Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 677–684

Some Applications of a Lorentz-Like Formulation of Galilean Invariance

Marc DE MONTIGNY †, *Faqir KHANNA* ‡ *and Ademir SANTANA* §

† *Faculté Saint-Jean, University of Alberta, Edmonton, Alberta, T6C 4G9, Canada*
E-mail: *montigny@phys.ualberta.ca*

‡ *Department of Physics, University of Alberta, Edmonton, Alberta, T6J 2J1, Canada*
E-mail: *khanna@phys.ualberta.ca*

§ *Instituto de Fisica, Universidade Federal da Bahia, Salvador, Bahia, 40210-340, Brazil*
E-mail: *santana@ufba.br*

We describe a metric formulation of Galilean covariance in $4+1$ dimensions. As a first example, we recover the two Galilean limits of electromagnetism investigated previously by Le Bellac and Lévy-Leblond. Then we describe the field theoretical formulation of some fluid and superfluid models. Finally the non-relativistic Bhabha equations for spin 0 and 1 particles, and the Dirac equation for spin 1/2 are considered.

1 Introduction

Almost one century has elapsed since Galilei relativity was superseded by Einstein's theory as a realistic framework for describing high velocity phenomena. Yet there exists a wealth of systems at low-energy, particularly in condensed matter physics and nuclear physics, where any new method involving Galilean invariance is likely to be useful. In fact, in most many-body theories, Galilean invariance simply cannot be ignored. Moreover, contrary to popular belief, the mathematical structure of the Galilei group is more intricate than that of the Lorentz group. A case in point is that the representation theory of the Galilei group was thoroughly investigated nearly twenty years after its relativistic counterpart. The general program discussed here consists of a metric formulation of Galilei-invariance, so that one can use Galilean covariance, tensor analysis, etc. as a guiding principle to devise many-body models. Essentially, we exploit the well-known fact that the central extension of the Galilei algebra in 3+1 dimensions is a subalgebra of the Poincaré algebra in $4+1$ dimensions. Hereafter we summarize the articles [1–5], where further details can be found. Our geometrical approach follows the articles of Takahashi and his collaborators [6]. Other five-dimensional formalisms can be found in [7, 8].

We define a *Galilei-vector* $(\boldsymbol{x}, t, s)$ such that a boost acts on it as

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t, \qquad t' = t,$$
$$s' = s - \boldsymbol{V}\cdot\boldsymbol{x} + \frac{1}{2}\boldsymbol{V}^2 t, \tag{1}$$

with relative velocity $\boldsymbol{V}$. Note that the units of s are $\frac{L^2}{T}$. The scalar product,

$$(A|B) = A^\mu B_\mu \equiv \boldsymbol{A}\cdot\boldsymbol{B} - A_4 B_5 - A_5 B_4, \tag{2}$$

of two Galilei-vectors A and B is invariant under transformation (1). This amounts to saying that we work on the light front in $4+1$ dimensions. The need for an additional coordinate may be explained in different ways: (1) as the phase required by the quantum wave function in order to keep the Schrödinger equation Galilei-invariant; (2) as a term added to the classical free Lagrangian so that it becomes Galilei-invariant rather than quasi-invariant; (3) as noted in [8], it may be understood as a control parameter that makes up for lack of a signal with a universal velocity. We expect this approach to be useful in field theories. However, we do not claim that any Galilei-invariant theory can be expressed in this way. Neither do we suggest that this formalism will lead to the respective non-relativistic limits (for instance, the Chaplygin gas model is obtained from the Nambu–Goto action [9]).

Equation (1) can be written as

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}, \tag{3}$$

where $\Lambda^{\mu'}{}_{\nu}$ is the $(\mu'\nu)$-entry, or

$$\begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \\ x^{4'} \\ x^{5'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -V_1 & 0 \\ 0 & 1 & 0 & -V_2 & 0 \\ 0 & 0 & 1 & -V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -V_1 & -V_2 & -V_3 & \frac{1}{2}\boldsymbol{V}^2 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}. \tag{4}$$

For a Galilei-oneform we have:

$$x_{\mu'} = \Lambda^{\nu}{}_{\mu'} x_{\nu}, \tag{5}$$

where now $\Lambda^{\nu}{}_{\mu'}$ is the $(\nu\mu')$-entry, with $\Lambda^{\nu}{}_{\mu'} x_{\nu}$ as in equation (4) with the change $V_j \to -V_j$.

Throughout this paper, except in Section 2, we utilize the Galilei-vectors $(x^1, \ldots, x^5)$ with each component having units of length:

$$(x^1, \ldots, x^5) = \left(\boldsymbol{x}, vt, \frac{s}{v}\right), \tag{6}$$

where v has units of velocity. For a real field $\tilde{\phi}$, the projection is defined as

$$\tilde{\phi}(x) \equiv \phi(\boldsymbol{x}, t) + a_0 s, \tag{7}$$

with a_0 a dimensionless constant. For a complex field $\tilde{\psi}$ we use the definition:

$$\tilde{\psi}(x) \equiv e^{ia_0 ms} \psi(\boldsymbol{x}, t), \tag{8}$$

with natural units, such that $\hbar = 1$. We use $a_0 = +1$ or -1.

If we use $(\boldsymbol{x}, t) \to x^{\mu} = (\boldsymbol{x}, t, s)$, then using the five-momentum $p_{\mu} \equiv -i\partial_{\mu} = (-i\boldsymbol{\nabla}, -i\partial_t, -i\partial_s)$ with $E = i\partial_t$ and $m = i\partial_s$, we obtain $p_{\mu} = (\boldsymbol{p}, -E, -m)$ and $p^{\mu} = g^{\mu\nu} p_{\nu} = (\boldsymbol{p}, m, E)$. Thereupon the mass does not enter as an external parameter, but rather as a remnant of the fifth component of the particle's momentum, starting from an apparently massless theory in $4+1$ dimensions!

2 Galilean electromagnetism

Here we recover the two 'Galilean limits' of electromagnetism obtained thirty years ago by Le Bellac and Lévy-Leblond [10].

The Lorentz transformations of a four-vector $(u^0, \boldsymbol{u})$,

$$\begin{aligned} u^{0\prime} &= \gamma\left(u^0 - \frac{1}{c}\boldsymbol{V}\cdot\boldsymbol{u}\right), \\ \boldsymbol{u}' &= \boldsymbol{u} - \gamma\frac{\boldsymbol{V}}{c}u^0 + \frac{\boldsymbol{V}}{\boldsymbol{V}^2}(\gamma - 1)\boldsymbol{V}\cdot\boldsymbol{u}, \end{aligned} \tag{9}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{V}^2/c^2}}$, admits two well defined Galilean limits [10]. One is related to largely timelike vectors, with $u^{0\prime} = u^0$ and $\boldsymbol{u}' = \boldsymbol{u} - \frac{1}{c}\boldsymbol{V}u^0$, and corresponds to the 'electric' limit. The second limit is for largely spacelike vectors, with $u^{0\prime} = u^0 - \frac{1}{c}\boldsymbol{V}\cdot\boldsymbol{u}$ and $\boldsymbol{u}' = \boldsymbol{u}$, and is associated with the 'magnetic' limit. Throughout this section, we define the embedding of the Newtonian space-time into the de Sitter space by

$$(\boldsymbol{x}, t) \hookrightarrow x = (\boldsymbol{x}, t, 0)\,, \tag{10}$$

so that $\partial_k = \boldsymbol{\nabla}_k$, $\partial_4 = \partial_t$ and $\partial_5 = 0$. The electric and magnetic limits will be obtained by considering two particular embeddings of the five-potential.

From equation (5), we find that

$$\begin{aligned} &\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{V}A_5, \\ &A_{4'} = A_4 + \boldsymbol{V}\cdot\boldsymbol{A} + \frac{1}{2}\boldsymbol{V}^2 A_5, \qquad A_{5'} = A_5. \end{aligned} \tag{11}$$

Let us denote the components of the five-dimensional electromagnetic antisymmetric tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & b_3 & -b_2 & c_1 & d_1 \\ -b_3 & 0 & b_1 & c_2 & d_2 \\ b_2 & -b_1 & 0 & c_3 & d_3 \\ -c_1 & -c_2 & -c_3 & 0 & a \\ -d_1 & -d_2 & -d_3 & -a & 0 \end{pmatrix}. \tag{12}$$

They are expressed in terms of the five-potential A as

$$\begin{aligned} &\boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}, \qquad \boldsymbol{c} = \boldsymbol{\nabla}A_4 - \partial_4\boldsymbol{A}, \\ &\boldsymbol{d} = \boldsymbol{\nabla}A_5 - \partial_5\boldsymbol{A}, \qquad a = \partial_4 A_5 - \partial_5 A_4. \end{aligned} \tag{13}$$

The external five-current, $j_\mu = (\boldsymbol{j}, j_4, j_5)$, also transforms as a five-vector and one writes the continuity equation as

$$\partial^\mu j_\mu = \boldsymbol{\nabla}\cdot\boldsymbol{j} - \partial_4 j_5 - \partial_5 j_4 = 0. \tag{14}$$

In terms of the components in equation (12), the Maxwell equations,

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0 \tag{15}$$

and

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{16}$$

become

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{c} + \partial_4\boldsymbol{b} = \boldsymbol{0}, \\ &\boldsymbol{\nabla}\times\boldsymbol{d} + \partial_5\boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{\nabla}a - \partial_4\boldsymbol{d} + \partial_5\boldsymbol{c} = \boldsymbol{0}, \end{aligned} \tag{17}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{b} - \partial_5\boldsymbol{c} - \partial_4\boldsymbol{d} = \boldsymbol{j}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{c} - \partial_4 a = -j_4, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{d} + \partial_5 a = -j_5, \tag{18}$$

respectively. Finally, the electromagnetic tensor transforms like $F_{\mu'\nu'} = \Lambda_{\mu'}{}^{\alpha}\Lambda_{\nu'}{}^{\beta}F_{\alpha\beta}$, so that its components transform as

$$\begin{aligned} &a' = a + \boldsymbol{V}\cdot\boldsymbol{d}, \qquad \boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{V}\times\boldsymbol{d}, \\ &\boldsymbol{c}' = \boldsymbol{c} + \boldsymbol{V}\times\boldsymbol{b} + \frac{1}{2}\boldsymbol{V}^2\boldsymbol{d} - a\boldsymbol{V} - \boldsymbol{V}(\boldsymbol{V}\cdot\boldsymbol{d}), \qquad \boldsymbol{d}' = \boldsymbol{d}. \end{aligned} \tag{19}$$

2.1 Electric limit

The electric limit corresponds to the embedding

$$(\boldsymbol{A}_e, \phi_e) \hookrightarrow A_e = \left(\boldsymbol{A}_e, 0, -\frac{1}{k_1}\phi_e\right), \tag{20}$$

and

$$(\boldsymbol{j}_e, \rho_e) \hookrightarrow j_e = (k_2\boldsymbol{j}_e, 0, -k_2\rho_e). \tag{21}$$

If we define $\boldsymbol{B}_e \equiv \boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}_e$ and take $\boldsymbol{E}_e \equiv k_1\boldsymbol{d} = \frac{1}{\mu_0\epsilon_0}\boldsymbol{d} = -\boldsymbol{\nabla}\phi_e$, then from equations (13) and (19), we find that the field components transform like

$$\boldsymbol{E}'_e = \boldsymbol{E}_e, \qquad \boldsymbol{B}'_e = \boldsymbol{B}_e - \mu_0\epsilon_0\boldsymbol{V}\times\boldsymbol{E}_e, \tag{22}$$

as in [10]. From equations (17) and (18), with $k_2 \equiv \mu_0$, we find the wave equations

$$\begin{aligned}
&\boldsymbol{\nabla}\times\boldsymbol{E}_e = \boldsymbol{0}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}_e = 0,\\
&\boldsymbol{\nabla}\times\boldsymbol{B}_e - \mu_0\epsilon_0\partial_t\boldsymbol{E}_e = \mu_0\boldsymbol{j}_e, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E}_e = \frac{1}{\epsilon_0}\rho_e,
\end{aligned} \tag{23}$$

as in equation (2.8) of [10]. Note that the Faraday term is missing in the first equation.

2.2 Magnetic limit

The magnetic limit corresponds to the embedding

$$(\boldsymbol{A}_m, \phi_m) \hookrightarrow A_m = (\boldsymbol{A}_m, \phi_m, 0), \tag{24}$$

and

$$(\boldsymbol{j}_m, \rho_m) \hookrightarrow j_m = (k_3\boldsymbol{j}_m, -k_4\rho_m, 0). \tag{25}$$

From equation (14), we find $\boldsymbol{\nabla}\cdot\boldsymbol{j} - \partial_4 j_5 - \partial_5 j_4 = \boldsymbol{\nabla}\cdot\boldsymbol{j}_m = 0$, which shows that the current $\boldsymbol{j}_m$ cannot be related to a transport of charge [10].

By defining $\boldsymbol{B}_m \equiv \boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}_m$ and taking $\boldsymbol{E}_m \equiv \boldsymbol{c} = -\boldsymbol{\nabla}\phi_m - \partial_t\boldsymbol{A}_m$, then from equation (19) we get

$$\boldsymbol{E}'_m = \boldsymbol{E}_m + \boldsymbol{V}\times\boldsymbol{B}_m, \qquad \boldsymbol{B}'_m = \boldsymbol{B}_m. \tag{26}$$

Finally, equations (17) and (18) show that the Maxwell equations reduce to

$$\begin{aligned}
&\boldsymbol{\nabla}\times\boldsymbol{E}_m = -\partial_t\boldsymbol{B}_m, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}_m = 0,\\
&\boldsymbol{\nabla}\times\boldsymbol{B}_m = \mu_0\boldsymbol{j}_m, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E}_m = \frac{1}{\epsilon_0}\rho_m
\end{aligned} \tag{27}$$

in agreement with [10]. The displacement current term is missing in the third equation.

3 Fluid and superfluid equations

3.1 Euler equation for fluids

Define the functional Lagrangian as

$$\tilde{\mathcal{L}}[\tilde{\rho}, \tilde{\phi}] = -\frac{1}{2}\tilde{\rho}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - V(\tilde{\rho}). \tag{28}$$

The Euler–Lagrange equation for $\tilde{\rho}$ leads to $\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + V'(\tilde{\rho}) = 0$. By defining the embedding as in equations (6) and (7), with $a_0 = -1$ and $\tilde{\rho}(x) \equiv \rho(\boldsymbol{x}, t)$, we find

$$\frac{1}{2}\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \partial_t\phi = -V'. \tag{29}$$

The gradient of this expression gives

$$(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla})\boldsymbol{\nabla}\phi + \partial_t(\boldsymbol{\nabla}\phi) = -\boldsymbol{\nabla}(V'). \tag{30}$$

With $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ (so that ϕ is a velocity potential) and $\boldsymbol{\nabla}(V') = \frac{1}{\rho}\boldsymbol{\nabla}p$ (where p is the pressure) we find the Euler equation,

$$\partial_t\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p, \tag{31}$$

which is a particular case of the Navier–Stokes equation, with viscosity and body force both equal to zero. The Lagrangian of equation (28) can be deduced from

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] \propto \partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}^* - V(|\tilde{\psi}|), \tag{32}$$

with complex field $\tilde{\psi}$, by defining the real fields $\tilde{\rho}$ and $\tilde{\phi}$ with the Madelung substitution $\tilde{\psi} \equiv \sqrt{\tilde{\rho}}e^{i\tilde{\phi}}$.

3.2 Generalized models for non-barotropic fluids

In [3] we noticed that the Takahashi model for compressible irrotational barotropic fluids with pressure proportional to the square of the mass density [6] can be expressed in a Galilean covariant form as

$$\tilde{\mathcal{L}} = \frac{\rho_0}{8v_0^2}\left(\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - 2v_0^2\right)^2. \tag{33}$$

In this section, we generalize equation (33) by relaxing $p \propto \rho^2$ (p: pressure, ρ: density of the fluid) to $p \propto \rho^\gamma$ ($\gamma \geq 1$). For $\gamma \neq 1$ we consider

$$\tilde{\mathcal{L}} \propto (\partial\tilde{\phi}\partial\tilde{\phi} - v_0^2)^\gamma, \tag{34}$$

so that variation of the field $\tilde{\phi}$ gives

$$\left(\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - v_0^2\right)\partial_\nu\partial^\nu\tilde{\phi} + (\gamma-1)\partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0. \tag{35}$$

Using equation (6) with $a_0 = -1$, it becomes

$$v_0^2\nabla^2\phi - (\gamma-1)\partial_t^2\phi = \nabla^2\phi\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t\phi\right) + (\gamma-1)\nabla\phi \cdot \nabla\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi + 2\partial_t\phi\right). \tag{36}$$

If $\gamma = 1$, it reduces further:

$$v_0^2\nabla^2\phi = \nabla^2\phi\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi + \partial_t\phi\right). \tag{37}$$

When $\gamma \neq 1$, we recover the Takahashi model [6].

Other equations relevant in condensed matter physics are obtained by generalizing equation (32). For instance, consider

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] \propto (\partial\tilde{\psi}\partial\tilde{\psi}^* - V(|\tilde{\psi}|))^p, \tag{38}$$

with a complex field $\tilde{\psi}$. The choices $p = 1$ and $V = \lambda|\tilde{\psi}|^4$, together with the embedding in equations (6) and (8), give us

$$\mathcal{L} \propto \nabla\psi \cdot \nabla\psi^* - im(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - \lambda|\psi|^4. \tag{39}$$

The Euler–Lagrange equation, with $a_0 = -1$, leads to the non-linear Schrödinger, or Gross–Pitaevski, equation:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + \frac{\lambda}{m}|\psi|^2\psi. \tag{40}$$

3.3 Model of non-viscous fluids and liquid helium

As a last example, let us consider equation (28) with a five-dimensional Clebsch transformation $\partial\tilde{\phi} \to \partial\tilde{\phi} + \tilde{\alpha}\partial\tilde{\beta}$:

$$\tilde{\mathcal{L}} = -\frac{\tilde{\rho}}{2v_0^2}(\partial_\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\beta})(\partial^\mu\tilde{\phi} + \tilde{\alpha}\partial^\mu\tilde{\beta}) - V(\tilde{\rho}). \tag{41}$$

Next we define $\tilde{\alpha}(x) = \alpha(\boldsymbol{x}, t)$, $\tilde{\beta}(x) = \beta(\boldsymbol{x}, t)$, and $\tilde{\rho}(x) = \rho(\boldsymbol{x}, t)$, with equation (7) for $\tilde{\phi}(x)$ and equation (6) for the coordinates. Here we take $a_0 = +1$. Then the Lagrangian in equation (41) becomes

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi - \frac{1}{2}\nabla\phi\cdot\nabla\phi + \alpha\left(\partial_t\beta - \frac{1}{2}\alpha\nabla\beta\cdot\nabla\beta\right) - \alpha\nabla\phi\cdot\nabla\beta\right) - V(\rho). \tag{42}$$

This may be expressed as

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi + \alpha\partial_t\beta - \frac{1}{2}\boldsymbol{v}^2\right) - V(\rho), \tag{43}$$

where $\boldsymbol{v} = -\boldsymbol{\nabla}\phi - \alpha\boldsymbol{\nabla}\beta$. This Lagrangian was employed by Thellung and Ziman to describe the rotational components of liquid helium (see section 4.3 of [2]).

4 Bhabha and Duffin–Kemmer–Petiau equations: spin zero and spin one

In this section, we briefly summarize the references [4] and [5]. The Duffin–Kemmer–Petiau (DKP) equation is

$$(\beta^\mu\partial_\mu + k)\Psi = 0, \tag{44}$$

with matrices β satisfying the DKP algebra:

$$\beta^\mu\beta^\lambda\beta^\nu + \beta^\nu\beta^\lambda\beta^\mu = g^{\mu\lambda}\beta^\nu + g^{\nu\lambda}\beta^\mu, \tag{45}$$

where $g^{\mu\nu}$ is the Galilean metric. The adjoint of Ψ is defined as $\overline{\Psi} \equiv \Psi^\dagger\eta$, where $\eta = (\beta^4+\beta^5)^2+\mathbf{1}$. In the following we use the momentum version of equation (44):

$$(\beta^\mu p_\mu - ik)\Psi = 0. \tag{46}$$

4.1 DKP equation for spin zero and spin one

For spinless particles, the β's can be taken as in reference [5]. The DKP oscillator is described by performing the non-minimal substitution

$$\boldsymbol{p} \rightarrow \boldsymbol{p} + i\omega\eta\boldsymbol{r}. \tag{47}$$

Then equation (46) results in the following equation [5]:

$$E\phi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega\right)\phi. \tag{48}$$

This equation has been obtained as the low-velocity limit of the energy of the DKP oscillator [11].

For spin one, we use a fifteen-dimensional representation of the DKP algebra [4]. We consider the DKP harmonic oscillator by first performing the non-minimal substitution, equation (47). Then equation (46) can be cast into the form [4]

$$E\boldsymbol{A} = \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{\omega}{\hbar}\boldsymbol{L}\cdot\boldsymbol{S}\right]\boldsymbol{A}. \tag{49}$$

This is the non-relativistic energy obtained in [11].

5 Dirac equation: spin 1/2

The details for this section are in [4]. Some recent developments, including the interaction with an external gauge field, are described in [1]. The non-relativistic Dirac equation is

$$(\gamma^\mu\partial_\mu + k)\,\Psi = 0, \qquad \mu = 1, \ldots, 5, \tag{50}$$

written in momentum space as equation (46) with the β's replaced by γ's. The gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{51}$$

and can be chosen as

$$\gamma^n = \begin{pmatrix} \sigma_n & 0 \\ 0 & -\sigma_n \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{52}$$

where each entry is a two-by-two matrix and the σ_n are the spin Pauli matrices. The adjoint spinor is defined as $\overline{\Psi} = \Psi^\dagger\zeta$, where $\zeta = \frac{-i}{\sqrt{2}}\left(\gamma^4 + \gamma^5\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Now let us consider the harmonic oscillator. If we perform the non-minimal substitution, equation (47), with η now replaced by ζ, for a spinor $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ we find the Lévy-Leblond equation [12]:

$$\begin{aligned} &(\boldsymbol{\sigma}\cdot\boldsymbol{p} - ik)\,\varphi + \left(\omega\boldsymbol{\sigma}\cdot\boldsymbol{r} + \sqrt{2}p_5\right)\chi = 0, \\ &(\boldsymbol{\sigma}\cdot\boldsymbol{p} + ik)\,\chi + \left(\sqrt{2}p_4 - \omega\boldsymbol{\sigma}\cdot\boldsymbol{r}\right)\varphi = 0. \end{aligned} \tag{53}$$

Defining $p_4 = p_5$ and $\chi = -i\varphi$ we find [4]

$$E\varphi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{2}{\hbar}\omega\boldsymbol{L}\cdot\boldsymbol{S}\right)\varphi, \tag{54}$$

where $\mathbf{S} \equiv \frac{1}{2}\hbar\boldsymbol{\sigma}$. This is in agreement with the low-velocity limit of the Dirac oscillator investigated in [13]. We plan to quantize the systems discussed in this paper following the same lines as the scalar field [14].

Acknowledgements

We thank the organizers of the Fifth “Symmetry in Nonlinear Mathematical Physics” Conference in Kyiv. Financial support has been provided by NSERC (Canada) and CNPq (Brazil).

[1] de Montigny M., Khanna F.C. and Santana A.E., Nonrelativistic wave equations with gauge fields, *Int. J. Theor. Phys.*, 2003, V.42, 649–671.

[2] de Montigny M., Khanna F.C. and Santana A.E., Lorentz-like covariant equations of non-relativistic fluids, *J. Phys. A: Math. Gen.*, 2003, V.36, 2009–2026.

[3] de Montigny M., Khanna F.C. and Santana A.E., On Galilei-covariant Lagrangian models of fluids, *J. Phys. A: Math. Gen.*, 2001, V.34, 10921–10937.

[4] de Montigny M., Khanna F.C., Santana A.E. and Santos E.S., Galilean covariance and non-relativistic Bhabha equations, *J. Phys. A: Math. Gen.*, 2001, V.34, 8901–8917.

[5] de Montigny M., Khanna F.C., Santana A.E., Santos E.S. and Vianna J.D.M., Galilean covariance and the Duffin–Kemmer–Petiau equation, *J. Phys. A: Math. Gen.*, 2000, V.33, L273–L278.

[6] Takahashi Y., Towards the many-body theory with the Galilei invariance as a guide. I, II, *Fortschr. Phys.*, 1988, V.36, 63–81, 83–96;
Omote M., Kamefuchi S., Takahashi Y. and Ohnuki Y., Galilean covariance and the Schrödinger equation, *Fortschr. Phys.*, 1989, V.37, 933–950;
Santana A.E., Khanna F.C. and Takahashi Y., Galilei covariance and (4,1) de Sitter space, *Prog. Theor. Phys.*, 1998, V.99, 327–336.

[7] Soper D.E., Classical field theory, New York, Wiley and Sons, 1976, Section 7.3;
Pinski G., Galilean tensor calculus, *J. Math. Phys.*, 1968, V.9, 1927–1930;
Duval C., Burdet G., Künzle H.P. and Perrin M., Bargmann structures and Newton–Cartan theory, *Phys. Rev. D*, 1985, V.31, 1841–1853.

[8] Kapuścik E., On the physical meaning of the Galilean space-time coordinates, *Acta Phys. Pol. B*, 1986, V.17, 569–575.

[9] Jackiw R., Lectures on fluid dynamics – a particle theorist’s view of supersymmetric, non-Abelian, noncommutative fluid mechanics and d-branes, Berlin, Springer, 2002; physics/0010042;
Hassaïne M. and Horváthy P.A., Field-dependent symmetries of a non-relativistic fluid model, *Ann. Phys.*, 2000, V.282, 218–246;
Hassaïne M. and Horváthy P.A., Relativistic Chaplygin gas with field-dependent Poincaré symmetry, *Lett. Math. Phys.*, 2001, V.57, 33–40.

[10] Le Bellac M. and Lévy-Leblond J.M., Galilean electromagnetism, *Nuov. Cim. B*, 1973, V.14, 217–233.

[11] Nedjadi Y. and Barrett R.C., The Duffin–Kemmer–Petiau oscillator, *J. Phys. A: Math. Gen.*, 1994, V.27, 4301–4315.

[12] Lévy-Leblond J.M., Nonrelativistic particles and wave equations, *Comm. Math. Phys.*, 1967, V.6, 286–311;
Fushchich W.I. and Nikitin A.G., Symmetries of equations in quantum mechanics, New York, Allerton Press, 1994.

[13] Moshinsky M. and Szczepaniak A., The Dirac oscillator, *J. Phys. A: Math. Gen.*, 1989, V.22, L817–L819.

[14] Abreu L.M., de Montigny M., Khanna F.C. and Santana A.E., Galilei-covariant path-integral quantization of non-relativistic complex scalar fields, *Ann. Phys.*, 2003, V.308, 244–262.