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On Construction of Zero-Curvature Representations for Some Chiral-Type Three-Field Systems

Dmitry K. DEMSKOI

Oryol State University, 95 Komsomolskaya Str., 302015 Oryol, Russia
E-mail: *demskoy@iname.com*

The problem of construction of matrix zero-curvature representations for some chiral-type three field systems is considered. The systems belong to the class described by the Lagrangian $L = \frac{1}{2} g_{ij}(u) u_x^i u_t^j + f(u)$, where g_{ij} is the metric of three-dimensional reducible Riemannian space. The investigation is based on the analysis of evolutionary system $u_t = S(u)$, where S is a higher symmetry.

1 Introduction

We consider here the systems belonging to the following class of two-dimensional fields

$$u^i_{tx} + \Gamma^i_{jk}(u) u^j_x u^k_t = f^i(u), \qquad f^i = g^{ij} \frac{\partial f}{\partial u^j}, \tag{1}$$

where g^{ij} is the metric tensor and Γ^i_{jk} are the Christoffel symbols of the configurational space with the coordinates u^i, the low indices x and t denote the partial derivatives, also we assume summation over repeating indices. The system (1) possesses the following Lagrangian

$$L = \frac{1}{2} g_{ij}(u) u^i_x u^j_t + f(u).$$

In the article [2] the systems with the Lagrangian

$$L = \frac{1}{2} \left(u_t u_x + \psi(v, w)(v_t w_x + v_x w_t)\right) + f(u, v, w), \tag{2}$$

were studied. There we found the systems possessing the nontrivial higher polynomial (with respect to derivatives $u^i_n = \partial u^i / \partial x^n$) symmetries of the 2nd, 3rd, 4th or 5th order. It was proved that the non-degenerate symmetries exist if and only if $\psi = (vw + c)^{-1}$, $c = \text{const}$, and f takes one of the following forms

$$f = ave^{\sqrt{2}u} + bwe^{-\sqrt{2}u}, \tag{3a}$$

$$f = av^2 e^{2u} + bw^2 e^{-2u}, \tag{3b}$$

$$f = av^2 e^{2u} + bwe^{-u}, \tag{3c}$$

$$f = ave^{u} + bwe^{-u}, \tag{3d}$$

$$f = (vw + c/2) \left[ae^{\sqrt{2}u} + be^{-\sqrt{2}u}\right], \tag{3e}$$

$$f = a\,(vw + c/2)\, e^{\sqrt{2}u} + be^{-\sqrt{2}u}, \tag{3f}$$

$$f = a\,(vw + c/2)\, e^{\sqrt{2}u} + be^{-2\sqrt{2}u}, \tag{3g}$$

$$f = \left(v^2 w + 2/3\, vc\right) e^{\sqrt{2}u}, \tag{3h}$$

$$f = ve^{\sqrt{2/3}u}, \tag{3i}$$

where a and b are arbitrary constants. We assume that $c \neq 0$, so the connection Γ^i_{jk} is nontrivial.

Evidently the simplest case for investigation is when function f is linear with respect to v and w. Among the two-exponential functions (3a)–(3g) there are only two functions of that kind, they are given by (3a), (3d). Hyperbolic systems (1) corresponding to these functions have the following forms

$$u_{tx} = \sqrt{2}\left[ave^{\sqrt{2}u} - bwe^{-\sqrt{2}u}\right], \qquad v_{tx} = b\psi^{-1}e^{-\sqrt{2}u} + \psi wv_x v_t,$$
$$w_{tx} = a\psi^{-1}e^{\sqrt{2}u} + \psi vw_x w_t, \tag{4}$$

and

$$u_{tx} = ave^{u} - bwe^{-u}, \qquad v_{tx} = b\psi^{-1}e^{-u} + \psi wv_t v_x, \qquad w_{tx} = a\psi^{-1}e^{u} + \psi vw_t w_x. \tag{5}$$

It was shown in [3] that each system posessing Lagrangian (2) can be represented in an explicit Hamiltonian form

$$u_t = D_x^{-1}\frac{\delta H}{\delta u}, \qquad v_t = e^{\varphi}D_x^{-1}e^{-\varphi}\psi^{-1}\frac{\delta H}{\delta w}, \qquad w_t = \psi^{-1}e^{-\varphi}D_x^{-1}e^{\varphi}\frac{\delta H}{\delta v},$$

which is possibly nonlocal. Existence of the Hamiltonian form gives us a hope that both systems (4) and (5) are the first nonlocal members of the corresponding sequences of integrable Hamiltonian evolution systems

$$\boldsymbol{u}_{t_n} = S_n(\boldsymbol{u}) = J^{-1}\frac{\delta H_n}{\delta \boldsymbol{u}}, \tag{6}$$

where J^{-1} is Hamiltonian operator, S_0 is a corresponding nonlocal vector field, $S_1(\boldsymbol{u}) = \boldsymbol{u}_x$ and S_n, $n > 1$ are higher order vector fields; H_n are the corresponding Hamiltonians.

2 Zero-curvature representations

If a nonlinear system can be represented as the compatibility condition of a linear system $\Psi_x = U\Psi$, $\Psi_t = V\Psi$, i.e. as

$$U_t - V_x + [U, V] = 0, \tag{7}$$

then the system is said to possess zero-curvature representation. In the equation (7) $[U, V]$ is the commutator of matrices. The matrices U, V depend on field functions, and finite set of theirs derivatives, and on a parameter λ, usually called the spectral parameter.

The zero-curvature representation can be constructed starting directly from equation (7). For simplicity, one can choose the matrices U, V in the form

$$U = U_i(u)u_x^i + \overline{U}(u), \qquad V = V_i(u)u_t^i + \overline{V}(u) \tag{8}$$

(this choice corresponds to the solutions given below). Substituting (8) to (7), and requiring the system (1) to be obtained, one can obtain a matrix system for U_i, V_i, $\overline{U}$, and $\overline{V}$ involving covariant derivatives:

$$\nabla_j U_i - \nabla_i V_j + [U_i, V_j] = 0, \qquad f^i(U_i - V_i) + [\overline{U}, \overline{V}] = 0,$$
$$\nabla_i \overline{U} + [\overline{U}, V_i] = 0, \qquad \nabla_i \overline{V} + [\overline{V}, U_i] = 0. \tag{9}$$

The direct computation of the matrices U and V from the equations (9) is a fundamentally difficult problem, and we therefore use a different approach. Existence of the explicit Hamiltonian form allows expecting that the systems (4) and (5) are nonlocal terms of hierarchies of evolutionary systems $u_t = S_n(u)$. Adopting this assumption we can try to find the zero curvature representation for any system (6) in the following form

$$U_{,t} - V_{n,x} + [U, V_n] = 0. \tag{10}$$

The matrix U must be one and the same for all equations of the hierarchy, but the matrices V_n are different. For the evolution systems such problems are usually solvable by the prolongation method [4, 5]. We assume that the matrix U for systems (4) and (5) has form (8). Replacing in (7) u_t, v_t, and w_t according to (6), we obtain matrices U, V in the form

$$U = \sum_i p_i(u)u_x^i A_i + \sum_j q_j(u)A_j, \qquad V_n = \sum_i g_i(u, u_1, \ldots, u_k)A_i, \tag{11}$$

where k is the order of a higher symmetry $S_n(u)$, A_i are some constant matrices which satisfy commutation relations

$$[A_i, A_j] = C_{ij}^k A_k. \tag{12}$$

Note, that table of commutators (12) is not closed. There are some ways to solve the systems like (12). For example, one can choose one of the matrices in the Jordan normal form and try to solve the equations directly. But this way leads to an excessive branching if the matrix size is large. Therefore we applied a modification of the prolongation method by H.D. Wahlquist and F.B. Estabrook [4, 5] to close the table of commutators (12). There are two possibilities for any unknown commutator $[A_i, A_j]$:

(i) $[A_i, A_j]$ is a linear combination of the known elements of the Lie algebra or (ii) $[A_i, A_j]$ is a new element linearly independent of the previous elements. In the first case we write $[A_i, A_j]$ as a linear combination of all elements $A_1, \ldots, A_n$ which we have for the current step and try to find the coefficients with the help of the Jacobi identity. In the second case we introduce the new element $A_{n+1} = [A_i, A_j]$ of the algebra. Then using the Jacobi identity we try to find the new commutational relations for A_{n+1} and so on. After a number of such steps we obtain a closed table of commutators. The obtained algebra can possess a centre. To obtain an algebra with the lowest dimension we construct a factor algebra, setting the elements of the centre as zeros. As a result, the matrices U and V were always embedded into simple classical algebras. To construct a representation of resulting algebra we use the standard algorithm: find a Cartan subalgebra, construct a Cartan–Weyl basis, and build all the matrices explicitly.

Zero-curvature representation for system (4). The simplest higher symmetry of this system is

$$\begin{gathered} u_{t_1} = \sqrt{2}\psi v_x w_x, \qquad v_{t_1} = v_{xx} - 2v\psi v_x w_x + \sqrt{2}u_x v_x, \\ w_{t_1} = -w_{xx} + 2w\psi v_x w_x + \sqrt{2}u_x w_x. \end{gathered} \tag{13}$$

Using prolongation technique of Whalquist–Estabrook we obtain the following matrices of the zero-curvature representation for system (13)

$$U = \begin{pmatrix} 0 & -v^{-1}(3vw + c)be^{-\sqrt{2}u} & \frac{2c^2}{3}v^{-1}v_x\psi \\ 0 & \frac{c}{3}v^{-1}v_x\psi & \lambda vae^{\sqrt{2}u} \\ \frac{1}{3}v^{-1}v_x\psi & v^{-1}be^{-\sqrt{2}u} & -\frac{c}{3}v^{-1}v_x\psi \end{pmatrix}, \tag{14}$$

$$V_1 = \begin{pmatrix} -\lambda ab & 3be^{-\sqrt{2}u}w_x & 2c^2g - \lambda ab(3vw + c) \\ \frac{\lambda}{3}ae^{\sqrt{2}u}v_x\psi & cg & (1 - \frac{2}{3}c\psi)\lambda ae^{\sqrt{2}u}v_x \\ g & 0 & -cg \end{pmatrix}, \tag{15}$$

here

$$g = v^{-1}\psi(v_{xx} + \sqrt{2}v_x u_x)/3 - \psi^2 v_x w_x/3.$$

Now we can obtain the matrix V_0 for original hyperbolic system (4). Substituting matrix (14) to equation (10), one can easily find that the matrix V_0 has the following form

$$V_0 = u_t B_1 + v^{-1} v_t B_2 + w_t v\psi B_3 + B_4.$$

The constant matrices B_i are of the form

$$B_1 = \sqrt{2}/3(\boldsymbol{e}_{22} - \boldsymbol{e}_{11} - \boldsymbol{e}_{33}), \qquad B_4 = \lambda^{-1}(2c\boldsymbol{e}_{12} + \boldsymbol{e}_{32}) - (\boldsymbol{e}_{21} + c\boldsymbol{e}_{23})/3,$$

$$B_2 = (\boldsymbol{e}_{11} + \boldsymbol{e}_{22} - 2\boldsymbol{e}_{33})/3 + c\boldsymbol{e}_{13}, \qquad B_3 = (2\boldsymbol{e}_{11} - \boldsymbol{e}_{22} - \boldsymbol{e}_{33})/3 - 2c\boldsymbol{e}_{13},$$

here $\boldsymbol{e}_{ij}$ are the Weyl matrices. Thus the matrix V is given by

$$V_0 = \begin{pmatrix} h_1 & 2c\lambda^{-1} & cv^{-1}v_t - 2cw_t v\psi \\ -1/3 & h_2 - h_1 & -c/3 \\ 0 & \lambda^{-1} & -h_2 \end{pmatrix}, \tag{16}$$

where $h_1 = \frac{1}{3}(v^{-1}v_t + 2w_t v\psi - \sqrt{2}u_t)$, $h_2 = \frac{1}{3}(2v^{-1}v_t + w_t v\psi + \sqrt{2}u_t)$.

Zero-curvature representation for system (5). The simplest higher symmetry of this system is

$$\begin{aligned} u_t &= -\frac{1}{2}u_{xxx} + \frac{3}{2}\psi(v_{xx}w_x - v_x w_{xx}) + \frac{1}{4}u_x^3 + \frac{9}{2}\psi u_x v_x w_x + \frac{3}{2}\psi^2 v_x w_x(v_x w - v w_x), \\ v_t &= v_{xxx} + \frac{3}{2}u_{xx}v_x + 3v_{xx}(u_x - \psi v w_x) + \frac{9}{4}u_x^2 v_x - 6\psi u_x v v_x w_x + 3\psi v_x w_x\left(\psi v^2 w_x - \frac{1}{2}v_x\right), \\ w_t &= w_{xxx} - \frac{3}{2}u_{xx}w_x - 3w_{xx}(u_x + \psi v_x w) + \frac{9}{4}u_x^2 w_x + 6\psi u_x v_x w w_x \\ &\quad + 3\psi v_x w_x\left(\psi w^2 v_x - \frac{1}{2}w_x\right). \end{aligned} \tag{17}$$

The matrices U and V_1 of the zero-curvature representation for the system (17) is embedded into $sl(4,\mathbb{C})$. The matrix U is given by

$$U = \begin{pmatrix} -\frac{1}{2}w\psi v_1 & -\lambda e^u & 0 & -we^{-u} \\ -\psi^{-1}e^{-u} & \frac{1}{2}w\psi v_1 & we^{-u} & 0 \\ 0 & -\lambda v e^u & \frac{1}{2}w\psi v_1 & -\psi^{-1}e^{-u} \\ -\lambda v e^u & 0 & \lambda e^u & -\frac{1}{2}w\psi v_1 \end{pmatrix}. \tag{18}$$

The matrix V_1 is too cumbersome, and we omit it. Let us consider now the zero-curvature representation for the hyperbolic system (5). We take the matrix U in the form (18), and moreover, we assume that the matrix V_0 is linear with respect to u_t, v_t and w_t

$$V_0 = f_1 u_t + f_2 v_t + f_3 w_t + f_4.$$

If one substitutes V_0 into the equation (10) and substitutes therein u_{tx}, v_{tx} and w_{tx} from (5), then this equation must be an identity. This implies, in particular

$$f_1 = C_1, \qquad f_2 = C_2, \qquad f_4 = C_4, \qquad f_3 = f_3(v,w),$$

where C_i are constant matrices. The matrix f_3 satisfies the system

$$\frac{\partial f_3}{\partial v} = cA_1\psi^2 + w\psi[A_1, f_3], \qquad \frac{\partial f_3}{\partial w} = -v\psi f_3,$$

which has the following general solution $f_3 = \psi(C_3 + vA_1)$, and we obtain

$$V_0 = C_1u_t + C_2v_t + \psi(C_3 + vA_1)w_t + C_4.$$

Now the equation (10) yields a considerable number of commutational relations for the constant matrices C_i and A_j. We solved these relations using (18) and found

$$C_1 = \boldsymbol{e}_{44} - \boldsymbol{e}_{22}, \qquad C_2 = \boldsymbol{e}_{31}, \qquad C_3 = \boldsymbol{e}_{13}, \qquad C_4 = \frac{b}{2}(\boldsymbol{e}_{32} - \boldsymbol{e}_{41}) - \frac{a}{2\lambda}(\boldsymbol{e}_{14} + \boldsymbol{e}_{23}),$$

and

$$V_0 = \begin{pmatrix} -\frac{1}{2}\psi w_t v & 0 & \psi w_t & -a/(2\lambda) \\ 0 & -u_t + \frac{1}{2}\psi w_t v & -a/(2\lambda) & 0 \\ v_t & b/2 & \frac{1}{2}\psi w_t v & 0 \\ -b/2 & 0 & 0 & u_t - \frac{1}{2}\psi w_t v \end{pmatrix}. \tag{19}$$

3 Conserved currents

To construct the conserved currents of the systems (4) and (5) we use the algorithm presented in [1]. Let the vector function Ψ satisfy the system

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi. \tag{20}$$

Introducing the function $\Phi = \Psi/(C, \Psi)$ where (C, Ψ) is the Euclidean scalar product and C is a constant vector, one can easily check that

$$\Phi_x = U\Phi - \Phi(C, U\Phi), \qquad \Phi_t = V\Phi - \Phi(C, V\Phi), \qquad (C, \Phi) = 1. \tag{21}$$

Multiplying (20) by vector $C/(C, \Psi)$ we obtain $(\log(C, \Psi))_x = (C, U\Phi)$, $(\log(C, \Psi))_t = (C, V\Phi)$. This implies the following conservation law

$$D_t(C, U\Phi) = D_x(C, V\Phi). \tag{22}$$

Expanding now Φ in a power series with respect to λ, one can obtain an infinite sequence of local conserved currents. Hence we may call the functions

$$R = (C, U\Phi) \qquad \text{and} \qquad T = (C, V_0\Phi) \tag{23}$$

the generating functions for the conserved densities and the fluxes accordingly.

Conserved currents for system (4). Let us consider equations (21) with matrices (14) and (16), where we set $a = 1$ and assume that $b \neq 0$. In this case, we can choose $c = (0, 1, 0)$. Then identity $(c, \varphi) = 1$ implies that $\varphi = (\varphi_1, 1, \varphi_3)$, and the first system in (21) takes the following form

$$\Phi_x = -c\frac{v_x}{v}\psi\Phi - 3b\frac{e^{-\sqrt{2}u}}{v\psi} - \lambda ve^{\sqrt{2}u}\Phi\varphi_3, \qquad \varphi_{3,x} = \frac{v_x}{3v}\psi\Phi + b\frac{e^{-\sqrt{2}u}}{v} - \lambda ve^{\sqrt{2}u}\varphi_3^2, \tag{24}$$

where $\Phi = \varphi_1 - 2c\varphi_3$. Generating functions (23) are then written as

$$\rho = \lambda ve^{\sqrt{2}u}\varphi_3 + \frac{cv_x}{3v}\psi, \qquad \theta = \frac{1}{3}\left(2\sqrt{2}u_t + \frac{v_t}{v} - vw_t\psi\right) - c\varphi_3 - \frac{1}{3}\Phi. \tag{25}$$

To obtain WKB-expansion for system (24), we set

$$\lambda = k^2/b, \qquad \Phi = gk^{-1}v^{-1}b\exp(-\sqrt{2}u), \qquad \varphi_3 = hk^{-1}v^{-1}b\exp(-\sqrt{2}u),$$

then equations (24) take simpler form

$$h_x = \left(\frac{v_x}{v} + \sqrt{2}u_x\right)h + \frac{v_x}{3v}\psi g + k(1-h^2), \quad g_x = (v_x w\psi + \sqrt{2}u_x)g - k(3\psi^{-1} + gh). \tag{26}$$

It is now clear that the expansions for g, and h must be given by

$$h = 1 + \sum_{i=1}^{\infty} h_i k^{-i}, \qquad g = -3\psi^{-1} + \sum_{i=1}^{\infty} g_i k^{-i}. \tag{27}$$

Substituting this expansions in (26), we get the recursion relations

$$\begin{aligned}
h_{i+1} &= \frac{1}{2}\left(\sqrt{2}u_x + \frac{v_x}{v}\right)h_i - \frac{1}{2}D_x h_i + \frac{v_x}{6v}\psi g_i - \frac{1}{2}\sum_{j=1}^{i} h_j h_{i-j+1},\\
g_{i+1} &= 3\psi^{-1}h_{i+1} + (\sqrt{2}u_x + v_x w\psi)g_i - D_x g_i - \sum_{j=1}^{i} h_j g_{i-j+1},\\
h_1 &= u_x/\sqrt{2}, \qquad g_1 = 3vw_x - 3/\sqrt{2}\,u_x\psi^{-1},
\end{aligned} \tag{28}$$

where $i \geqslant 1$. Applying all the substitutions to (25), we obtain the series

$$\rho = k + \sum_{i=0}^{\infty} \rho_i k^{-i}, \qquad \theta = \sum_{i=0}^{\infty} \theta_i k^{-i},$$

which determine canonical conserved currents (ρ_i, θ_i) of the system (4):

$$\begin{aligned}
\rho_0 &= \frac{\sqrt{2}}{2}u_x + \frac{cv_x}{3v}\psi, \qquad \rho_i = h_{i+1}, \qquad i \geq 1,\\
\theta_0 &= \frac{1}{3}\left(2\sqrt{2}u_t + \frac{v_t}{v} - vw_t\psi\right), \qquad \theta_1 = wb\exp(-\sqrt{2}u),\\
\theta_{i+1} &= -v^{-1}b\exp(-\sqrt{2}u)\left(ch_i + g_i/3\right), \qquad i \geq 1.
\end{aligned} \tag{29}$$

With the help of relations (28), and (29) one can easily obtain any number of conserved currents. For example

$$\begin{aligned}
\rho_1 &= -\frac{\sqrt{2}}{4}\omega_1, \qquad \rho_2 = \frac{\sqrt{2}}{8}D_x(\omega_1) - \frac{1}{2}\omega_3,\\
\rho_3 &= \frac{3}{4}D_x(\omega_3) - \frac{\sqrt{2}}{16}D_x^2(\omega_1) - \frac{1}{16}\omega_1^2 - \frac{1}{2}\omega_2\omega_3, \qquad \theta_2 = -b\exp(-\sqrt{2}u)\left(w_x - \frac{\sqrt{2}}{2}wu_x\right),\\
\theta_3 &= bw\exp(-\sqrt{2}u)\left(-\frac{\sqrt{2}}{4}\omega_1 + (v_x w\psi)^{-1}\omega_3\right),
\end{aligned}$$

where the functions ω_i are given by

$$\begin{aligned}
\omega_1 &= (\sqrt{2}u_2 - u_1^2 - 2v_1w_1\psi)/6, \qquad \omega_2 = v_2v_1^{-1} + u_1/\sqrt{2} - \psi w_1 v,\\
\omega_3 &= \psi v_1(w_2 - v_1w_1\psi w - \sqrt{2}u_1w_1).
\end{aligned} \tag{30}$$

Conserved currents for system (5). To construct the conserved currents for the system (5), let us simplify the matrix U with the help of the gauge transformation $\widetilde{U} = S^{-1}(US - S_x)$, $\widetilde{V}_0 = S^{-1}(V_0 S - S_t)$. We choose the matrix S in the following form

$$S = \exp(-\varphi/2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-u} & 0 & 0 \\ v & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{u} \end{pmatrix},$$

where $\varphi = D_x^{-1}\psi w v_1$ is the quasi-local variable: $D_t\varphi = \psi v w_t - u_t$. Then the transformed matrices are

$$\widetilde{U} = \begin{pmatrix} 0 & -\lambda & 0 & -w \\ -c & \psi w v_1 + u_1 & w & 0 \\ -c\psi v_1 & 0 & \psi w v_1 & -c \\ 0 & 0 & \lambda & -u_1 \end{pmatrix}, \tag{31}$$

$$\widetilde{V}_0 = \begin{pmatrix} \psi v w_t - \frac{1}{2}u_t & 0 & \psi w_t & -\frac{a}{2\lambda}e^{u} \\ -\frac{av}{2\lambda}e^{u} & \psi v w_t - \frac{1}{2}u_t & -\frac{a}{2\lambda}e^{u} & 0 \\ 0 & \frac{b}{2}e^{-u} & -\frac{1}{2}u_t & \frac{av}{2\lambda}e^{u} \\ -\frac{b}{2}e^{-u} & 0 & 0 & -\frac{1}{2}u_t \end{pmatrix}. \tag{32}$$

Let us set $C = (0, 0, 0, 1)$, then the first of the systems (21) takes the following form

$$\Phi_{1,x} = u_1\Phi_1 - \lambda\Phi_2 - w - \lambda\Phi_1\Phi_3, \qquad \Phi_{2,x} = -c\Phi_1 + (2u_1 + \psi w v_1)\Phi_2 + w\Phi_3 - \lambda\Phi_2\Phi_3,$$
$$\Phi_{3,x} = -c\psi v_1\Phi_1 + (u_1 + \psi w v_1)\Phi_3 - c - \lambda\Phi_3^2. \tag{33}$$

The generating functions (23) take now the simplest form

$$R = \lambda\Phi_3 - u_x, \qquad T = -(be^{-u}\Phi_1 + u_t)/2. \tag{34}$$

To obtain the local conserved densities we set $\lambda = -k^2/c$ and adopted the following formal series for Φ_i:

$$\Phi_1 = \frac{1}{k}\left(w + \sum_{i=1}^{\infty} h_i k^{-i}\right), \quad \Phi_2 = \frac{1}{k^3}\left(\frac{cw_1}{2} + \sum_{i=1}^{\infty} g_i k^{-i}\right), \quad \Phi_3 = \frac{c}{k}\left(1 + \sum_{i=1}^{\infty} \rho_i k^{-i}\right). \tag{35}$$

Substituting these series into the equations (33) we found that $\rho_1 = -u_x/2$ and the first conservation law is trivial $D_t u_x = D_x u_t$. The next conservation laws are given by

$$D_t\rho_i = D_x \frac{b}{2} e^{-u} h_{i-2}, \qquad i \geq 2, \tag{36}$$

where ρ_i and h_i satisfy the following relations

$$\rho_{i+1} = \frac{1}{2}\psi v_1 h_i - \frac{1}{2}(u_1 + \psi w v_1)\rho_i + \frac{1}{2}D_x\rho_i - \frac{1}{2}\sum_{j=0}^{i-1}\rho_{j+1}\rho_{i-j}, \qquad i \geq 1,$$

$$h_{i+1} = D_x h_i - w\rho_{i+1} - u_1 h_i - \frac{1}{c}g_i - \sum_{j=0}^{i-1}\rho_{j+1}h_{i-j}, \qquad i \geq 1,$$

$$g_{i+1} = \frac{c}{4}\delta_{i0}(w_2 - 2u_1 w_1 - \psi v_1 w w_1) + \frac{1}{2}D_x g_i - cw\rho_{i+2} - \left(u_1 + \frac{1}{2}\psi w v_1\right)g_i - \frac{c}{4}w_1\rho_{i+1}$$

$$-\frac{1}{2}\sum_{j=0}^{i-1}\rho_{j+1}g_{i-j}+\frac{c}{2}D_xh_{i+1}-\frac{c}{2}u_1h_{i+1}-\frac{c}{2}\sum_{j=0}^{i}\rho_{j+1}h_{i+1-j}, \qquad i\geq 0,$$

$$\rho_1=-u_1/2, \qquad g_0=0, \qquad h_0\equiv w, \qquad h_1=(w_1-u_1w)/2. \tag{37}$$

Using relations (37) we found in particular

$$\rho_2=\omega_1/4, \qquad \rho_3=D_x(\rho_2)/2, \qquad \rho_4=(4\omega_2-\omega_1^2+2D_x^2(\omega_1))/32,$$
$$\omega_1=u_1^2/2+\psi v_1w_1-u_2,$$
$$\omega_2=-\psi v_1w_3+\psi u_2v_1w_1+\psi^2v_1v_2ww_1+3\psi u_1v_1w_2+2\psi^2v_1^2ww_2$$
$$-3\psi^2u_1v_1^2ww_1-2\psi u_1^2v_1w_1+c\psi^3v_1^2w_1^2+\psi^2v_1^2w_1^2/2-2\psi^3v_1^3w^2w_1. \tag{38}$$

It can be seen that $\omega_2/8-\omega_1^2/32$ is not a total derivative, hence the densities ρ_1 and ρ_3 are trivial, and ρ_2, ρ_4 are non-trivial.

It is easy to see from (29), and (36) that if we set $b=0$, then all conserved densities of the systems (4), and (5) become pseudo-constants and the systems become Liouvillian ones. To prove this statement we must present three independent pseudo-constants along each characteristic (see for instance [6]).

Obviously, functions (30) constitute complete set of pseudo-constants for system (4) along the characteristic $D_t\omega=0$. Lorentz invariance $x\longleftrightarrow t$ of the system allows us to obtain the pseudo-constants along the characteristic $D_x\omega=0$ simply by the substitutions $u_x\to u_t$, $u_{xx}\to u_{tt}$, … from ω_1, ω_2 and ω_3.

The two obvious independent pseudo-constants of the system (5) are ω_1 and ω_2 given by (38). To find the third pseudo-constant let us rewrite the second equation of (5) under the condition $b=0$ in the following form

$$D_t\log v_x=\psi v_tw.$$

This implies $D_x(D_t\log v_x+u_t)=D_x(\psi v_tw+u_t)$. Using the conservation law $D_t\psi vw_x=D_x(\psi v_tw+u_t)$ we find $D_t(D_x\log v_x+u_x-\psi vw_x)=0$. Thus, the function

$$\omega_3=D_x\log v_x+u_x-\psi vw_x \tag{39}$$

is the third independent pseudo-constant along the characteristic $D_t\omega=0$ for system (5).

In the case when $a=0$ (and $b\neq 0$), the complete set of the pseudo-constants can be obtained with the help of the discrete symmetry of the system (5) $v\longleftrightarrow w$, $u\longrightarrow -u$.

4 Conclusion

We believe that the modification of the Wahlquist and Estabrook method applied here may be used for construction of the zero curvature representations for other hyperbolic systems too.

[1] Alberty J.M., Koikawa T. and Sasaki R., Canonical structure of soliton equations. I, *Physica D*, 1982, V.5, 43–65; Canonical structure of soliton equations. II. The Kaup–Newell system, *Physica D*, 1982, V.5, 66–74.

[2] Demskoi D.K. and Meshkov A.G., New integrable string-like fields in 1+1 dimensions, in Proc. Second Int. Conf. "Quantum Field Theory and Gravity" (July 28 – August 2, 1997, Tomsk, Russia), Editors I.L. Bukhbinder and K.E. Osetrin, Tomsk, Tomsk Pedagogical University, 1998, 282–285.

[3] Demskoi D.K. and Meshkov A.G., Lax representation for a triplet of scalar fields, *Theor. and Math. Phys.*, 2003, V.134, N 3, 351–364.

[4] Wahlquist H.D. and Estabrook F.B., Prolongation structures of nonlinear evolution equations, *J. Math. Phys.*, 1975, V.16, 1–7.

[5] Wahlquist H.D. and Estabrook F.B., Prolongation structures of nonlinear evolution equations. II, *J. Math. Phys.*, 1976, V.17, 1293–1297.

[6] Zhiber A.V. and Sokolov V.V., Exactly integrable hyperbolic equations of Liouville type, *Uspekhi Mat. Nauk*, 2001, V.56, N 1, 63–106 (in Russian); translation in *Russian Math. Surveys*, 2001, V.56, N 1, 61–101.