The Generalized Leray Transformation for Finite Time Singular Vortices in Fluids

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The finite time singularity solution for a single vortex field in both viscous and non-viscous fluids is discussed. The Leray transformation, which gives self similar solutions for (local) inner region of incompressible fluids, is generalized to a dynamical time dependent case. A new generalized time T is introduced to modify the Leray equation. Two important examples are generalized to produce both decreasing and constant areas of singularity instead of an exact line of singularity in the self-similar solutions. This is done by assuming a "line source" of the matter in the core of the singularity.

1 Introduction

The problem of singularity is generally of importance in many domains. One of the interesting cases is finite time singularity in the Euler and/or Navier–Stokes equation. General mathematical investigations about the formation and creation of singularities can be found in [1–3]. The blow up solution for vortices is a fundamental problem in fluid dynamics which firstly was noticed by Leray [4]. In this solution one finds a self-similar collapsing behavior towards a singularity at a finite time t^* while the length scale is decreasing like $(t^* - t)^{1/2}$ and velocity is diverging as $(t^* - t)^{-1/2}$. The above singular solution however does not allow the fluid physical parameters like energy or momentum to be finite [5,6]. Hence, the singular solution must be considered as an inner solution which must be appropriately matched with a non-singular outer solution [6]. There exist significant numerical results about the singular vortices and their interaction (see for instance [7–15]).

The existence of a smooth and bounded solution for the Leray equation is a rather difficult problem which is still under consideration [16]. Usually (as appears below) a singular strain field is considered to yield a (local) collapsing behavior. Although this strain field gives an infinite energy but it is a useful phenomenological model for this singularity. Following this strategy in this paper, a "line source" is assumed to exist in the core of the singularity. This line source can spread out the singularity to a non-zero area. So, it will be possible to have an area of singularity instead of a line. Therefore, our solution does not satisfy the finite energy condition and enters a phenomenological model showing the tendency of the singularity. This is due to the line source which is injecting matter and energy to the singular area. Again because of this source, it is difficult to talk about any matching between the inner and outer regions since these regions have no significant meaning here. Hence, the present solution is only a new type of singularity affected by an external line source. In doing this, we generalize the Leray transformation leading to a time dependent version. We consider two original solutions of the old Leary equation and generalize them to obtain their dynamical versions.

To introduce the Leray problem we must start from the Navier–Stokes equation for an incompressible ($\boldsymbol{\nabla}\cdot \boldsymbol{u} = 0$) fluid

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}, \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is the vorticity and ν is a (dimensional) constant representing the viscosity effect. Although the singularity has probably a close connection with the turbulence but it is convenient to consider a local idealized unsteady "strain" velocity field $\boldsymbol{u}'(\boldsymbol{x},t)$ which satisfies

$$\boldsymbol{\nabla} \times \boldsymbol{u}' = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0. \tag{2}$$

So the total local fluid velocity is

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', \tag{3}$$

where

$$\boldsymbol{\nabla} \times \boldsymbol{u}_0 = \boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}_0 = 0. \tag{4}$$

Moreover, $\boldsymbol{u}_0$ is the solution of the equilibrium (non-viscous) Euler equation

$$\boldsymbol{\nabla} \times (\boldsymbol{u}_0 \times \boldsymbol{\omega}) = 0 \tag{5}$$

and has no direct influence on the creation of $\boldsymbol{\omega}$ but it can change the dynamics of $\boldsymbol{\omega}$ because

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (u' \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}. \tag{6}$$

Equation (6) is a direct result from (1)–(5).

2 Self-similar solution

The Leray transformation changes the variables $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{\omega}$ to new dimensionless variables $\boldsymbol{X}$, $\boldsymbol{U}$, $\boldsymbol{\Omega}$. For an inviscid fluid ($\nu = 0$) the Leray transformation

$$\boldsymbol{x} \longrightarrow \boldsymbol{X} \equiv \frac{\boldsymbol{x}}{\sqrt{\Gamma(t^* - t)}}, \qquad \boldsymbol{u} \longrightarrow \boldsymbol{U}(\boldsymbol{X}) \equiv \sqrt{\frac{t^* - t}{\Gamma}}\boldsymbol{u}, \qquad \boldsymbol{\omega} \longrightarrow \boldsymbol{\Omega}(\boldsymbol{X}) \equiv (t^* - t)\boldsymbol{\omega} \tag{7}$$

changes equation (1) ($\nu = 0$) to

$$\boldsymbol{\nabla} \times \left[\left(\boldsymbol{U} + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] = 0. \tag{8}$$

In (7) the constant Γ is an arbitrary scale for the circulation for example, it can be the surface integration $\int \boldsymbol{\omega} \cdot \boldsymbol{n} da$ in an area perpendicular to $\boldsymbol{\omega}$. In a viscous fluid ($\nu \neq 0$) we may write

$$\boldsymbol{x} \longrightarrow \boldsymbol{X} \equiv \frac{\boldsymbol{x}}{\sqrt{\nu(t^* - t)}}, \qquad \boldsymbol{u} \longrightarrow \boldsymbol{U}(\boldsymbol{X}) \equiv \sqrt{\frac{t^* - t}{\nu}}\,\boldsymbol{u}, \tag{9}$$

with $\boldsymbol{\Omega}(\boldsymbol{X})$ the same as in (7) which yields

$$\boldsymbol{\nabla} \times \left[\left(\boldsymbol{U} + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] + \nabla^2 \boldsymbol{\Omega} = 0. \tag{10}$$

In equations (8) and (10) the operator $\boldsymbol{\nabla}$ denotes the derivatives with respect to $\boldsymbol{X}$. Also $\boldsymbol{\Omega}(\boldsymbol{X}) = \boldsymbol{\nabla} \times \boldsymbol{U}(\boldsymbol{X})$ and $\boldsymbol{\nabla} \cdot \boldsymbol{U}(\boldsymbol{X}) = 0$.

Denoting the strain field by $\boldsymbol{U}'(\boldsymbol{X})$ one can similarly write

$$\begin{gathered} \boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{U}_0(\boldsymbol{X}) + \boldsymbol{U}'(\boldsymbol{X}), \qquad \boldsymbol{\nabla} \times \boldsymbol{U}_0 = \boldsymbol{\Omega} = \boldsymbol{\nabla} \times \boldsymbol{U}, \\ \boldsymbol{\nabla} \times \boldsymbol{U}' = \boldsymbol{0}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{U}' = \boldsymbol{\nabla} \cdot \boldsymbol{U}_0 = 0, \end{gathered} \tag{11}$$

and

$$\boldsymbol{\nabla}\times(\boldsymbol{U}_0\times\boldsymbol{\Omega})=0,\qquad \boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}'+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]+\nabla^2\boldsymbol{\Omega}=0. \tag{12}$$

For an inviscid fluid the last term of the second equation of (12) vanishes.

It is important to stress again that $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are only functions of $\boldsymbol{X}$, i.e. we are sitting on a collapsing reference frame with a scale proportional to $\sqrt{t^*-t}$.

Two important singular solutions of (11) and (12) will be considered here. For both of these solutions the cylindrical coordinates $\boldsymbol{x}=(r,\theta,z)$ for usual real coordinates corresponding to $X=(R,\theta,Z)$ for new Leray coordinates are used. So $\boldsymbol{U}=(U_R,U_\theta,U_Z)$ and $\boldsymbol{\Omega}=(\Omega_R,\Omega_\theta,\Omega_z)$. The strain field for both cases is assumed to be [6]

$$\boldsymbol{U}'=c\left(-\frac{1}{2}R,0,Z\right),\qquad \text{or}\qquad \boldsymbol{u}'=\frac{c}{t^*-t}\left(-\frac{1}{2}r,0,z\right)\qquad (c=\text{const}) \tag{13}$$

which clearly satisfy (11).

The first solution. Suppose $\boldsymbol{\Omega}=(0,0,\Omega(R))$ is in the Z-direction and $\boldsymbol{U}_0=(0,U_0(R),0)$ is in the $\hat{\boldsymbol{\theta}}$ direction. So

$$\boldsymbol{\Omega}(R)=\frac{1}{R}\frac{d}{dR}[RU_0(R)]\hat{\boldsymbol{Z}}. \tag{14}$$

The first equation of (12) is automatically satisfied and its second equation regarding (13) yields

$$(c-1)\Omega+\frac{1}{2}(c-1)R\frac{d\Omega}{dR}+\frac{1}{R}\frac{d}{dR}\left(R\frac{d\Omega}{dR}\right)=0. \tag{15}$$

The solution of (15) initialing from a Gaussian form is [6]

$$\Omega=\Omega_0 e^{-\frac{c-1}{4}R^2},\qquad \text{or}\qquad \omega=\frac{\Omega_0}{t^*-t}\,e^{-\frac{(c-1)r^2}{4\nu(t^*-t)}}. \tag{16}$$

In [6] a complete discussion is given for the value of c. For a viscous fluid $c>1$ and for an inviscid fluid $c=1$. Here we restrict to $\nu\neq 0$.

Another solution of the linear equation (15) can be obtained to be

$$\Omega=Ae^{-\frac{c-1}{4}R^2}\int^R\frac{e^{\frac{c-1}{4}R'^2}}{R'}\,dR'\qquad (A=\text{const}),$$

which is divergent at $R\to 0$.

The second solution. In this case we may imagine a cylindrical beam of fluid symmetric around the Z-axis and moving parallel to it. So, $\boldsymbol{U}_0=(0,0,U_0(R))$ and $\boldsymbol{\Omega}=(0,\Omega(R),0)$. For simplicity let us neglect the viscosity term in (12) and use (13) to obtain

$$-\left(1+\frac{c}{2}\right)\Omega+\frac{c-1}{2}\,R\frac{d\Omega}{dR}=0. \tag{17}$$

The solution of (17) is

$$\Omega=AR^{\frac{c+2}{c-1}},\qquad \omega=A\Gamma^{\frac{c+2}{2(1-c)}}\frac{r^{\frac{c+2}{c-1}}}{(t^*-t)^{\frac{3c}{2(c-1)}}}\qquad (A=\text{const}). \tag{18}$$

The case of $c=1$ in (17) gives $\Omega=0$. If $\frac{c+2}{c-1}<0$ $(-2<c<1)$ then ω diverges when $r\to 0$ which is not physical. If on the other hand $0<c<1$, then for a fixed point $r\neq 0$, $\omega\to 0$ when $t\to t^*$ which does not contain singularity. Hence the best physical case is when $c>1$ or $c<-2$. It is clear that the radious of the beam must be finite.

3 Generalization of the Leray transformation

As mentioned before in the Leray transformation similarity solutions were sought and hence $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ were assumed to be only functions of $\boldsymbol{X}$. Obviously this is not the only possibility for finite time singularity. For example see [17]. The point is that when t^* is long enough, $\boldsymbol{\Omega}$ and $\boldsymbol{U}$ may change significantly and deviate from the self-similar solution, i.e. $\boldsymbol{\Omega}$ and $\boldsymbol{U}$ are not necessarily only functions of $\boldsymbol{X}$. In other words we want to allow the smooth functions $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ to change arbitrarily and not to be restricted to decreasing length scale $\sqrt{t^*-t}$. So, generally we can find another dimensionless time T such as

$$t \longrightarrow T \equiv \ln \frac{t^*}{t^*-t}\,, \tag{19}$$

where $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X},T)$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{X},T)$. If we add (19) to (7) or (9), equations (10) and (12) will be generalized to

$$\frac{\partial \boldsymbol{\Omega}}{\partial T} = \boldsymbol{\nabla} \times \left[\left(\boldsymbol{U} + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] + \nabla^2 \boldsymbol{\Omega}, \tag{20}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{U}_0 \times \boldsymbol{\Omega}) = 0, \qquad \frac{\partial \boldsymbol{\Omega}}{\partial T} = \boldsymbol{\nabla} \times \left[\left(\boldsymbol{U}' + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] + \nabla^2 \boldsymbol{\Omega}. \tag{21}$$

Clearly $T = 0$ when $t = 0$ and $T \to +\infty$ when $t \to t^*$.

To generalize the examples given in the previous section we again consider the strain field (13) in cylindrical coordinates. It can be seen that this strain field has a tendency to collect the singularity to the Z-axis ($R = 0$). This is of course the nature of a self-similar solution. Searching another symmetric strain field $\boldsymbol{U}''$ added to $\boldsymbol{U}'$ in equation (13) having the opposite effect of $\boldsymbol{U}'$, yields

$$\boldsymbol{U}'' = F(T)\left(R^{-1},0,0\right), \qquad \text{or} \qquad \boldsymbol{u}'' = \nu f(t)\left(r^{-1},0,0\right), \tag{22}$$

where $F(T) = f(t)$ are some functions to be determined. The θ-dependence of the strain field is not considered here. For this case see [6] and [17]. It can be seen that

$$\boldsymbol{\nabla} \times \boldsymbol{U}'' = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{U}'' = 2\pi F(T)\delta(R\hat{\boldsymbol{R}}), \tag{23}$$

so we do not consider the Z-axis ($R = 0$). Indeed $\boldsymbol{U}''$ represents a line source exactly on the Z-axis which is adding new mass with a fixed density (equal to the fixed fluid density) to the existing large amount of the fluid. This line source then near the Z-axis is very strong and prevents the singularity to collapse on the Z-axis.

Now let us generalize the two mentioned solutions.

The first solution. Again suppose $\boldsymbol{U}_0(R,T)$ to be only in the $\hat{\boldsymbol{\theta}}$ direction and $\boldsymbol{\Omega}(R,T)$ parallel to the Z-axis. Regarding the total strain field $\boldsymbol{U}' + \boldsymbol{U}''$ in (21) results in

$$(c-1)\Omega + \frac{1}{2}(c-1)R\frac{\partial \Omega}{\partial R} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Omega}{\partial R}\right) - \frac{F(T)}{R}\frac{\partial \Omega}{\partial R} = \frac{\partial \Omega}{\partial T}\,. \tag{24}$$

Three solution of (24) can be obtained.

i) We suggest the following solution (outside the z-axis)

$$\Omega = \Omega_0 \exp\left(Ae^T - \frac{c-1}{4}R^2\right), \qquad \omega = \frac{\Omega_0}{t^*-t}\,\exp\frac{r_0^2 - r^2}{t^*-t}\,, \tag{25}$$

where

$$r_0 \equiv \sqrt{\frac{4At^*\nu}{c-1}}\,, \tag{26}$$

and A and Ω_0 are constant. In the hole domain of $0 < r \le r_0$ the vorticity is singular while for $r > r_0$, $\omega \to 0$ when $t \to t^*$. Substituting (25) into (24) one finds.

$$F(T) = \frac{2A}{c-1}e^T, \qquad f(t) = \frac{2A}{c-1}\frac{t^*}{t^*-t}, \tag{27}$$

which means that the source is also singular in harmony with other singular parameters. This harmony which is locally in the inner region may come from complicated nonlinear interactions connected to turbulence.

ii) Assuming

$$\Omega = \Omega_0 e^{G(T)}\frac{R^2}{4}e^{-\frac{c-1}{4}R^2}$$

and substituting it in (24) results in $F(T) = 2$, $G(T) = 0$, and so

$$\Omega = \Omega_0\frac{R^2}{4}\,e^{-\frac{c-1}{4}R^2}, \qquad \omega = \frac{\Omega_0}{4\nu(t^*-t)^2}\,r^2 e^{-\frac{c-1}{4\nu}\frac{r^2}{t^*-t}} \quad (r \neq 0). \tag{28}$$

For the convergence of ω at large radiuses, it is necessary to have $c > 1$. The form of (28) introduces a finite radius

$$R_{\max} \equiv \sqrt{\frac{4}{c-1}}\,, \qquad r_{\max}(t) \equiv \sqrt{\frac{4\nu}{c-1}(t^*-t)}\,, \tag{29}$$

at which ω is maximum. This radius of course goes to zero for $t \to t^*$. Hence, $\omega \to 0$ as $r \to 0$ and is maximum on a cylindrical "shell" around the Z-axis which its radius is decreasing. If we sit on the Leray (collapsing) coordinates i.e. when $R = R_{\max} = \text{const}$, then $\omega = \omega_{\max} \propto (t^*-t)^{-1}$.

iii) Let us generally assume

$$\Omega = \Omega_0 e^{H(T)} e^{-\frac{c-1}{4}(R-R_m(T))^2}, \tag{30}$$

which from (24) one can get

$$H(T) = \frac{c-1}{2}T, \qquad R_m(T) = R_0 e^{-\frac{c-1}{2}T} \quad (R_0 = \text{const}), \qquad F(T) = 1 = f(t). \tag{31}$$

Hence

$$\Omega = \Omega_0 e^{\frac{c-1}{2}T} e^{-\frac{c-1}{4}(R-R_m(T))^2}, \qquad \omega = \frac{\Omega_0}{t^*}\left(\frac{t^*}{t^*-t}\right)^{\frac{c+1}{2}} e^{-\frac{c-1}{4\nu(t^*-t)}[r-r_m(t)]^2} \quad (r \neq 0), \tag{32}$$

where

$$r_m(t) \equiv R_0\sqrt{\nu t^*}\left(\frac{t^*-t}{t^*}\right)^{\frac{c}{2}} \tag{33}$$

and $R_m(T)$ is defined in (31). Moving with $r_m(t)$ a singularity of the order of $1/(t^*-t)^{\frac{c+1}{2}}$ can be seen. Since $c > 1$ this singularity is stronger than $\frac{1}{t^*-t}$. Also $r_m(t)$ is decreasing as $(t^*-t)^{c/2}$ which is faster than the Leray coordinates. Finally it is interesting to note that if only the strain field $\boldsymbol{U}''$ is considered (without $\boldsymbol{U}'$) the solutions seem to be divergent as $r \to \infty$.

The second solution. We can naturally generalize this solution to $\boldsymbol{U}_0(\boldsymbol{X},T) = (0, 0, U_0(R,T))$ and $\boldsymbol{\Omega}(\boldsymbol{X},T) = -\frac{\partial U_0}{\partial R}\hat{\boldsymbol{\theta}}$. Again neglection of the viscosity yields

$$-\left(1+\frac{c}{2}\right)\Omega + \frac{c-1}{2}R\frac{\partial\Omega}{\partial R} = \frac{\partial\Omega}{\partial T}, \tag{34}$$

which is the generalization of (17). The general solution of (34) has the form

$$\Omega = e^{-(1+\frac{c}{2})T} W\left(Re^{\frac{c-1}{2}T}\right),$$

where W is an arbitrary function. Choosing a power form for W one obtains

$$\Omega(R,T) = Ae^{(\frac{k}{2}-1)T} R^{\frac{c+k}{c-1}}, \qquad \omega(r,t) = \frac{A}{t^*(\Gamma t^*)^{\frac{e+k}{2(c-1)}}}\left(\frac{t^*}{t^*-t}\right)^{\frac{c}{2}\frac{k+1}{c-1}} r^{\frac{c+k}{c-1}}. \tag{35}$$

The necessary condition for the existence of singularity is

$$\frac{c(k+1)}{2(c-1)} > 0. \tag{36}$$

When $r \to 0$ for a fixed time $t < t^*$ we have $\boldsymbol{\omega} \to 0$ if

$$\frac{c+k}{c-1} > 0. \tag{37}$$

In the case of $k = 2$, (35) reduces to (18) and (36), (37) give $c > 1$ or $c < -2$. It should be again mentioned that the above solution is obtained in the presence of $\boldsymbol{U}'$ (equation (13)) only and without $\boldsymbol{U}''$.

4 Time reversibility for the inviscid limit

For the inviscid limit the last term of (20) vanishes to yield

$$\frac{\partial\boldsymbol{\Omega}}{\partial T} = \boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]. \tag{38}$$

Because of the time reversibility of Euler equation, we may think about an "expanding" solution for $-t^* < t \le 0$ [6, 10]. Indeed, the change $t \to -t$, offers the new variables $\boldsymbol{X}$, $\boldsymbol{T}_1$, $\boldsymbol{U}_1$, $\boldsymbol{\Omega}_1$ as

$$\boldsymbol{x} \longrightarrow \boldsymbol{X}_1 \equiv \frac{\boldsymbol{x}}{\sqrt{\Gamma(t^*+t)}}, \qquad t \longrightarrow T_1 \equiv \ln\frac{t_*}{t^*+t},$$
$$\boldsymbol{u} \longrightarrow \boldsymbol{U}_1 \equiv \sqrt{\frac{t^*+t}{\Gamma}}\,\boldsymbol{u}, \qquad \boldsymbol{\omega} \longrightarrow \boldsymbol{\Omega}_1 \equiv (t^*+t)\,\boldsymbol{\omega}, \tag{39}$$

which convert the Euler equation to

$$-\frac{\partial\boldsymbol{\Omega}_1}{\partial T_1} = \boldsymbol{\nabla}_1\times\left[\left(\boldsymbol{U}_1-\frac{1}{2}\boldsymbol{X}_1\right)\times\boldsymbol{\Omega}_1\right], \tag{40}$$

where $\boldsymbol{\nabla}_1$ denotes the derivative with respect to $\boldsymbol{X}_1$. It can be seen that if $\boldsymbol{\Omega}_1(\boldsymbol{X}_1,T_1)$, $\boldsymbol{U}_1(\boldsymbol{X}_1,T_1)$ is a solution of (40), then $\boldsymbol{\Omega}(\boldsymbol{X},T) = -\boldsymbol{\Omega}_1(\boldsymbol{X},T)$, $\boldsymbol{U}(\boldsymbol{X},T) = -\boldsymbol{U}_1(\boldsymbol{X},T)$ will be a solution of (38). This result is also covered in the self-similar Leray variables [6] and so, our generalization does not change this symmetry. An important conclusion is that when a singular inner solution passes through the critical time t^*, then it may change to an expanding solution [6, 10].

5 Summary

The problem of finite time singularity in fluid dynamics which was firstly noticed by Leray in 1934, has a close relation with the turbulence. Instead of complicated turbulence domains, one may assume a singular unsteady strain field which can produce singularly increasing velocities and vorticities upon a decreasing length scale. This picture that obviously violates the energy and momentum conservations cannot be global so in an inner level a singular solution must match an outer regular solution.

In the Leray transformation it is possible to find only the self-similar inner solutions when the length scale is decreasing as $\sqrt{t^* - t}$ while the fluid velocity is increasing as $(t^* - t)^{-1/2}$. In the present paper the Leray transformation, was generalized to obtain dynamical transformed Navier–Stokes and Euler equations. In this picture the finite time t^* for singularity is assumed to be long enough so that the deviation from the self-similar solution is significant. Two important cylindrically symmetric self-similar solutions were both generalized to the "dynamical" finite time singular case in which the singular areas enlarged to decreasing or constant cylindrical shells. Also the time reversibility of the Euler equation turned out to recover the same results for the generalized Leray parameters as in the usual self-similar solutions.

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