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Irreducible Killing Tensors from Conformal Killing Vectors

S. Brian EDGAR †, *Rafaelle RANI* ‡ *and Alan BARNES* §

† *Department of Mathematics, Linköping University, Linköping, Sweden S581-83*
E-mail: *bredg@mai.liu.se*

‡ *Institute of Astronomy and Astrophysics, Department of Theoretical Astrophysics, University of Tübingen, D-72076 Tübingen, Germany*
E-mail: *rani@tat.physik.uni-tuebingen.de*

§ *School of Engineering & Applied Science, Aston University, Birmingham B4 7ET, UK*
E-mail: *barnesa@aston.ac.uk*

Koutras has proposed some methods to construct reducible proper conformal Killing tensors and Killing tensors (which are, in general, irreducible) when a pair of orthogonal conformal Killing vectors exist in a given space. We give the completely general result demonstrating that this severe restriction of orthogonality is unnecessary. In addition we correct and extend some results concerning Killing tensors constructed from a single conformal Killing vector. We give as an example a Kimura metric and demonstrate how it is possible to construct a larger class of reducible proper conformal Killing tensors and Killing tensors than permitted by the Koutras algorithms. In addition, from our new result that all conformal Killing tensors are reducible in conformally flat spaces, we have a method of constructing all conformal Killing tensors and hence all the Killing tensors (which will in general be irreducible) of conformally flat spaces using their conformal Killing vectors.

1 Introduction

In this paper we shall consider an indirect method of constructing irreducible Killing tensors via conformal Killing vectors which has been proposed by Koutras [5], and also used recently by Amery and Maharaj [1]. However, in these two papers the underlying principle is not completely transparent nor are the algorithms obtained the most general; this is partly due to a distraction caused by the trace-free requirement in the definitions of conformal Killing tensors which is used in these two papers. Also in a paper by O'Connor and Prince [6] there has been an independent related discussion, but in the narrower context of a particular metric. We shall show that the arguments in these papers can be made more general than in the original presentations; in particular, we shall show that our more general approach enables us to obtain more conformal Killing tensors and hence more irreducible Killing tensors than those which can be obtained by the algorithms in [5, 1]. We extend a result of Weir [8] for flat spaces to conformally flat spaces and obtain the maximum number of conformal Killing tensors, which shows that they are all reducible in conformally flat spaces.

We begin with some notation and known results for Killing vectors and Killing tensors of order 2.

A *Killing vector* ξ satisfies $\xi_{(a;b)} = 0$.

A *Killing tensor of order 2* is a symmetric tensor K_{ab} such that $K_{(ab;c)} = 0$.

Reducible Killing tensors are built from Killing vectors ξ_J and the metric g_{ab}

$$K_{ab} = a_0 g_{ab} + \sum_{I=1}^{N}\sum_{J=I}^{N} a_{IJ}\xi_{I(a}\xi_{|J|b)},$$

where a_0 and a_{IJ} for $J \geq I$ are constants.

All other Killing tensors are called *irreducible (degenerate, trivial).*

- Physically the interest in Killing tensors is due to their connection with quadratic first integrals of geodesic motion and separability of classical partial differential equations.
- Irreducible Killing tensors yield quadratic first integrals which are not simply linear combinations of products of the linear first integrals associated with the Killing vectors.
- The maximum number of linearly independent Killing tensors in an n-dimensional Riemannian space is $n(n+1)^2(n+2)/12$.
- The existence of this maximum number is a necessary and sufficient condition for spaces of constant curvature.
- In n-dimensional spaces of constant curvature all Killing tensors are reducible (built from Killing vectors and the metric).

We have analogous notation and results for conformal Killing vectors and conformal Killing tensors of order 2.

A *conformal Killing vector* χ satisfies $\chi_{(a;b)} = \vartheta g_{ab}$.

A *conformal Killing tensor of order 2* is a symmetric tensor Q_{ab} such that $Q_{(ab;c)} = q_{(a}g_{bc)}$.

Reducible conformal Killing tensors are built from conformal Killing vectors χ_J and a scalar times the metric g_{ab}

$$Q_{ab} = \sum_{I=1}^{M}\sum_{J=I}^{M} a_{IJ}(\chi_{I(a}\chi_{|J|b)} + \lambda g_{ab}.$$

All other conformal Killing tensors are called *irreducible.*

A *trace-free conformal Killing tensor of order 2* is a symmetric trace-free tensor P_{ab} such that $P^i{}_i = 0$ and $P_{(ab;c)} = p_{(a}g_{bc)}$

Reducible trace-free conformal Killing tensors are built from conformal Killing vectors χ_J and a scalar times the metric g_{ab}

$$P_{ab} = \sum_{I=1}^{M}\sum_{J=I}^{M} a_{IJ}\left(\chi_{I(a}\chi_{|J|b)} - \frac{1}{n}\chi_I{}^c\chi_{Jc}g_{ab}\right).$$

All other trace-free conformal Killing tensors are called *irreducible.*

- Proper conformal Killing tensors do not generate quadratic first integrals for geodesic motion in general, but they do so for null geodesics.
- Irreducible proper conformal Killing tensors yield quadratic first integrals which are not simply linear combinations of products of the linear first integrals associated with the proper conformal Killing vectors.
- The maximum number of trace-free conformal Killing tensors in an n-dimensional Riemannian space is $(n-1)(n+2)(n+3)(n+4)/12$, [8].

2 To find all Killing tensors in conformally flat spaces

It is well known that, if χ^a is a conformal Killing vector of the metric g_{ab} with conformal factor ϑ, then it is also a conformal Killing vector of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$ with conformal factor $\tilde{\vartheta} = \vartheta + \Omega_{,c}\chi^c$. We now obtain the analogous result for conformal Killing tensors:

Theorem 1. *If Q^{ab} is a conformal Killing tensor satisfying $\nabla^{(a}Q^{bc)} = q^{(a}g^{bc)}$, then Q^{ab} is also a conformal Killing tensor of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$. Q^{ab} satisfies $\tilde{\nabla}^{(a}Q^{bc)} = \tilde{q}^{(a}\tilde{g}^{bc)}$, where $\tilde{q}^a = q^a + 2\Omega_{,d}Q^{da}$.*

Proof. The proof is straightforward involving an evaluation of $\tilde{\nabla}^{(a}Q^{bc)}$ using the result that

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b\Omega_{,c} + \delta^a_c\Omega_{,b} - \Omega^{,a}g_{bc}. \qquad \blacksquare$$

We cannot determine the number of linearly independent conformal Killing tensors because of the freedom in their trace; but we can consider the number of linearly independent trace-free conformal Killing tensors. From the above theorem and the analogous result for conformal Killing vectors we have,

Corollary 1. *The number of linearly independent trace-free conformal Killing tensors is invariant under conformal change of the metric. The number of linearly independent reducible trace-free conformal Killing tensors is similarly invariant.*

The maximum number of trace-free conformal Killing tensors in an $n\,(>2)$-dimensional Riemannian space has been found by Weir [8] to be $(n-1)(n+2)(n+3)(n+4)/12$, and he has shown that this number is attained in flat space.

For M conformal Killing vectors there are in general $M(M+1)/2$ symmetrised products of pairs of conformal Killing vectors; hence, in conformally flat spaces, we can construct $M(M+1)/2$ reducible trace-free conformal Killing tensors. In an n-dimensional Riemannian space there exist at most $(n+1)(n+2)/2$ linearly independent conformal Killing vectors, and the maximum number can be attained only in conformally flat spaces. Hence by substituting $M = (n+1)(n+2)/2$ we can obtain the maximum possible number of reducible conformal Killing tensors in an n-dimensional Riemannian space; but of course these need not all be linearly independent. (For example, in 4 dimensions there are 120 reducible trace-free conformal Killing tensors which can be constructed from the metric and the conformal Killing vectors, while the theoretical upper limit of linearly independent trace-free conformal Killing tensors is only 84.) However, Weir [8] has shown explicitly, in $n\,(>2)$-dimensional flat spaces, that of the $M(M+1)/2$ possible trace-free conformal Killing tensors constructed as above, only $(n-1)(n+2)(n+3)(n+4)/12$ are linearly independent. So, in $n\,(>2)$-dimensional flat spaces all $(n-1)(n+2)(n+3)(n+4)/12$ trace-free conformal Killing tensors are reducible [8].

Applying the above corollary we can extend Weir's results to conformally flat spaces:

Corollary 2. *The maximum number of linearly independent trace-free conformal Killing tensors in $n\,(>2)$ dimensions is $(n-1)(n+2)(n+3)(n+4)/12$ and is attained in conformally flat spaces. In this case all the trace-free conformal Killing tensors are reducible.*

So to find all the Killing tensors for conformally flat spaces, we simply investigate all the reducible conformal Killing tensors; these we can build up from all the conformal Killing vectors.

In four dimensions, conformally flat space-times necessarily admit 15 independent conformal Killing vectors from which 84 independent reducible conformal Killing tensors can be constructed. Hence in such space-times there is a rich supply of 'candidate' conformal Killing tensors which may satisfy the gradient condition and so be associated with possibly irreducible Killing tensors. The large number of candidate tensors means that a direct approach by hand calculation would be lengthy and error-prone. However the calculations involved, though lengthy, are routine and this enables them to be automated by using of a computer algebra package such as Reduce. Work is in progress on investigating a number of conformally flat space-times including the perfect fluid solutions [7] and the pure radiation solutions [2], the Robertson–Walker metrics and the interior Schwarzschild solution; these results will be presented elsewhere. In this paper we will restrict ourselves to a few preliminary remarks.

The generic perfect fluid solutions [7] and the pure radiation solutions [2] admit no Killing vectors and so if any gradient conformal Killing tensors are found then the associated Killing tensors will necessarily be irreducible (unless they are simply constant multiples of the metric).

Amery and Maharaj [1] found a number of conformal Killing tensors and Killing tensors in Robertson–Walker space-times using Koutras' algorithms, but because they used only mutually orthogonal conformal Killing vectors in their construction, they were only able to construct 39 'candidate' conformal Killing tensors. However, the Robertson-Walker metrics, being conformally flat, admit the maximal number, namely 84, of reducible conformal Killing tensors and so Amery and Maharaj's results are incomplete.

A generic Robertson–Walker metric admits 6 independent Killing vectors and so 22 (= 1 + 6.7/2) reducible Killing tensors can be constructed from the metric and the Killing vectors – of which 21 are linearly independent. Similarly for the special case of the static Einstein universe which admits a seventh Killing vector, we can construct 30 (= 1 + 7.8/2) reducible Killing tensors from the metric and the Killing vectors – of which 27 are linearly independent. Hence, after finding the gradient conformal Killing tensors and their associated Killing tensors of the generic Robertson–Walker metric (or of the Einstein universe), we need to determine whether they are irreducible by checking if they are independent of these 21 (or 27) reducible Killing tensors. Again the high dimension of these linear subspaces involved and the routine nature of the calculations means that the computations can be automated by use of the computer algebra system Reduce.

3 Finding irreducible Killing tensors in arbitrary spaces

Consider the most general reducible conformal Killing tensor Q_{ab}

$$Q_{ab} = \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ}(\chi_{I(a}\chi_{|J|b)} + \lambda g_{ab})$$

satisfying

$$Q_{(ab:c)} = q_{(a}g_{bc)}.$$

When the vector q_a is a gradient vector $q_{,a}$ then Q_{ab} is said to be *of gradient type*, and has an *associated Killing tensor* K_{ab} where

$$K_{ab} = Q_{ab} - q g_{ab}.$$

Hence, we have an indirect method – using conformal Killing tensors – to find examples of Killing tensors, most of which we expect to be irreducible:

(i) From the conformal Killing vectors of a given metric, we can find all the reducible conformal Killing tensors.

If the metric admits N independent Killing vectors $\xi_1, \ldots, \xi_N$ then the linear space of all reducible conformal Killing tensor contains a linear subspace of reducible Killing tensors. We can exclude these from consideration if we choose the basis of the conformal Killing vectors $\xi_1, \ldots, \xi_N,\ \chi_{N+1}, \ldots, \chi_M$, where the ξ_I's are Killing vectors and we consider only reducible conformal Killing tensors of the form

$$Q_{ab} = \sum_{I=1}^{N} \sum_{J=N+1}^{M} a_{IJ}\xi_{I(a}\chi_{|J|b)} + \sum_{I=N+1}^{M} \sum_{J=I}^{M} a_{IJ}\chi_{I(a}\chi_{|J|b)}.$$

(ii) Next test to see if any of these conformal Killing tensors are of gradient type (including $q_a = 0$) and if so, construct the associated Killing tensors.

(iii) Check directly which of these Killing tensors are irreducible by comparison with the definition in Section 1.

4 Some sample theorems

For individual spaces we can always work from 'first principles' by systematically examining all reducible conformal Killing tensors. However, there are some general theorems which we can easily deduce, and then exploit:

Theorem 2. *Any space which admits a proper homothetic Killing vector χ_a with homothetic constant h as well as a gradient Killing vector $\xi_{,a}$ also admits a Killing tensor $K_{ab} = \chi_{(a}\xi_{,b)} - \xi g_{ab}/h$.*

Theorem 3. *Any space which admits a conformal Killing vector field χ_a which is a gradient also admits the Killing tensor $K_{ab} = \chi_a\chi_b - \chi^2 g_{ab}$, where $\chi^2 = \chi_a\chi^a$.*

Theorem 4. *Any space which admits a proper non-null conformal Killing vector field χ_a which is geodesic (that is $\chi_{a;b}\chi^b = \lambda\chi_a$) also admits the Killing tensor $K_{ab} = \chi_a\chi_b - \chi^2 g_{ab}$.*

These theorems (and others in [3]) generalise the results in [5] and [1]. It is important to note that, in all such cases the Killing tensors have to be investigated directly to determine whether they are irreducible Killing tensors, i.e. not constructed from Killing vectors and metric.

5 Example: A Kimura metric

A Kimura metric (type I) [4] given by

$$ds^2 = \frac{r^2}{b}dt^2 - \frac{1}{r^2b^2}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2,$$

is of Petrov type D with a non-zero energy momentum tensor.

Four Killing vectors

$$\xi_1{}^a = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \qquad \xi_2{}^a = \frac{\partial}{\partial\phi},$$
$$\xi_3{}^a = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \qquad \xi_4{}^a = \frac{\partial}{\partial t},$$

which in covariant form are

$$\xi_{1a} = -r^2\sin\phi\theta_{,a} - r^2\sin\theta\cos\theta\cos\phi\,\phi_{,a}, \qquad \xi_{2a} = -r^2\sin^2\theta\phi_{,a},$$
$$\xi_{3a} = r^2\cos\phi\theta_{,a} - r^2\sin\theta\cos\theta\sin\phi\phi_{,a}, \qquad \xi_{4a} = \frac{r^2}{b}t_{,a}.$$

None are gradient vectors, but ξ_{2a} and ξ_{4a} are hypersurface orthogonal.

Two proper conformal Killing vectors

$$\chi_1{}^a = r^2\frac{\partial}{\partial r} \qquad \text{and} \qquad \chi_2{}^a = r^2 t\frac{\partial}{\partial r} - \frac{1}{br}\frac{\partial}{\partial t}$$

with conformal factors r and rt respectively.

Both are gradient vectors $\chi_{1a} = -\left(r/b^2\right)_{,a}$ and $\chi_{2a} = -\left(rt/b^2\right)_{,a}$.

Eleven reducible proper conformal Killing tensors.

We can immediately write down 11 reducible proper conformal Killing tensors from the symmetrised products of each proper conformal Killing vector with each Killing vector, together with the symmetrised products of the proper conformal Killing vectors

$$\xi_{1(a}\chi_{1b)},\quad \xi_{1(a}\chi_{2b)},\quad \xi_{2(a}\chi_{1b)},\quad \xi_{2(a}\chi_{2b)},\quad \xi_{3(a}\chi_{1b)},\quad \xi_{3(a}\chi_{2b)},\quad \xi_{4(a}\chi_{1b)},\quad \xi_{4(a}\chi_{2b)},$$
$$\chi_{1a}\chi_{1b},\quad \chi_{2a}\chi_{2b},\quad \chi_{1(a}\chi_{2b)}.$$

Killing tensors.

χ_{1a} and χ_{2a} are gradient vectors and so by Theorem 3 we obtain respectively two Killing tensors with non-zero components,

$$K_1^{tt} = \frac{1}{b},\qquad K_1^{\theta\theta} = -\frac{1}{b^2},\qquad K_1^{\phi\phi} = -\frac{1}{b^2\sin^2\theta}$$

and

$$K_2^{tt} = b^2 + \frac{1}{r^2},\qquad K_2^{tr} = -btr,\qquad K_2^{\theta\theta} = -t^2,\qquad K_2^{\phi\phi} = -\frac{t^2}{\sin^2\theta}.$$

Noting that for the case $Q_{ab} = \chi_{1(a}\chi_{2b)} + \lambda g_{ab}$,

$$q_a = r\chi_{2a} + rt\chi_{1a} = r\left(\frac{rt}{b^2}\right)_{,a} + rt\left(\frac{r}{b^2}\right)_{,a} = \left(\frac{r^2t}{b^2}\right)_{,a}$$

is a gradient vector, we find the associated Killing tensor

$$K_3^{tt} = 2\frac{t}{b},\qquad K_3^{tr} = -\frac{r}{b},\qquad K_3^{\theta\theta} = -2\frac{t}{b^2},\qquad K_3^{\phi\phi} = -\frac{t^2}{\sin^2\theta}b^2.$$

These are the only Killing tensors which can be found by this method.

A comparison of the Killing tensor $\boldsymbol{K}_1$ with the Killing vectors shows that it is in fact reducible since,

$$K_{1ab} = \frac{1}{b}\xi_{4a}\xi_{4b} - \frac{1}{b^2}(\xi_{1a}\xi_{2b} + \xi_{2a}\xi_{2b} + \xi_{3a}\xi_{3b}).$$

$\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are irreducible Killing tensors since it is clearly impossible to obtain, using the Killing vectors and metric, those terms in which are explicit functions of t. It is easy to confirm from observation that these three tensors are linearly independent of each other and of the metric.

In Kimura's original work [4] he sought directly for irreducible Killing tensors, and found the two tensors $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$. Koutras [5] only found 8 reducible trace-free conformal Killing tensors and only the 2 Killing tensors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, because he used his less general algorithms. O'Connor and Prince [6] obtained all three Killing tensors since they used the same more general argument as we have done.

6 Summary

We have clarified the concept and definition of *reducible* conformal Killing tensors of order 2 and their trace-free counterparts; this enables us to write down immediately all the reducible conformal Killing tensors in a space where the conformal Killing vectors are known. By identifying those reducible conformal Killing tensors of gradient type we are able to construct associated Killing tensors, most of which we expect to be irreducible. For conformally flat spaces we have shown that all conformal Killing vectors are reducible and so they can all (including both reducible and irreducible Killing tensors) be found by this indirect method.

The full details of the results summarised here are given in [3].

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