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Extensions of Physical Theories: Quantum Mechanics with Internal Degrees of Freedom and Nonlinear Transformations

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We review several nonlinear mathematical models for physical systems which are based on a gauge-like structure. For systems with internal degrees of freedom like spin, a nonlinearity leads to coupling of the internal coordinates; this kind of interaction appears as an addition to the coupling through Pauli or Dirac matrices. We briefly discuss a family of nonlinear and relativistically invariant extensions of the Dirac equation.

1 Introduction

Construction of mathematical models for physical systems is often connected with the problem of extending a partly successful theory. The motivations are different: e.g. the parameter regions and the space time domains of a model are too narrow or the chosen mathematical framework is not suitable for a realistic model. Such extensions are expected to explain new observable effects and to provide corrections to the "old" theory. For their development different options are at hand: generalisations of the theory, changes of the present mathematical formalism or inventions of new mathematical methods.

2 An example

2.1 Gauge like models

We illustrate this problem for an example in the following (gauge theoretic) design: A system is modelled through linear partial differential operator (PDO) $\mathbb{D}$ on a suitable k-vector valued function space $\mathcal{G}$ over space-time $\mathbb{R}^3_x \times \mathbb{R}^1_t$. The solution variety

$$\mathcal{B} = \{f(x,t)/\mathbb{D}f(x,t) = 0,\ f(x,t) \in \mathcal{G}\}$$

serves as a *basic* background structure for the set $\mathcal{O} = \{o\}$ of (all or a part of the) physical observables o. This means that the elements of $\mathcal{O}$ are modelled through functions F_α, $\alpha = 1, \ldots, s$, of the solution variety. These *utility* functions form a set $B = \{F_\alpha(f),\ \alpha = 1, \ldots, s, f \in \mathcal{B}\}$ with a map

$$\mathcal{O} \ni o \mapsto F_\alpha(f),$$

where α depends on o. Hence the system is modelled through $\{\mathcal{G}, \mathbb{D}, B\}$. In general this description is not complete. Further properties have to be included and the physical interpretation is essential. The characterisation of the models as 'gauge like' stems from the example of electrodynamics and quantum mechanics [1].

2.2 Extensions of gauge like models

There are only few plausible and manageable possibilities to extend a gauge like theory based on $\{\mathcal{G}, \mathbb{D}, B\}$. We mention here two lines: the *deformation* of possible symmetries of $\mathbb{D}$ and the construction of nonlinear $\mathbb{D}$ through extensions with nonlinear transformations in $\mathcal{G}$.

A. Deformations of symmetries. If $\mathbb{D}$ has a group theoretical or an algebraic symmetry S, one can extend this symmetry through deformations S_{def} of S. Use the function space $\mathcal{G}'$ which carries a realisation of S_{def}, construct from $\mathbb{D}$ the corresponding deformed operator $\mathbb{D}_{\text{def}}$ and take the same utility functions as before. The result is a deformed extension $\{\mathcal{G}', \mathbb{D}_{\text{def}}, B\}$ of $\{\mathcal{G}, \mathbb{D}, B\}$.

B. Nonlinear extensions. Another method is based on an interplay between the linear operator $\mathbb{D}$ and B; it leads to nonlinear extensions. Consider a group $\mathfrak{N}$ of linear or nonlinear and invertible transformations in $\mathcal{B}$

$$\mathfrak{N} \ni N : \mathcal{B} \ni f \mapsto N[f] = N(f)f \in \mathcal{B}$$

The partial differential equation (PDE) $\mathbb{D}f = 0$ implies

$$\mathbb{D}_N = \mathbb{D}N(f) \quad \text{with} \quad \mathbb{D}_N f' = 0, \quad f' = N(f)f.$$

The operator $\mathbb{D}_N$ is nonlinear (for nonlinear N) and leads together with the utility functions to the same description as before *if* N is chosen such that the utility functions are invariant under N, namely

$$F_\alpha(N[f]) = F_\alpha(f), \qquad F_\alpha \in B.$$

Transformations with this *Equivalence Condition* [1] also generate a transformation group $\mathfrak{N}^e \subset \mathfrak{N}$ (e denotes equivalence). The theory based on $\{\mathcal{G}, \mathbb{D}, B\}$ and $\{\mathcal{G}, \mathbb{D}_N, B\}$ describes the same physical system; both are *physically equivalent*. However, through the description with $\mathbb{D}_N$ some *additional properties* A of the linear system which are not encoded in $\{\mathcal{G}, \mathbb{D}, B\}$, could be lost, e.g. the space-time symmetry or the separability of $\mathbb{D}$. To restore such properties, restrict $\mathfrak{N}$ to subgroups $\mathfrak{N}^A$ such that they are invariant and use $\mathfrak{N}^A \cap \mathfrak{N}^e = \mathfrak{N}^{A,e}$. Nonlinear extensions through $\mathfrak{N}^e \in \mathfrak{N}$ are trivial reformulations of the linear system. To get something "new", i.e. a non trivial extension, change $\mathbb{D}_N$ to $\mathbb{D}_{\text{ext}} = (\mathbb{D}_N)_{\text{ext}}$ if there is an obvious and plausible method to do this [1]. This is the case if the coefficients (functions, numbers) in $\mathbb{D}_N$ are related among each other: Break this relations, i.e. consider the coefficients as independent or free, get $\mathbb{D}_{\text{ext}}$, use the same $\mathcal{G}$ and B and find a nonequivalent description $\{\mathcal{G}, \mathbb{D}_{\text{ext}}, B\}$; hence a "new" system appears which contains the "old" system.

2.3 Quantum mechanics as a gauge like model

We specialise this design to the time evolution of a quantum mechanical observable. Here the function space $\mathcal{G}$ is a suitable Hilbert space $\mathcal{H}$, its elements are k-vector valued wave functions $\psi(x,t) = (\psi_1(x,t), \ldots, \psi_k(x,t))$, $\mathbb{D}_{\mathsf{Q}}$ is the quantum mechanical time evolution, e.g. the Schrödinger operator $\mathbb{D}_S$, and the (only, i.e. $s = 1$) utility function is the positional density for all x and t

$$F_\varrho \overline{\psi}\psi = \sum_{j=1}^{k} \psi_j^* \psi_j$$

with $\overline{\psi} = (\psi_1^*, \ldots, \psi_k^*)^T$, T denotes the transpose. All properties of the system are encoded in one utility function F_ϱ and the evolution operator (see e.g. [3]). The system is characterised through $(\mathcal{H}, \mathbb{D}_{\mathsf{Q}}, F_q)$. A similar design appears in gauge theories like electrodynamics; the situation in relativistic quantum field theory is different.

2.4 Extensions of quantum mechanical evolutions

We sketch some application of the lines A and B (Section 2.2) and present in Sections 3 and 4 the detailed version of line B.

A. Deformations of symmetries. A symmetry of a quantum system is often connected with algebraic symmetries through a linear (possibly integrable) representation of a finite or infinite dimensional (space time or kinematical) Lie algebra; examples in $\mathbb{R}^3_x \times \mathbb{R}^1_t$ are: the central extension of the Galilei algebra, the Poincaré algebra or the Lie algebra of a subgroup of the inhomogeneous diffeomorphism group of the configuration space (Borel quantisation [4, 5]); inhomogeneous current algebras [6, 7]. There are physical and mathematical reasons to deform (extend or contract) this algebra (in the sense of Gerstenhaber) in the category of Lie algebras or, more general, in the category of Hopf algebras. In the first case one gets a relation between Poincaré and Galilei algebras. In the latter case one finds a q-algebra or q symmetry, the evolution operator appears as a difference operator, the space-time has discrete properties with a corresponding behaviour of the wave functions and the utility function. Another related extension is choice of a noncommutative space-time. Algebraic symmetries can be formulated in the usual (linear) quantum mechanical framework. Properties and behaviour of the corresponding systems are available and a comparison with undeformed symmetries is possible. The interest in this mathematical and physical field is strong. However, there is up to now only "weak" experimental evidence for extended symmetries.

B. Nonlinear extensions. Quantum mechanics is based deeply on linear structures (the above mentioned gauge like design is e.g. a linear construction). A physically acceptable nonlinear extension of quantum mechanics – e.g. with nonlinear evolutions equations – such that the linear theory appears as a linear approximation is not known. The reason is that the linear framework does not allow for nonlinear operators (Hamiltonians) [8]. It is in principle difficult to develop such extensions; also the usual physical interpretation breaks down. On the other hand a deeper reason for a linear structure of quantum mechanics is unknown. Therefore a reasonable first ansatz for a nonlinear extension is of interest, especially a derivation of nonlinear evolutions equations from first (physical) principles. The known methods for calculation of physical properties can be applied in an extended theory as approximations for deviations from the linear theory. In recent quantum mechanical precision experiments such nonlinear deviations were not found [9]. But future experiments and a new experimental design could reveal quantum mechanical nonlinearities.

3 Nonlinear Schrödinger, Pauli and Dirac equations

3.1 Linear evolution equation and a locality condition

We present and partly review nonlinear extensions for the evolution of a quantum mechanical observable in the $(\mathcal{H}, \mathbb{D}_{\mathrm{Q}}, F_\varrho)$ design (see Section 2.3). For $k = 1$, $\mathbb{D}_{\mathrm{Q}}$ is the Schrödinger operator $\left(\nu_1 = -\frac{\hbar}{2m},\ \mu_0 = \frac{1}{\hbar}\right)$

$$\mathbb{D}_S = \left(-i\partial_t + \nu_1 \Delta + \mu_0 V(x) \right);$$

for $k = 2$, $\mathbb{D}_{\mathrm{Q}}$ is the Pauli operator

$$\mathbb{D}_P = \left(-i\partial_t + V_1 \Delta + \mu_0 V(x) + \alpha \vec{B}(x,t) \times \vec{\sigma} + \beta \vec{L} \times \vec{\sigma} \right),$$

with $\vec{\sigma}$ as Pauli matrices, $\vec{B}(x,t)$ as external field and $\vec{L}$ as angular momentum; for $k = 4$, $\mathbb{D}_{\mathrm{Q}}$ is the Dirac operator ($\hbar = 1$)

$$\mathbb{D}_D = \left(j_\mu p^\mu - m \right), \qquad p^\mu = i \frac{\partial}{\partial x^\mu};$$

we use the notation of [13]; the component index k runs from 0 to 3, the time component is labelled by 0; the solutions depend on x_0, x_1, x_2, x_3.

As a first *additional simplifying property A* for $\mathbb{D}$ (to avoid e.g. boundary value discussions) we want that the highest order of $\mathbb{D}_{\mathrm{Q}}$ and of $\mathbb{D}_{\mathrm{Q},N}$ is the same. This implies a locality condition for N

$$N(\psi) = (N_1[\psi], \ldots, N_k[\psi]) = (N_1(\psi)\psi_1, \ldots, N_k(\psi)\psi_k).$$

The nonlinear transformations $N_i(\psi)$, $i = 1, \ldots, k$ act on $\mathcal{H}$ as multiplicative operators and depend on ψ only. They form a group $\mathfrak{N}^l \subset \mathfrak{N}$ (l stands for locality).

3.2 A separability condition

A next additional property, a *separability condition*, stems from the fact that quantum mechanical systems are composed of n one-particle systems with Hilbert spaces $\mathcal{H}_j$ and self adjoint operators $\mathbb{A}_j$ for the same observable o' in each system $j = 1, \ldots, n$. The Hilbert space for the n-particle system is chosen as

$$\mathcal{H}^{(n)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$$

with a dense set of product states

$$P^{(n)} = \left\{\psi^1 \otimes \cdots \otimes \psi^n,\ \psi^j \in \mathcal{H}_j,\ j = 1, \ldots, n\right\}.$$

The corresponding observable o' of the n-particle system is represented on the product states as

$$\mathbb{A}^{(n)} = \left(\mathbb{A}_1\psi^1 \otimes \cdots \otimes \mathbb{A}_n\psi^n\right)$$

and extend to $\mathcal{H}^{(n)}$ through linear completion.

For a nonlinear operator, like N, this procedure is not possible. As in the linear case $N^{(n)}$ can be defined on product states $\mathcal{P}^{(n)}$ but an extension to $\mathcal{H}^{(n)}$ is not defined; some further guideline is needed. Here one of the difficulties for nonlinear extensions shows up. It is plausible to require as a minimal assumption that for a 2-particle system the relation

$$N^{(2)}\left[\psi^1 \otimes \psi^2\right] = N\left[\psi^1\right] \otimes N\left[\psi^2\right]$$

holds. This is a *separability condition* [1], it implies that a k -component function $N(\psi)$ on $P^{(2)}$ exists such that for $N \ni \mathfrak{N}^l$ and any two ψ^1, ψ^2, $i = 1, 2$ the condition holds. Such N form a group $\mathfrak{N}^{l,s}$ (s for separability). Its elements are calculated in [10]. We quote the results for $\mathcal{G}$ (which may be specified for the corresponding Hilbert space) and use the (not unique) polar decomposition for $\psi_j(x,t) = R_j(x,t)\exp iS_j(x,t)$.

The transformations

$$N : \mathcal{G} \mapsto \mathcal{G}$$

which satisfy the locality and separability condition are given through

$$N_j(\psi)_{[a,b,F]} = (R_j)^{a-1}\exp\left((b-i)S_j\right) \cdot F_j\left(\frac{R_2}{R_1}, \ldots, \frac{R_n}{R_1}, S_2 - S_1, \ldots, S_n - S_1\right)$$

The N are labelled by two complex parameter a, b and k functions

$$F(\psi) = (F_1(\psi), \ldots, F_k(\psi)).$$

In order to derive the group multiplication law, arrange the parameter $a = \widetilde{a} + i\widehat{a}$, $b = \widetilde{b} + i\widehat{b}$ as

$$K = \begin{pmatrix} \widetilde{a} & \widehat{a} \\ \widetilde{b} & \widehat{b} \end{pmatrix}$$

and find for the product

$$N_{[K_2,F_2]}\left(N_{[K_1,F_1]}(\psi)\,\psi\right) N_{[K_1,F_1]}(\psi)\,\psi = N_{[K_3,F_3]}\psi$$

with

$$K_3 = K_2 K_1,$$
$$F_{31} = |F_{1j}|^{a_2} \exp\left(b_2 \arg F_{1j}\right) F_{2j}\left(u_2, \ldots, u_n, v_2, \ldots, v_n\right).$$

The variables in F_{1j} and in F_{2j} are as before resp. given through $(i = 2, \ldots, k)$

$$u_i = \left(\frac{R_i}{R_1}\right)^{\widetilde{a}_i} \exp \widetilde{b}_1 \left(S_1 - S_i\right) |F_{1i}||F_{11}|^{-1},$$
$$v_i = \widehat{a}_1 \ln\left(\frac{R_i}{R_1}\right) + \widehat{b}_1 \left(S_1 - S_i\right) \arg\left(F_{1i} F_{11}\right).$$

respectively. The group $\mathfrak{N}^{l,s}$ is a local (infinite) parameter group; the element N_{11} is the identity, $N_{[K,F]}$ is locally invertible in a neighbourhood of the identity; the associativity is respected. For $n > 2$ the corresponding result holds.

3.3 Quantum mechanical equivalence condition

We combine the essential equivalence condition with the locality condition (more technical) and the separability condition (necessary to build n-particle states) and determine

$$\mathfrak{N}^{l,s} \cap \mathfrak{N}^{l,e} = \mathfrak{N}^{l,s,e}.$$

We quote the result of [10]: The transformations $N \in \mathfrak{N}^{l,s,e}$ which satisfy the locality, separability and equivalence condition are given through

$$N_j(\psi)_{[a,b,F^e]} = R_j^{i\widehat{a}} \exp i\left(\widehat{b} - 1\right) S_j \cdot F_j^e\left(\frac{R_2}{R_1}, \ldots, \frac{R_n}{R_1}, S_2 - S_1, \ldots, S_n - S_1\right).$$

with the condition for the k components of F^e:

$$\sum_{j=1}^{k} R_j^2 \left(|F_j^e|^2 - 1\right) = 0.$$

The $\mathfrak{N}^{l,s,e}$ form a group.

3.4 Application for $k = 1$: Schrödinger equation

In the Schrödinger case we have in the configuration space $\mathbb{R}^3_x$ with wave function $\psi = R \exp is$ for

$$\mathfrak{N}^{l,s,e} \ni N[\psi]_{[\widehat{a},\widehat{b}]} = R^{1+i\widehat{a}} \exp i\widehat{b} s \cdot F^e.$$

F^e is a constant phase factor; we choose $F^e = 1$. This transformation is written as $(\widehat{a} = \gamma,\ \widehat{b} = \Lambda)$

$$N[\psi]_{[\gamma,\Lambda]} = \exp i\left(\gamma \ln R + (\Lambda - 1)\, s\right) \psi.$$

Following our construction we calculate [1] a nonlinear Schrödinger equation $\mathbb{D}_{S,N}\psi' = 0$ with $\psi' = N[\psi]_{[\gamma,\Lambda]}$ and find

$$
\begin{aligned}
&i\partial_t\psi = \left(\nu'\Delta + \mu_0' V + F_{DG}^{(0)}[\psi]\right)\psi,\\
&F_{DG}^{(0)}[\psi] = \mu_1'\left(R_1[\psi] - R_4[\psi]\right) - \frac{1}{2}\mu_1' R_2[\psi] + \kappa'\left(R_2[\psi] - \frac{1}{2}R_5[\psi]\right),\\
&R_1[\psi] = \frac{\nabla\cdot\mathbf{J}}{\rho}, \qquad R_2[\psi] = \frac{\Delta\rho}{\rho},\\
&R_3[\psi] = \frac{\mathbf{J}^2}{\rho^2}, \qquad R_4[\psi] = \frac{\mathbf{J}\cdot\nabla\rho}{\rho}, \qquad R_5[\psi] = \frac{(\nabla\rho)^2}{\rho^2},
\end{aligned}
$$

where $\rho = \overline{\psi}\psi$ and $\mathbf{J} = \frac{1}{2}\left(\overline{\psi}\nabla\psi - \left(\nabla\overline{\psi}\right)\psi\right)$.

The coefficients in this PDO are not independent; the constraint relations are

$$
\nu_1' = \frac{1}{\Lambda}\nu_1, \qquad \mu_0' = \Lambda\mu_0, \qquad \mu_1' = -\frac{\gamma}{\Lambda}\nu_1, \qquad \kappa' = \frac{\gamma^2 + \Lambda^2 - 1}{2\Lambda}\nu_1.
$$

Following the method mentioned in Section 2.2 we extend $\mathbb{D}_{S,N}$ through a breaking of this constraints. An extended evolution operator $(\mathbb{D}_{S,N})_{\text{ext}}^{(1)}$ appears. One can continue this procedure: Apply $N \in \mathfrak{N}^{l,s,e}$ to $\mathbb{D}_{\text{ext}}^{(1)}$ one gets an evolution operator with constraints and different from $\mathbb{D}_{\text{ext}}^{(1)}$. Break this constraints and continue. The process stops after 4 steps and yields a 8-parameter nonlinear Schrödinger equation [1] (DG equation)

$$
\begin{aligned}
i\partial_t\psi = {} & \left(\nu_1\Delta + \mu_0 V\right)\psi + i\nu_2 R_2[\psi]\psi + \mu_1 R_1[\psi]\psi + \left(\mu_2 - \frac{1}{2}\nu_1\right)R_2[\psi]\psi\\
& + \left(\mu_3 + \nu_1\right)R_3[\psi]\psi + \mu_4 R_4[\psi]\psi + \left(\mu_5 + \frac{1}{4}\nu_1\right)R_5[\psi]\psi.
\end{aligned}
$$

The DG equation was first derived [11] with a mathematically completely different method (representation theory of infinite dimensional Lie algebras) and motivated physically through an analysis of geometrical details of the kinematics of the system.

3.5 Application for $k = 2$: Pauli equation

Here, already the special complications connected with vector-valued wave functions appear. This is because a two component function $F = (F_1, F_2)$ reduce to two equivalent constant phase factors only for a special type of solutions (type 0); they have the property that different (spin-) components of the wave function are not coupled through nonlinearities in the evolution equation; only the spin matrices yields a coupling. For all other types a coupling through the nonlinear term and the spin matrices occurs. This remains valid also for extension of the non linear Pauli operator $\mathbb{D}_P$. The corresponding solutions are involved and clumsy. Even if $\alpha, \beta = 0$, i.e. if the term with the Pauli matrices is absent, the nonlinearities yield a coupling of the (spin-) components which resembles in some sense an analogue for a spin-like degree of freedom. A more detailed analysis of extensions of nonlinear Pauli equations is in preparation [12].

4 Applications for $k = 4$: Dirac equation

4.1 Poincaré invariance

We mentioned already that a description through $(\mathcal{G}, \mathbb{D}, B)$ is in general not complete. Additional properties of D could be physically significant for the interpretation and for an extension, e.g. the

behaviour under space-time translations. For the Dirac operator $\mathbb{D}_D$ the spin $\frac{1}{2}$ representation U of the Poincaré group is such an essential property. Hence, it is reasonable to choose N from $\mathfrak{N}^l$ such that $\mathbb{D}_{D,N}$ has the same transformation property as $\mathbb{D}_D$. This is guaranteed if (local) N transform under $\mathcal{U}$ as

$$N[U\psi] = UN[\psi]$$

The general form of N with the above property follows from a theorem of Fushchych and Zhdanov [13, 14] (in connection with nonlinear Dirac equation with local nonlinearities)

$$N(\psi)\psi = (f_1(X,Y) + f_2(X,Y)\gamma_5)\psi,$$

f_1, f_2 are independent complex functions depending on the invariant quantities $X = \overline{\psi}\psi$ and $Y = \overline{\psi}\gamma_5\psi$. Such N form a group $\mathfrak{N}^{l,P}$ (P for Poincaré invariance).

4.2 Consideration of separability and equivalence conditions

In this section (see also [10]) we are interested in the *quantum mechanical* Dirac equation. Therefore we assume for $N \in \mathfrak{N}^{l,P}$ also the separability and the equivalence condition. The situation in quantum field theory needs another approach and is not related to the later constructions of nonlinear extensions. With the previous result it is straightforward to calculate the corresponding transformation groups through

$$\mathfrak{N}_2 \equiv \mathfrak{N}^{l,s,P,e} = \mathfrak{N}^{l,s,e} \cap \mathfrak{N}^{l,P}.$$

We quote the result: Elements of $\mathfrak{N}_2$ are labelled through a real parameter a and two real functions ϕ and ϱ depending on the invariant Z. The transformation is given through

$$N(\psi)_{[\widehat{a},\phi,\varrho]}\psi = \exp\left(i\frac{\widehat{a}}{2}\ln\left(\overline{\psi}\psi\right) + i\phi(z) + \varrho(z)\gamma_5\right)\psi.$$

The corresponding group relations are involved.

4.3 Some extended nonlinear Dirac equations

The gauge like model yields with $\mathfrak{N}_2$ and a family $\mathbf{F}_2$ of nonlinear Dirac equations depending on m (mass) and the parameter and functions which label the transformations. We calculate

$$(\gamma_\mu p^\mu - m)N(\psi)\psi = (\gamma_\mu p^\mu + H_2(\psi))\psi.$$

For the nonlinear term $H_2(\psi)$ we find ($'$ denotes the derivative; $Z = \frac{X}{Y}$):

$$H_2(\psi) = (\gamma_\mu p^\mu X)\, i\frac{\widehat{a}}{2X} + (\gamma_\mu p^\mu Z)\left(i\phi'(Z) + \varrho'(Z)\gamma_5\right) - m\exp 2\varrho(Z)\gamma_5.$$

As explained before the families are *reformulations of the linear theory* with certain additional properties (Poincaré invariance, separability, equivalence). To get new evolutions, which are related to the linear one, we applied in Section 3.4 for $k = 1$ the method of gauge generalisation. This method can be applied also to $\mathbf{F}_2$: The coefficient function $\varrho(Z)$ and $\varrho'(Z)$ are obviously not independent. Break the relation between them, generalise the term proportional to $\frac{\widehat{\alpha}}{2X}$ and get an extension of $\mathbf{F}_2$

$$H_{2,\mathrm{ex}} = (\gamma_\mu p^\mu X)\, ig(X) + (\gamma_\mu p^\mu Z)\cdot(ik(Z) + l(Z)\gamma_5) - m\exp 2n(Z)\gamma_5.$$

This family with functions $g(X)$, $k(Z)$, $l(Z)$ and $n(Z)$ is Poincaré invariant, local and separable; the equivalence condition does not hold.

5 Conclusions

We presented and reviewed mathematical models for physical systems which are based on a gauge-like structure, i.e. with a description of observable through (utility) functions on the solution manifold of a family of PDO's. We explained some methods to extend this models, especially on extensions with nonlinear transformations on the solution manifold and a subsequent "gauge generalisation". Quantum mechanical evolutions of observable and states in a Hilbert space spanned through k-vector valued wave functions have a gauge-like structure. We applied the extension method through physically motivated nonlinear transformations which behave like nonlinear gauge transformations. In the Schrödinger case $k = 1$ some mathematical and physical details of an extended nonlinear theory and its possible relevance are known [5, 8]. For systems with $k > 1$, i.e. with internal degrees of freedom like spin, a nonlinearity leads to a coupling of the internal coordinates; this kind of interaction appears as an addition to the coupling through Pauli or Dirac matrices. For $k = 2$ the Pauli equation was mentioned [12]; for $k = 4$ quantum mechanical Dirac equations were discussed and we constructed a family of nonlinear and relativistic invariant extensions [10]. This extensions are not relevant in connection with the relativistic quantum field theory for spin $\frac{1}{2}$ particles.

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