u_0 - u_4}, \quad J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_3^2 - u_4^2;$$

3. $\langle L_3 - X_4,\ P_1,\ P_2,\ X_3\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \left(x_1 + u_1\frac{x_0 + x_4}{u_0 - u_4}\right)^2 + \left(x_2 + u_2\frac{x_0 + x_4}{u_0 - u_4}\right)^2,$$
$$J_4 = x_0^2 - x_1^2 - x_2^2 - x_4^2 + (x_0 + x_4)\arctan\left(\frac{x_1(u_0 - u_4) + u_1(x_0 + x_4)}{x_2(u_0 - u_4) + u_2(x_0 + x_4)}\right),$$
$$J_5 = u_3, \quad J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

4. $\langle L_3 - P_3 + \alpha_0X_0,\ X_1,\ X_2,\ X_4,\ \alpha_0 < 0\rangle$,

$$J_1 = 2x_3\alpha_0 + (x_0 + x_4)^2, \quad J_2 = u, \quad J_3 = \alpha_0\frac{u_3}{u_0 - u_4} - x_0 - x_4, \quad J_4 = u_0 - u_4,$$
$$J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2, \quad J_7 = \arctan\frac{u_1}{u_2} - \frac{u_3}{u_0 - u_4}.$$

Five-dimensional subalgebras

1. $\langle L_3,\ P_1 + X_2,\ P_2 - X_1,\ P_3 + \tilde{h}_3X_3,\ X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (u_0 - u_4)x_3 + (x_0 + x_4)u_3 - u_3\tilde{h}_3,$$
$$J_4 = (x_1(u_0 - u_4) + u_2 + u_1(x_0 + x_4))^2 + (x_2(u_0 - u_4) - u_1 + u_2(x_0 + x_4))^2,$$
$$J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle G + a_2X_2 + a_3X_3,\ P_1,\ P_2,\ X_1,\ X_4,\ a_2 < 0,\ a_3 < 0\rangle$,

$$J_1 = x_3 - a_3\ln(x_0 + x_4), \quad J_2 = u, \quad J_3 = \frac{x_0 + x_4}{u_0 - u_4},$$
$$J_4 = x_2 - a_2\ln(x_0 + x_4) + \frac{x_0 + x_4}{u_0 - u_4}u_2, \quad J_5 = u_3, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

3. $\langle L_3 + dG + \alpha_3X_3,\ P_3,\ X_1,\ X_2,\ X_4,\ d > 0,\ \alpha_3 < 0\rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = d\arctan\frac{u_2}{u_1} - \ln(x_0 + x_4),$$

$$J_4 = dx_3 - \alpha_3 \ln(x_0 + x_4) + du_3 \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle L_3 + d_3 X_3,\ P_1,\ P_2,\ P_3,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \left(\frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4} \right)^2 + \left(\frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4} \right)^2,$$
$$J_4 = x_3 + \frac{x_0 + x_4}{u_0 - u_4} u_3 - d_3 \arctan \left(\frac{x_2(u_0 - u_4) + u_2(x_0 + x_4)}{x_1(u_0 - u_4) + u_1(x_0 + x_4)} \right),$$
$$J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2.$$

Six-dimensional subalgebras

1. $\langle G,\ L_3 + dX_3,\ P_1,\ P_2,\ P_3,\ X_4,\ d < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = \left(\frac{u_0 - u_4}{x_0 + x_4} x_1 + u_1 \right)^2 + \left(\frac{u_0 - u_4}{x_0 + x_4} x_2 + u_2 \right)^2,$$
$$J_4 = x_3 + \frac{x_0 + x_4}{u_0 - u_4} u_3 + d \arctan \left(\frac{x_1(u_0 - u_4) + u_1(x_0 + x_4)}{x_2(u_0 - u_4) + u_2(x_0 + x_4)} \right),$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle G + \tilde{a}_2 X_2,\ P_1,\ P_2,\ P_3,\ X_1,\ X_4,\ \tilde{a}_2 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = x_2 + \frac{x_0 + x_4}{u_0 - u_4} u_2 - \tilde{a}_2 \ln(x_0 + x_4),$$
$$J_4 = x_3 + u_3 \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle L_3 - P_3 + \alpha_0 X_0,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ \alpha_0 < 0 \rangle$,

$$J_1 = 2\alpha_0 x_3 + (x_0 + x_4)^2, \quad J_2 = u, \quad J_3 = \alpha_0 u_3 - (x_0 + x_4)(u_0 - u_4),$$
$$J_4 = u_0 - u_4, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

4. $\langle G + aX_3,\ L_3 + dX_3,\ X_0,\ X_1,\ X_2,\ X_4,\ a < 0,\ d < 0 \rangle$,

$$J_1 = u, \quad J_2 = x_3 - d \arctan \frac{u_2}{u_1} + a \ln(u_0 + u_4), \quad J_3 = u_3,$$
$$J_4 = u_0^2 - u_4^2, \quad J_5 = u_1^2 + u_2^2.$$

Seven-dimensional subalgebras

1. $\langle P_1 + X_0,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4 \rangle$,

$$J_1 = u, \quad J_2 = u_1 + (x_0 + x_4)(u_0 - u_4), \quad J_3 = u_0 - u_4, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle L_3 + d_3 X_3,\ P_1,\ P_2,\ P_3 + X_0,\ X_1,\ X_2,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = (x_0 + x_4)(u_0 - u_4) + u_3, \quad J_3 = u_0 - u_4, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle G + a_3 X_3,\ L_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ a_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = x_3 - a_3 \ln(u_0 - u_4), \quad J_4 = u_3, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

4. $\langle G,\ L_3 + d_3 X_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_3 = u_3, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_4^2.$$

Eight-dimensional subalgebras

1. $\langle G + aX_3,\ L_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_4,\ a < 0\rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = a\ln(u_0 - u_4) - \frac{x_0 + x_4}{u_0 - u_4}u_3 - x_3,$$
$$J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle P_1,\ P_2,\ P_3 + X_0,\ L_3 + \beta X_0,\ X_1,\ X_2,\ X_3,\ X_4,\ \beta < 0\rangle$,

$$J_1 = u, \quad J_2 = u_0 - u_4, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle G,\ L_3 + bX_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_4,\ b < 0\rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

4. $\langle L_3,\ P_1,\ P_2,\ P_3 + X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$$J_1 = u, \quad J_2 = x_0 + x_4 + \frac{u_3}{u_0 - u_4}, \quad J_3 = u_0 - u_4,$$
$$J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2.$$

The results obtained can be used, for example, to construct the first-order differential equations in the space $M(1,4) \times R(u)$ with non-trivial symmetry groups. In particular, among them, there will be equations, which are invariant under the non-splitting subgroups of the Poincaré group $P(1,3)$, as well as equations which are invariant under the non-splitting subgroups of the extended Galilei group $\widetilde{G}(1,3)$.

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