Bi-Hamiltonian Formulation of Generalized Toda Chains

Pantelis A. DAMIANOU

Department of Mathematics and Statistic, University of Cyprus, 1678, Nicosia, Cyprus
E-mail: *damianou@ucy.ac.cy*

In this paper we prove that the Bogoyavlensky–Toda lattices corresponding to simple, classical Lie groups are bi-Hamiltonian. We illustrate in detail the case of B_N Toda systems.

1 Introduction

In this paper we establish for the first time the bi-Hamiltonian nature of the Toda lattices corresponding to classical simple Lie groups. These are systems that generalize the usual finite, non-periodic Toda lattice (which corresponds to a root system of type A_n). This generalization is due to Bogoyavlensky [2]. These systems were studied extensively in [10] where the solution of the system was connected intimately with the representation theory of simple Lie groups. There are also studies by Adler, van Moerbeke [1] and Olshanetsky, Perelomov [15]. We call such systems the Bogoyavlensky–Toda lattices. We make the following more general definition which involves systems with exponential interaction: Consider a Hamiltonian of the form

$$H = \frac{1}{2}(\boldsymbol{p}, \boldsymbol{p}) + \sum_{i=1}^{m} e^{(\boldsymbol{v}_i, \boldsymbol{q})}, \tag{1}$$

where $\boldsymbol{q} = (q_1, \ldots, q_N)$, $\boldsymbol{p} = (p_1, \ldots, p_N)$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are vectors in $\mathbb{R}^N$ and $(\, , \,)$ is the standard inner product in $\mathbb{R}^N$. The set of vectors $\Delta = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is called the spectrum of the system. In this paper we limit our attention to the case where the spectrum is a system of simple roots for a simple Lie algebra $\mathcal{G}$. In this case $m = l = \operatorname{rank} \mathcal{G}$. It is worth mentioning that the case where m, N are arbitrary is an open and unexplored area of research. The main exception is the work of Kozlov and Treshchev [11] where a classification of system (1) is performed under the assumption that the system possesses N polynomial (in the momenta) integrals. Such systems are called Birkhoff integrable.

Example. The usual Toda lattice corresponds to a Lie algebra of type A_{N-1}. In other words, $l = N-1$ and we choose Δ to be the set:

$$\boldsymbol{v}_1 = (1, -1, 0, \ldots, 0), \quad \ldots\ldots\ldots, \quad \boldsymbol{v}_{N-1} = (0, 0, \ldots, 0, 1, -1).$$

The Hamiltonian becomes:

$$H(q_1, \ldots, q_N, \, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}, \tag{2}$$

which is the well-known classical Toda lattice.

The results of this paper were announced in [5] where it was pointed out that the higher Toda flows of the B_N, C_N and D_N systems are bi-Hamiltonian, but there was no similar result for the initial flow. Usually, it is more convenient to work instead in the space of the natural (q, p) variables, with the Flaschka variables (a, b). We end-up with a new set of polynomial equations in the variables (a,b). One can write the equations in Lax pair form. The Lax pair $(L(t),B(t))$ in $\mathcal{G}$ can be described in terms of the root system. The symplectic bracket is transformed in (a,b) coordinates into a Lie–Poisson bracket denoted by π_1. The construction of the bi-Hamiltonian pair may be summarized as follows: Define a recursion operator $\mathcal{R}$ in (a,b) space by finding a second bracket, π_3, and inverting the initial Poisson bracket π_1. Define the negative recursion operator $\mathcal{N}$ by inverting the second Poisson bracket π_3. This recursion operator is the inverse of the operator $\mathcal{R}$. Finally, define a new rational bracket π_{-1} by $\pi_{-1} = \mathcal{N}\pi_1 = \pi_1\pi_3^{-1}\pi_1$. We obtain a bi-Hamiltonian formulation of the system:

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4,$$

where $H_i = \frac{1}{i}\operatorname{tr} L^i$. The brackets π_1 and π_{-1} are compatible and Poisson. In this paper we treat in detail the B_N case. For complete results and full proofs for all classical simple Lie groups, see [6]. We begin with a review of the classical Toda lattice.

2 A_N Toda lattice

Equation (2) is the classical, finite, nonperiodic Toda lattice. This system was investigated in [7–9, 13, 14] and in numerous of other papers that are impossible to list here. To determine the constants of motion, one uses Flaschka's transformation:

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \qquad b_i = -\frac{1}{2} p_i.$$

Then Hamilton's equations become

$$\dot{a}_i = a_i(b_{i+1} - b_i),$$
$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2).$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix},$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric). This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{tr} L^i$ are constants of motion.

Consider $\mathbb{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right),$$

and the mapping $F : \mathbb{R}^{2N} \to \mathbb{R}^{2N-1}$ defined by

$$F : (q_1, \ldots, q_N, p_1, \ldots, p_N) \to (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N).$$

There exists a bracket on $\mathbb{R}^{2N-1}$, which satisfies

$$\{f, g\} \circ F = \{f \circ F, g \circ F\}_s \ .$$

This bracket (up to a constant multiple) is given by

$$\{a_i, b_i\} = -a_i, \qquad \{a_i, b_{i+1}\} = a_i, \tag{3}$$

all other brackets are zero. $H_1 = b_1 + b_2 + \cdots + b_N$ is the only Casimir. The Hamiltonian in this bracket is $H_2 = \frac{1}{2} \operatorname{tr} L^2$. The Lie algebraic interpretation of this bracket can be found in [10]. We denote this bracket by π_1. The invariants H_i are in involution with respect to this Lie Poisson bracket π_1.

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let λ be an eigenvalue of L with normalized eigenvector v. Standard perturbation theory shows that

$$\nabla\lambda = (2v_1v_2, \ldots, 2v_{N-1}v_N, v_1^2, \ldots, v_N^2)^T := U^\lambda \ ,$$

where $\nabla\lambda$ denotes $\left(\frac{\partial\lambda}{\partial a_1}, \ldots, \frac{\partial\lambda}{\partial b_N}\right)$. Some manipulations show that U^λ satisfies

$$\pi_2 U^\lambda = \lambda \pi_1 U^\lambda,$$

where π_1 and π_2 are skew-symmetric matrices. It turns out that π_1 is the matrix of coefficients of the Poisson tensor (3), and π_2, whose coefficients are quadratic functions of the a's and b's, that can be used to define a new Poisson tensor. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. The defining relations are:

$$\{a_i, a_{i+1}\} = \frac{1}{2} a_i a_{i+1}, \qquad \{a_i, b_i\} = -a_i b_i, \qquad \{a_i, b_{i+1}\} = a_i b_{i+1}, \qquad \{b_i, b_{i+1}\} = 2a_i^2,$$

all other brackets are zero. This bracket, due to Adler, has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of L (and therefore the H_i as well) are still in involution. Furthermore, π_2 is compatible with π_1. We also have

$$\pi_2 \nabla H_l = \pi_1 \nabla H_{l+1}, \qquad l = 1, 2, \ldots . \tag{4}$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations. Taking $l = 1$ in (4), we conclude that the Toda lattice is bi-Hamiltonian. In fact, using results from [5], we can prove that the Toda lattice is multi-Hamiltonian:

$$\pi_2 \nabla H_1 = \pi_1 \nabla H_2 = \pi_0 \nabla H_3 = \pi_{-1} \nabla H_4 = \cdots .$$

The notion of bi-Hamiltonian system is due to Magri [12].

The sequence of Poisson tensors can be extended to form an infinite hierarchy. In order to produce the hierarchy of Poisson tensors one uses master symmetries. The first two Poisson brackets are precisely the linear and quadratic brackets we mentioned above. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables (a, b)) both operators are non-invertible and therefore this method fails. Recursion operators were introduced by Olver [16].

In the case of Toda equations, the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. We quote the results from refs. [3, 4].

Theorem 1. *There exists a sequence of vector fields X_i, for $i \geq -1$, and a sequence of contravariant 2-tensors π_j, $j \geq 1$, satisfying:*

i) *π_j are all Poisson;*

ii) *the functions H_i, $i \geq 1$, are in involution with respect to all of the π_j;*

iii) *$X_i(H_j) = (i+j)H_{i+j}$, $i \geq -1$, $j \geq 1$;*

iv) *$L_{X_i}\pi_j = (j-i-2)\pi_{i+j}$, $i \geq -1$, $j \geq 1$;*

v) *$[X_i, X_j] = (j-i)X_{i+j}$, $i \geq 0$, $j \geq 0$;*

vi) *$\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$, where π_j denotes the Poisson matrix of the tensor π_j.*

Remark 1. Theorem 1 was extended for all integer values of the index in [5].

Remark 2. To define the vector fields X_n one considers expressions of the form

$$\dot{L} = [B, L] + L^{n+1}.$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^{n+1}$ instead of $\dot{\lambda} = 0$.

3 B_N Toda systems

In this section, we show that higher polynomial brackets exist also in the case of B_n Toda systems. We will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution. Finally, we show that these systems are bi-Hamiltonian. The Hamiltonian for B_n is

$$H = \frac{1}{2}\sum_{1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}.$$

We make a Flaschka-type transformation

$$a_i = \frac{1}{2}e^{\frac{1}{2}(q_i-q_{i+1})}, \qquad a_n = \frac{1}{2}e^{\frac{1}{2}q_n}, \qquad b_i = -\frac{1}{2}p_i.$$

Then

$$\dot{a}_i = a_i(b_{i+1} - b_i), \qquad \dot{b}_i = 2(a_i^2 - a_{i-1}^2), \qquad i = 1, \ldots, n,$$

with the convention that $a_0 = b_{n+1} = 0$.

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$L = \begin{pmatrix} b_1 & a_1 & & & & & & \\ a_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & \\ & & a_{n-1} & b_n & a_n & & & \\ & & & a_n & 0 & -a_n & & \\ & & & & -a_n & -b_n & \ddots & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix},$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric). We note that the determinant of L is zero.

In the new variables a_i, b_i, the canonical bracket on $\mathbb{R}^{2N}$ is transformed into a bracket π_1 which is given by

$$\{a_i, b_i\} = -a_i, \qquad \{a_i, b_{i+1}\} = a_i.$$

It is easy to show by induction that

$$\det \pi_1 = a_1^2 a_2^2 \cdots a_n^2.$$

The invariant polynomials for B_n, which we denote by

$$H_2, \quad H_4, \quad \ldots, \quad H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i} \operatorname{Tr} L^{2i}$.

We look for a bracket π_3 which satisfies

$$\pi_3 \; \nabla \; H_2 = \pi_1 \; \nabla \; H_4 \; .$$

We define the following homogeneous cubic bracket π_3:

$$\begin{aligned}
&\{a_i, a_{i+1}\} = a_i a_{i+1} b_{i+1}, \qquad \{a_i, b_i\} = -a_i b_i^2 - a_i^3, \qquad \{a_n, b_n\} = -a_n b_n^2 - 2a_n^3, \\
&\{a_i, b_{i+2}\} = a_i a_{i+1}^2, \qquad \{a_i, b_{i+1}\} = a_i b_{i+1}^2 + a_i^3, \qquad \{a_i, b_{i-1}\} = -a_{i-1}^2 a_i, \\
&\{b_i, b_{i+1}\} = 2a_i^2 (b_i + b_{i+1}), \qquad i = 1, 2, \ldots, n-1
\end{aligned}$$

We summarize the properties of this new bracket in the following:

Theorem 2. *The bracket π_3 satisfies the following:*

1. *π_3 is Poisson;*
2. *π_3 is compatible with π_1;*
3. *H_{2i} are in involution.*

Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \; \pi_3, \; \pi_5, \; \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4. $\pi_{j+2} \operatorname{grad} H_{2i} = \pi_j \operatorname{grad} H_{2i+2} \; \forall \; i, \; j.$

The proof of this result is in [4].

Following the procedure outlined in the introduction we obtain a bi-Hamiltonian formulation of the system. In other words, we define $\pi_{-1} = \mathcal{N} \pi_1 = \pi_1 \pi_3^{-1} \pi_1$ and use it to obtain the desired formulation. We illustrate with the B_2 Toda system. In this case $\det \pi_1 = a_1^2 a_2^2$ and $\det \pi_3 = a_1^2 a_2^2 \Delta^2 = \det \pi_1 \Delta^2$, where

$$\Delta = a_1^4 + 2a_2^2 a_1^2 + 2a_2^2 b_1^2 + b_1^2 b_2^2 - 2a_1^2 b_1 b_2.$$

The explicit formula for π_{-1} is

$$\pi_{-1} = \frac{1}{\Delta} A,$$

where

$$A = \begin{pmatrix} 0 & -a_1a_2b_2 & -a_1(b_2^2+a_1^2+2a_2^2) & a_1(b_1^2+a_1^2+2a_2^2) \\ a_1a_2b_2 & 0 & a_1^2a_2 & -a_2(b_1^2+2a_1^2) \\ a_1(b_2^2+a_1^2+2a_2^2) & -a_1^2a_2 & 0 & -2a_1^2(b_1+b_2) \\ -a_1(b_1^2+a_1^2+2a_2^2) & a_2(b_1^2+2a_1^2) & 2a_1^2(b_1+b_2) & 0 \end{pmatrix}.$$

This bracket is Poisson by construction. It is also compatible with π_1. We note that $\Delta = \sqrt{\det \mathcal{R}}$ and it is also equal to the product of the non-zero eigenvalues of L. Using the rational bracket π_{-1} we establish the bi-Hamiltonian nature of the system, i.e.

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4.$$

In fact the system is multi-Hamiltonian:

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 = \pi_{-3} \nabla H_6 = \cdots ,$$

where

$$\pi_{-(2i-1)} = \mathcal{N}^i \pi_1.$$

[1] Adler M. and van Moerbeke P., Completely integrable systems, Euclidean Lie algebras, and curves, *Adv. in Math.*, 1980, V.38, 267–317.

[2] Bogoyavlensky O.I., On perturbations of the periodic Toda lattices, *Comm. Math. Phys.*, 1976, V.51, 201–209.

[3] Damianou P.A., Master symmetries and R-matrices for the Toda lattice, *Lett. Math. Phys.*, 1990, V.20, 101–112.

[4] Damianou P.A., Multiple Hamiltonian structures for Toda-type systems, *J. Math. Phys.*, 1994, V.35, 5511–5541.

[5] Damianou P.A., The negative Toda hierarchy and rational Poisson brackets, *Journal of Geometry and Physics*, 2003, V.45, 184–202.

[6] Damianou P.A., On the bi-Hamiltonian structure of Bogoyavlensky–Toda lattices, *Nonlinearity*, 2004, V.17, 397–413.

[7] Flaschka H., The Toda lattice I. Existence of integrals, *Phys. Rev. B*, 1974, V.9, 1924–1925.

[8] Flaschka H., On the Toda lattice II. Inverse-scattering solution, *Progr. Theor Phys.*, 1974, V.51, 703–716.

[9] Henon M., Integrals of the Toda lattice, *Phys. Rev. B*, 1974, V.9, 1921–1923.

[10] Kostant B., The solution to a generalized Toda lattice and representation theory, *Adv. Math.*, 1979, V.34, 195–338.

[11] Kozlov V.V. and Treshchev D.V., Polynomial integrals of Hamiltonian systems with exponential interaction, *Math. USSR-Izv.*, 1990, V.34, 555–574.

[12] Magri F., A simple model of the integrable Hamiltonian equations, *J. Math. Phys.*, 1978, V.19, 1156–1162.

[13] Manakov S., Complete integrability and stochastization of discrete dynamical systems, *Zh. Exp. Teor. Fiz.*, 1974, V.67, 543–555.

[14] Moser J., Finitely many mass points on the line under the influence of an exponential potential–an integrable system, *Lect. Notes Phys.*, 1976, V.38, 97–101.

[15] Olshanetsky M.A. and Perelomov A.M., Explicit solutions of classical generalized Toda models, *Invent. Math.*, 1979, V.54, 261–269.

[16] Olver P.J., Evolution equations possessing infinitely many symmetries, *J. Math. Phys.*, 1977, V.18, 1212–1215.