New Lumps of Veselov–Novikov Integrable Nonlinear Equation and New Exact Rational Potentials of Two-Dimensional Stationary Schrödinger Equation via $\overline{\partial}$-Dressing Method

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Recently calculated via $\overline{\partial}$-dressing method new lumps of 2+1-dimensional integrable nonlinear Veselov–Novikov (VN) evolution equation and exact rational potentials of two-dimensional stationary Schrödinger (2DSchr) equation with multiple pole of order two wave functions are reviewed. Among the constructed rational solutions and rational potentials are both nonsingular and singular.

1 Introduction

Exact solutions of differential equations of physics are very important for the understanding of various physical phenomena. The generation and application of new methods of calculation of exact solutions was and is actual task in all times for human scientific civilization. In the last two decades the Inverse Spectral Transform (IST) method has been generalized and successfully applied to various two-dimensional nonlinear evolution equations such as Kadomtsev–Petvashvili, Davey–Stewartson, Nizhnik–Veselov–Novikov, Zakharov–Manakov system, Ishimori, two-dimensional integrable sine-Gordon and others (see books [1–4] and references therein). The nonlocal Riemann–Hilbert problem [5], $\overline{\partial}$-problem [6] and more general $\overline{\partial}$-dressing method of Zakharov and Manakov [7–10] are now basic tools for solving two-dimensional integrable nonlinear evolution equations. Great task for mathematicians and physicists now is the generalization of IST method to multidimensional differential equations of mathematical physics.

In the present short note the results of recent calculations [19] of new exact rational so called multiple pole solutions of the famous two-dimensional integrable Veselov–Novikov (VN) nonlinear equation [11]

$$u_t + \kappa u_{zzz} + \overline{\kappa} u_{\overline{z}\,\overline{z}\,\overline{z}} + 3\kappa \left(u\partial_z^{-1} u_{\overline{z}} \right)_{\overline{z}} + 3\overline{\kappa} \left(u\partial_{\overline{z}}^{-1} u_z \right)_z = 0 \tag{1}$$

via $\overline{\partial}$-dressing method of Zakharov and Manakov are summarized. Here and in what follows $z = x + iy$, $\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\overline{z}} \equiv \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$; $u_z \equiv \frac{\partial u}{\partial z}$, $\partial_z^{-1}\partial_z = \partial_{\overline{z}}^{-1}\partial_{\overline{z}} = 1$; κ is arbitrary complex constant; $u(z, \overline{z}, t)$ is scalar function of space-time variables. It is well known that VN equation can be represented as compatibility condition for two linear auxiliary problems

$$L_1\psi = \left(\partial_{z\overline{z}}^2 + u(z, \overline{z}, t) \right) \psi = 0,$$
$$L_2\psi = \left(\partial_t + \kappa\partial_z^3 + \overline{\kappa}\partial_{\overline{z}}^3 + 3\kappa \left(\partial_z^{-1} u_{\overline{z}} \right) \partial_z + 3\overline{\kappa} \left(\partial_{\overline{z}}^{-1} u_z \right) \partial_{\overline{z}} \right) \psi = 0 \tag{2}$$

in the form of Manakov's triad representation

$$[L_1, L_2] = BL_1, \qquad B = 3 \left(\kappa\partial_z^{-1} u_{\overline{z}\,\overline{z}} + \overline{\kappa}\partial_{\overline{z}}^{-1} u_{zz} \right). \tag{3}$$

The $\overline{\partial}$-dressing method of Zakharov and Manakov [7–10] allows

- To construct integrable nonlinear equations together with corresponding linear auxiliary problems.

- Using the solution of linear auxiliary problems via reconstruction formula to calculate broad classes of exact solutions and to solve Cauchy problem for integrable nonlinear evolution equations.

- To construct explicitly broad classes of exactly solvable variable coefficients-fields and corresponding wave functions of linear auxiliary problems.

In conclusion of this introduction let us mention that exact integration of VN equation has remarkable history, more detailed information about all known cases of exact integration of (1) one can found in [11–15] and in doctorate dissertation of Grinevich [16], see also the review [17] and books [1–4]. The calculated in the paper [19] and reviewed in the present note multiple pole of order two rational solutions of VN equation (1) are completely new.

2   General formulas for calculating of exact multiple pole rational solutions

The basic equation of $\overline{\partial}$-dressing method in our case is the following scalar non-local $\overline{\partial}$-problem [7–9]:

$$\frac{\partial \chi}{\partial \overline{\lambda}} = (\chi * R)(\lambda, \overline{\lambda}) = \iint d\mu \wedge d\overline{\mu}\, \chi(\mu, \overline{\mu})\, R(\mu, \overline{\mu}; \lambda, \overline{\lambda}) \tag{4}$$

stated here in the auxiliary space of spectral variables μ, λ. Via "long" derivatives $D_1 = \partial_\xi + i\lambda$, $D_2 = \partial_\eta - i\frac{i\epsilon}{\lambda}$, $D_3 = \partial_t + i\left(\kappa_1 \lambda^3 - \kappa_2 \frac{\epsilon^3}{\lambda^3}\right)$ containing the "spectral" variable λ one can construct the auxiliary linear problems for VN equation (1):

$$L_1\chi = (D_1 D_2 + V_1 D_1 + V_2 D_2 + u)\chi = 0,$$
$$L_2\chi = \left(D_3 + \kappa D_1^3 + \overline{\kappa} D_2^3 + W_1 D_1^2 + W_2 D_2^2 + W_3 D_1 + W_4 D_2 + W\right)\chi = 0 \tag{5}$$

explicitly containing the "spectral" parameter λ. In terms of the wave function

$$\psi = \chi e^{F(\lambda)}, \qquad F(\lambda) := i\left(\lambda z - \frac{\epsilon}{\lambda}\overline{z}\right) - i\left(k\lambda^3 - \overline{k}\frac{\epsilon^3}{\lambda^3}\right)t \tag{6}$$

the linear auxiliary problems (5) in potential case for the operator L_1 (when $V_1 = V_2 = 0$) coincide with the problems (2). In the $\overline{\partial}$-dressing method the non-analytic wave functions χ are explored. The detailed derivation of reconstruction formulas for the variable coefficients V_1, V_2, u and W_1, W_2, W_3, W_4, W of auxiliary linear problems (5) and of the conditions of reality and potentiality in the framework of $\overline{\partial}$-dressing method in the paper [19] are presented (see also [20]).

The multiple pole solutions u of VN equation (1) correspond to special analytic structure of wave function χ (or ψ): the solution u is multiple pole solution of order m if the maximum order of pole terms in Loran expansion of χ near some points λ_k is equal to m. The solution of $\overline{\partial}$-problem (4) in the case of canonical normalization, $\chi \to \infty$ as $\lambda \to 1$, is equivalent to the solution of singular integral equation:

$$\chi(\lambda) = 1 + \iint_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)}) \iint_C d\mu \wedge d\overline{\mu}\, \chi(\mu, \overline{\mu})\, R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'}) e^{F(\mu) - F(\lambda')}, \tag{7}$$

where $F(\lambda)$ is given by the formula (6). The kernel R_0 of $\overline{\partial}$-problem (4) in the following form corresponds to the multiple pole rational solutions u of VN equation (1):

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \frac{\pi}{2} \sum_p \sum_{k,m}^{N_p} r_k^{(p)}(\mu) l_m^{(p)}(\lambda) \delta^{(k)}(\mu - \lambda_p) \delta^{(m)}(\lambda - \lambda_p). \tag{8}$$

The insertion of (8) into (7) gives the Loran expansion of wave function χ near the points $\lambda = \lambda_p$ and this leads to multiple pole solutions u of VN equation. The scheme for calculating of 2+1-dimensional integrable nonlinear evolution equations via $\overline{\partial}$-dressing method of multiple pole solutions was developed at first in the paper [18].

Let us review here some recent results [19] of calculation of exact order two multiple pole rational solutions of VN equation (1) with constant asymptotic values $-\epsilon$ at infinity:

$$u = \widetilde{u}(x, y, t) - \epsilon, \qquad \widetilde{u}(x, y, t) \underset{|z| \to \infty}{\longrightarrow} 0. \tag{9}$$

In this case the first linear auxiliary problem (5) (expressed in terms of wave function ψ (6)) in potential case ($V_1 = V_2 = 0$) coincides with problem (2a) or with two-dimensional stationary Schrödinger equation

$$(\partial_{z\bar{z}}^2 + \widetilde{u})\psi = \epsilon\psi. \tag{10}$$

So the construction of exact rational multiple pole solutions u with constant asymptotic values at infinity $u \underset{|z| \to \infty}{\longrightarrow} -\epsilon$ of VN equation (1) means simultaneously construction of exact multiple pole rational potentials $\widetilde{u} := u + \epsilon$ of two-dimensional stationary Schrödinger equation (10). Recently the calculations of exact rational multiple pole of order two solutions of VN equations for the kernel R_0 of the form

$$R_0 = \frac{\pi}{2} \sum_{k=1}^N \left[A_k \delta_\mu(\mu - \lambda_k) \delta_\lambda(\lambda - \lambda_k) + B_k \delta_\mu(\mu + \lambda_k) \delta_\lambda(\lambda + \lambda_k) \right. \tag{11}$$

$$\left. + \frac{\epsilon^3}{|\mu|^2 |\lambda|^2 \bar{\mu}\bar{\lambda}} \left(\overline{A}_k \delta_{\frac{\epsilon}{\mu}} \left(\frac{\epsilon}{\mu} + \bar{\lambda}_k \right) \delta_{\frac{\epsilon}{\lambda}} \left(\frac{\epsilon}{\lambda} + \bar{\lambda}_k \right) + \overline{B}_k \delta_{\frac{\epsilon}{\mu}} \left(\frac{\epsilon}{\mu} - \bar{\lambda}_k \right) \delta_{\frac{\epsilon}{\lambda}} \left(\frac{\epsilon}{\lambda} - \bar{\lambda}_k \right) \right) \right]$$

containing the terms with first derivatives of delta-functions and satisfying to the reality condition $u = \overline{u}$ for the solutions have been performed by us (D.V.G. and F.I.B.) [19]. The obtained results for nonsingular and singular rational multiple pole of order two solutions u in the following two sections are stated.

3 Rational nonsingular solutions of VN equation corresponding to double pole wave functions

Let us consider at first the case of negative energies $E = -\epsilon = -|\lambda_k|^2 < 0$ in the standard representation $(-\frac{1}{4}\Delta - \widetilde{u})\psi = E\psi$ of stationary Schrödinger equation. Some difficulties (really this is the main problem in calculations of exact solutions of VN equation) presents the fulfillment of the potentiality condition $V_2 = 0$ for the operators L_1 of the first linear auxiliary problem (5). Long calculations [19] lead in the considered case $\epsilon = |\lambda_k|^2 > 0$ to the following restrictions from potentiality on the constants A_k, B_k of the kernel R_0 (11):

$$\frac{\lambda_k^3}{B_k} - \frac{\lambda_k^3}{A_k} = \frac{i}{2}, \qquad \frac{\bar{\lambda}_k^3}{\overline{B}_k} = -\frac{\lambda_k^3}{B_k}, \qquad \frac{\bar{\lambda}_k^3}{\overline{A}_k} = -\frac{\lambda_k^3}{A_k}, \qquad k = 1, \ldots, N. \tag{12}$$
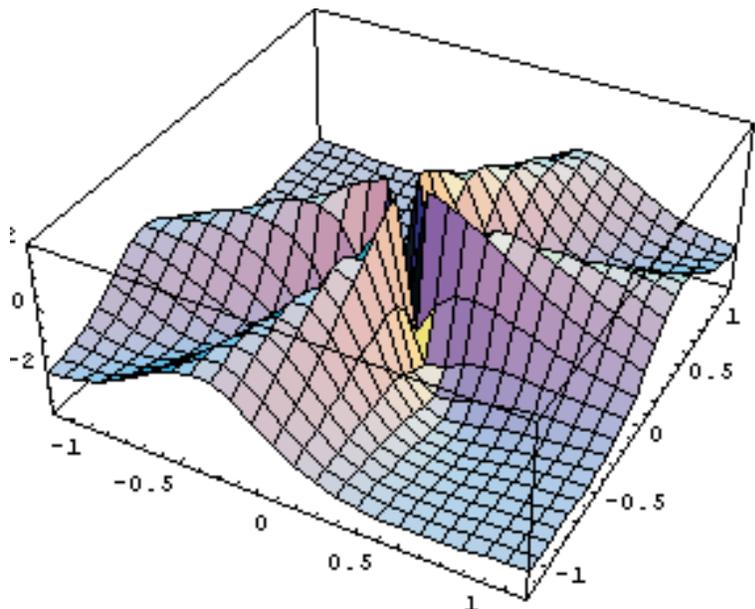
Figure 1. Nonsingular multiple order two pole rational (lump) solution of VN equation.

For the case $N = 1$ in the sum (11) under the additional condition on κ and λ_1 of the form $\kappa\lambda_1^3 = -\bar{\kappa}\bar{\lambda}_1^3$ one obtains the following exact rational nonsingular order two multiple pole solution [19] with constant asymptotic value $-\epsilon$ at infinity of VN equation (1):

$$u = -\epsilon - 32\epsilon\frac{(\lambda_R\hat{x} - \lambda_I\hat{y})^2[4(\lambda_R\hat{x} - \lambda_I\hat{y})^4 - 9(\lambda_I\hat{x} + \lambda_R\hat{y})^2]}{[4(\lambda_R\hat{x} - \lambda_I\hat{y})^4 + 3\,|\lambda_1|^2\,(\hat{x}^2 + \hat{y}^2)]^2}. \tag{13}$$

Here $\lambda_1 = \lambda_R + i\lambda_I$ and the wave variables $\hat{x}$ and $\hat{y}$ are introduced: $\hat{x} = x - V_1t - x_0$, $\hat{y} = y - V_2t - y_0$ with real $V_1 = 18i\frac{\lambda_I}{|\lambda_1|^2}k\lambda_1^3$, $V_2 = 18i\frac{\lambda_R}{|\lambda_1|^2}k\lambda_1^3$. This solution represents a new rational nonsingular lump, with behavior at infinity as $u \underset{|z|\to\infty}{\longrightarrow} -\epsilon + O\left(\frac{1}{|z|^2}\right)$, of VN equation propagating on the plane x, y with the velocity $\overrightarrow{V} = (V_1, V_2)$. One can see an illustration for this new lump solution of VN equation on the Fig. 1[1].

Corresponding new exact rational nonsingular potential of two-dimensional stationary Schrödinger equation (10) with multiple pole of order two wave function ψ (6) gives the formula $\widetilde{u} = u + \epsilon$ with u from (13).

4 Rational singular solutions of VN equation corresponding to double pole wave functions

Let us consider now the case of positive energies $E = -\epsilon = |\lambda_k|^2 > 0$ in the standard representation $\left(-\frac{1}{4}\Delta - \widetilde{u}\right)\psi = E\psi$ of stationary Schrödinger equation. Some difficulties (as in the previous case $\epsilon > 0$ of Section 3) presents fulfillment of the potentiality condition $V_2 = 0$ for the operators L_1 of the first linear auxiliary problem (5). Long calculations [19] in the considered case $\epsilon = -|\lambda_k|^2 < 0$ lead to the following restriction from potentiality on the constants A_k, B_k of the kernel R_0 (11):

$$\frac{\lambda_k^3}{B_k} - \frac{\lambda_k^3}{A_k} = \frac{i}{2}, \qquad \frac{\bar{\lambda}_k^3}{\bar{A}_k} = -\frac{\lambda_k^3}{B_k}, \qquad k = 1, \dots, N. \tag{14}$$

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[1]Figures in colour will be available only in electronic version.
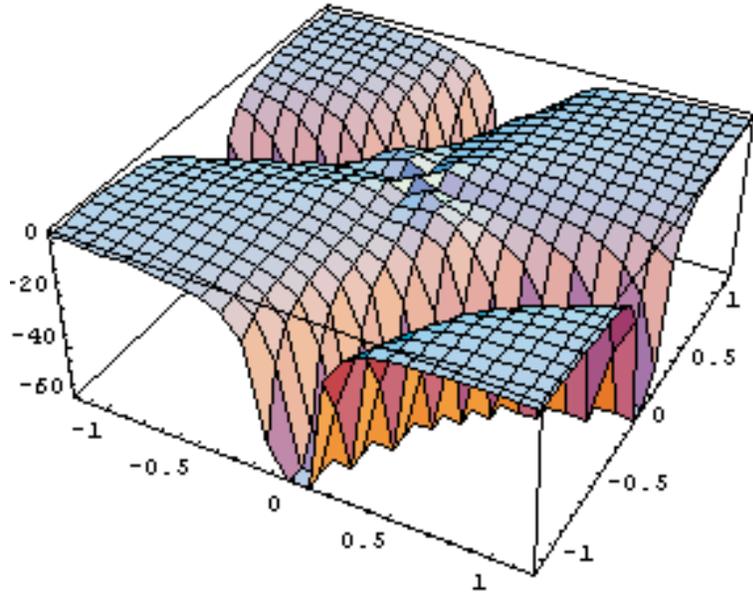
Figure 2. Singular multiple order two pole rational solution of VN equation.

For the case $N = 1$ in the sum (11) under the additional condition on κ and λ_1 of the form $\kappa\lambda_1^3 = \overline{\kappa}\overline{\lambda}_1^3$ one obtains the following exact rational singular multiple pole of order two solution with constant asymptotic value $-\epsilon$ at infinity of VN equation (1):

$$u = -\epsilon - 32\epsilon\frac{(\lambda_I\hat{x} + \lambda_R\hat{y})^2[4(\lambda_I\hat{x} + \lambda_R\hat{y})^4 + 9(\lambda_R\hat{x} - \lambda_I\hat{y})^2]}{[4(\lambda_I\hat{x} + \lambda_R\hat{y})^4 - 3\,|\lambda_1|^2\,(\hat{x}^2 + \hat{y}^2)]^2}. \tag{15}$$

Here $\lambda_1 = \lambda_R + i\lambda_I$ and the wave variables $\hat{x} = x - V_1 t - x_0$, $\hat{y} = y - V_2 t - y_0$ with real $V_1 = 9\frac{\lambda_R}{|\lambda_1|^2}k\lambda_1^3$, $V_2 = -9\frac{\lambda_I}{|\lambda_1|^2}k\lambda_1^3$ are introduced. The singular rational solution (15) propagates on the plane x, y with the velocity $\overrightarrow{V} = (V_1, V_2)$ One can see an illustration for this new singular rational solution of VN equation on the Fig. 2.

The corresponding new exact rational singular potential of two-dimensional stationary Schrödinger equation (10) gives the formula $\widetilde{u} = u + \epsilon$ with u from (15).

In conclusion of this note let us mention that for VN nonlinear integrable equation (1) the first auxiliary linear problem (2), i.e. two-dimensional stationary Schrödinger equation (5), is self-adjoint, but multiple pole solutions exist as the present calculations show [19]. For the KP or mKP equations the first auxiliary linear problems are non-self-adjoint, and with the non-self-adjoint character of these problems the existence of multiple pole solutions has been connected [18]. By VN equation we have the first explicit example which shows that multiple pole solutions may exist even for cases with self-adjoint first auxiliary linear problems. Our work on multiple pole rational solutions of VN equation and multiple pole rational potentials of 2DSchr equation for more general choices of kernel R_0 (8) and its parameters in general position is in progress and the results will be published elsewhere.

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