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Permutations in Tensor Products of Factors, and L^2 Betti Numbers of Configuration Spaces

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We prove that the natural action of permutations in a tensor product of type II factors is free, and compute the von Neumann trace of the projection onto the space of symmetric and antisymmetric elements respectively. We apply this result to computation of von Neumann dimensions of spaces of harmonic forms (L^2-Betti numbers) of N-point configuration spaces of infinite coverings of compact manifolds.

1 Introduction

It is difficult to overestimate the role of the theory of von Neumann algebras and their traces in different areas of mathematical and theoretical physics. One of the important applications is the definition of regularized dimensions of certain infinite-dimensional Hilbert modules, in particular of the spaces of harmonic forms over certain non-compact manifolds possessing infinite discrete groups of isometries (L^2-Betti numbers, see [3, 7] and references given there).

Thus an important problem is construction of von Neumann algebras containing particular operators, and computation of the corresponding traces of these operators. In this note, we describe the structure of the von Neumann algebra $\{\mathcal{M}^{\otimes n}, U\}''$, where $\mathcal{M}$ is a von Neumann algebra acting in a separable Hilbert space H and U is the natural action of the symmetric group S_n by permutations in $H^{\otimes n}$, $\otimes n$ denoting the n-th tensor power.

It is clear that the answer to this question is spatial dependent, i.e. it depends on the choice of the concrete $\mathcal{M}$-module H. For example, if $H = \mathbb{C}^m$ is a module over the I_m factor $\mathcal{M} = M_m(\mathbb{C})$, then (for $n = 2$) $\mathcal{M} \otimes \mathcal{M}$ coincides with the space of all linear operators in $\mathbb{C}^m \otimes \mathbb{C}^m$. Therefore the permutation operator U belongs to $\mathcal{M} \otimes \mathcal{M}$ and $\{\mathcal{M} \otimes \mathcal{M}, U\}'' = \mathcal{M} \otimes \mathcal{M}$. On the contrary, if the same I_m factor $\mathcal{M}$ acts on its standard form $H = \mathbb{C}^m \otimes \mathbb{C}^m$ by operators $x(f \otimes g) = xf \otimes g,\ x \in \mathcal{M},\ f, g \in \mathbb{C}^m$, then U does not belong to $\mathcal{M} \otimes \mathcal{M}$. Thus U induces an outer action α of the symmetric group S_2 on the factor $\mathcal{M} \otimes \mathcal{M}$ and the von Neumann algebra $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ is isomorphic to the cross-product $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$.

In Section 2, we consider the case where H is a separable module over a type II factor $\mathcal{M}$. We prove that the action α of the group S_n in $\mathcal{M}^{\otimes n}$ generated by the representation U is outer and free and thus the von Neumann algebra $\{\mathcal{M}^{\otimes n}, U\}''$ is isomorphic to the cross-product $\mathcal{M}^{\otimes n} \times_\alpha S_n$. We compute the von Neumann trace of the projection onto the space of symmetric and antisymmetric elements of $H^{\otimes n}$ respectively. In Section 3, we apply this result to computation of the von Neumann dimensions of the spaces of harmonic forms of the spaces of N-point configurations in infinite coverings of compact manifolds (L^2-Betti numbers of configuration spaces).

In the particular case where $\mathcal{M}$ is the commutant of a free action of an infinite discrete group, the results of Section 2 were obtained (by different methods) in [5] ($n = 2$) and [2] (n arbitrary).

In the latter work, the L^2-Betti numbers of infinite configuration spaces equipped with Poisson measures were computed.

In what follows we denote by $\mathcal{L}(H)$ the algebra of all bounded operators in Hilbert space H. We refer to [4, 9] for general notions of the theory of von Neumann algebras.

2 Permutations in a tensor product of type II factors

Let $L^2(\mathcal{M})$ be the standard form of a finite factor $\mathcal{M}$. Denote by Ω the corresponding cyclic and separating vector for $\mathcal{M}$. Let τ be a faithful normal trace on $\mathcal{M}$. Since $(\mathcal{M}\otimes\mathcal{M})' = \mathcal{M}'\otimes\mathcal{M}'$, $\mathcal{M}\otimes\mathcal{M}$ is a finite factor acting in Hilbert space $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$. Let U be the permutation operator in $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$. Denote by α_U the corresponding automorphism of the factor $\mathcal{M}\otimes\mathcal{M}$,

$$\alpha_U(x\otimes y) = U(x\otimes y)U^* = y\otimes x, \qquad x,y\in\mathcal{M}. \tag{1}$$

This automorphism generates a natural action α of the group S_2 on $\mathcal{M}\otimes\mathcal{M}$. Recall that the action of an automorphism β on $\mathcal{M}$ is called free, if each element $x\in\mathcal{M}$ satisfying the equality $xy=\beta(y)x$ for all $y\in\mathcal{M}$ is zero. It is well known that an automorphism of a factor is free iff it is outer. If $\alpha: G\to \mathrm{Aut}(\mathcal{M})$ is a free action of a discrete group G on $\mathcal{M}$ then $\mathcal{M}\times_\alpha G$ is a factor (see, for example, [6, Proposition 1.4.4]).

We have the following statement.

Proposition 1. *(i) The Hilbert space $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$ is the standard form of the factor $\mathcal{M}\otimes\mathcal{M}$;*

(ii) The action α of the group S_2 on $\mathcal{M}\otimes\mathcal{M}$ is free;

(iii) There exists a natural isomorphism of the finite factor $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ and the von Neumann algebra $\{\mathcal{M}\otimes\mathcal{M},U\}''$.

Proof. Note that the vector $\Omega_1=\Omega\otimes\Omega$ is cyclic for both $\mathcal{M}\otimes\mathcal{M}$ and $(\mathcal{M}\otimes\mathcal{M})'=\mathcal{M}'\otimes\mathcal{M}'$. Hence Ω_1 is separating for $\mathcal{M}\otimes\mathcal{M}$ (that is, $z\Omega_1=0$, $z\in\mathcal{M}\otimes\mathcal{M}$ implies $z=0$).

Denote $\tau_1=\tau\otimes\tau$. It is obvious that τ_1 is a trace on $\mathcal{M}\otimes\mathcal{M}$. Moreover the trace τ_1 is faithful on $\mathcal{M}\otimes\mathcal{M}$. Indeed, since Ω_1 is separating for $\mathcal{M}\otimes\mathcal{M}$, for $x=\sum_i x_i\otimes y_i\in\mathcal{M}\otimes\mathcal{M}$ we have

$$\begin{aligned}\tau_1(x^*x) &= \sum_i \tau(x_i^*x_k)\tau(y_i^*y_k) = \sum_i (x_i^*x_k\Omega,\Omega)_{L^2(\mathcal{M})}(y_i^*y_k\Omega,\Omega)_{L^2(\mathcal{M})}\\ &= (x^*x\Omega_1,\Omega_1)_{L^2(\mathcal{M})\otimes L^2(\mathcal{M})} = \|x\Omega_1\|^2\neq 0.\end{aligned} \tag{2}$$

Since

$$\tau_1(x) = (x\Omega_1,\Omega_1)_{L^2(\mathcal{M})\otimes L^2(\mathcal{M})}$$

for any $x\in\mathcal{M}\otimes\mathcal{M}$ we can conclude that $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$ is the standard form of $\mathcal{M}\otimes\mathcal{M}$.

Let us show that α_U given by (1) is a nontrivial outer automorphism of $\mathcal{M}\otimes\mathcal{M}$, i.e. that the operator U does not belong to $(\mathcal{M}\otimes\mathcal{M})\cup(\mathcal{M}\otimes\mathcal{M})'$. Suppose that $U\in\mathcal{M}\otimes\mathcal{M}$. Rewrite the equality $U\Omega\otimes\Omega=\Omega\otimes\Omega$ in the form $(U-1)\Omega_1=0$. Moreover Ω_1 is a separating vector for $\mathcal{M}\otimes\mathcal{M}$, which implies that $U=1$. Thus $U\notin\mathcal{M}\otimes\mathcal{M}$. It can be shown by similar arguments that $U\notin(\mathcal{M}\otimes\mathcal{M})'$.

Since $\mathcal{M}\otimes\mathcal{M}$ is a factor, it follows from Proposition 1.4.4 of [6] that

$$(\mathcal{M}\otimes\mathcal{M})'\cap(\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2)=\mathbb{C}.$$

This implies in particular that the crossed product $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ is also a finite factor. We conclude from the equality $\alpha_U(x)\Omega_1=UxU^{-1}\Omega_1$ that there exists a natural homomorphism of

$\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ onto $\{\mathcal{M} \otimes \mathcal{M}, U\}''$. Since a finite factor does not contain two-sided ideals, any normal homomorphism of it is either identically zero or injective. Hence factors $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ and $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ are isomorphic. ■

Now we can consider the case where $\mathcal{M}$ is a type II factor. Let H be a separable $\mathcal{M}$-module. Denote by U the operator of permutation in $H \otimes H$ and let α_U be the corresponding (nontrivial) automorphism of $\mathcal{M} \otimes \mathcal{M}$.

Theorem 1. *The automorphism α_U defines an outer action of the group S_2 on the II-factor $\mathcal{M} \otimes \mathcal{M}$, and there exists an isomorphism of factors*

$$\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2 \simeq \{\mathcal{M} \otimes \mathcal{M}, U\}''. \tag{3}$$

Proof. *Case II_1.* Let $\mathcal{M}$ be a II_1-factor. Denote by K the standard form of $\mathcal{M}$. Using the theorem on the structure of normal isomorphisms of von Neumann algebras [9] we can conclude that H as $\mathcal{M}$-module is isomorphic to $\mathcal{M}$-module

$$H_d = p(K \otimes l_2) \tag{4}$$

for some $d \in [0, \infty]$, where $p \in \mathcal{M}' \otimes \mathcal{L}(l_2)$ is a projection with $\mathrm{Tr}\, p = d$. Here Tr denotes the natural trace in $\mathcal{M}' \otimes \mathcal{L}(l_2)$, with the normalization $\mathrm{Tr}\,(1_{\mathcal{M}} \otimes q) = 1$, where q is a projection of rank 1 in $\mathcal{L}(l_2)$. The action of $\mathcal{M}$ on H_d is given by

$$x(p(f \otimes \xi)) = p(xf \otimes \xi), \qquad x \in \mathcal{M}, \quad f \in K, \quad \xi \in l_2.$$

Let us remark that the Hilbert spaces $K \otimes l_2 \otimes K \otimes l_2$ and $K \otimes K \otimes l_2 \otimes l_2$ are isomorphic. Thus there exists a projection $\tilde{p}$ such that $\mathcal{M} \otimes \mathcal{M}$-modules

$$H_d \otimes H_d = (p \otimes p)(K \otimes l_2 \otimes K \otimes l_2)$$

and

$$\tilde{p}(K \otimes K \otimes l_2 \otimes l_2)$$

are isomorphic, where the action of $\mathcal{M} \otimes \mathcal{M}$ on the latter space is defined by

$$(x \otimes y)(\tilde{p}(f \otimes g \otimes \xi \otimes \eta)) = \tilde{p}(xf \otimes yg \otimes \xi \otimes \eta),$$

$x, y \in \mathcal{M}$, $f, g \in K$, $\xi, \eta \in l_2$. The operator U of permutation in $H_d \otimes H_d$ is unitarily isomorphic to the operator $U_1 \otimes U_2$ in $\tilde{p}(K \otimes K \otimes l_2 \otimes l_2)$, where U_1 is the operator of permutation in $K \otimes K$ and U_2 is the operator of permutation in $l_2 \otimes l_2$. It follows from Proposition 1 that $U_1 \notin \mathcal{M} \otimes \mathcal{M}$. Hence the operator U does not belong to $\mathcal{M} \otimes \mathcal{M}$ and thus α_U is outer (and consequently, free) automorphism of $\mathcal{M} \otimes \mathcal{M}$. Repeating the arguments from the proof of Proposition 1 we conclude that the factors $\mathcal{M} \otimes \mathcal{M} \times S_2$ and $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ are isomorphic.

Case II_∞. Let $\mathcal{M}$ be a II_∞ factor. Fix an arbitrary finite projection $p \in \mathcal{M}$. Then there exists [9] a spatial isomorphism of $\mathcal{M}$ and the II_∞ factor $\mathcal{M}_p \otimes \mathcal{L}(l_2)$, where $\mathcal{M}_p = p\mathcal{M}p$ (the so-called "corner" of $\mathcal{M}$) is a II_1 factor. Denote $H_p = pH$. Then the II_∞ factor $\mathcal{M} \otimes \mathcal{M}$ is isomorphic to $\mathcal{M}_p \otimes \mathcal{M}_p \otimes \mathcal{L}(l_2 \otimes l_2)$ and the permutation operator U in $H \otimes H$ is unitarily equivalent to the operator $U_1 \otimes U_2$, where U_1 is the operator of permutation in $H_p \otimes H_p$ and U_2 is the operator of permutation in $l_2 \otimes l_2$. Note that the operator U_2 belongs to $\mathcal{L}(l_2 \otimes l_2)$.

It follows from the arguments presented above that the operator U_1 does not belong to $\mathcal{M}_p \otimes \mathcal{M}_p$. Thus the operator U does not belong to $\mathcal{M} \otimes \mathcal{M}$ and as above α is free automorphism of $\mathcal{M} \otimes \mathcal{M}$. Therefore $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ is a II_∞ factor. It follows from the arguments presented in the proof of Proposition 1 that there exists a normal homomorphism of $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$ onto $\{\mathcal{M} \otimes \mathcal{M}, U\}''$. Since any normal homomorphism of a factor is either identically zero or injective we have that $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ is also a II_∞ factor isomorphic to $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$. ■

The following result is the extension of the theorem above to the case of the symmetric group S_n acting in $\mathcal{M}^{\otimes n}$, $n \geq 2$.

Theorem 2. *Let $\mathcal{M}$ be a type II factor and H be a separable $\mathcal{M}$-module. Let U_{ij} $(i, j = 1, \ldots, n)$ be the operator in $H^{\otimes n}$ permuting i-th and j-th components. Then the family of operators $\{U_{ij}\}_{i,j=1,\ldots,n}$ defines an outer action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$, and there exists an isomorphism*

$$\mathcal{M}_n \times_\alpha S_n \simeq \{\mathcal{M}_n, \{U_{ij}\}_{i,j=1,\ldots,n}\}''. \tag{5}$$

Proof. It follows from Theorem 1 that the operator U_{ij} does not belong to the factor $\mathcal{M}^{\otimes n}$ and therefore determines the outer automorphism $\alpha_{U_{ij}}$ of $\mathcal{M}^{\otimes n}$. Since automorphisms $\alpha_{U_{ij}}$, $i, j = 1, \ldots, n$ generate the action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$, we conclude that this action is free. Therefore the factors $\mathcal{M}_n \times_\alpha S_n$ and $\{\mathcal{M}_n, \{U_{ij}\}_{i,j=1,\ldots,n}\}''$ are isomorphic (see the proof of Theorem 1). ∎

Let P_s and P_a be projections in $H^{\otimes n}$ onto the symmetric tensor power $H^{\hat{\otimes} n}$ and the anti-symmetric tensor power $H^{\wedge n}$ respectively,

$$P_s = \frac{1}{n!} \sum_{g \in S_n} U_g \tag{6}$$

and

$$P_a = \frac{1}{n!} \sum_{g \in S_n} (-1)^{\mathrm{sign}\,(g)} U_g. \tag{7}$$

It is obvious that P_s and P_a belong to $\mathcal{M}_n \times_\alpha S_n$.

Denote

$$\mathcal{M}_s = \{\mathcal{M}^{\otimes n}, P_s\}'', \qquad \mathcal{M}_a = \{\mathcal{M}^{\otimes n}, P_a\}'' \tag{8}$$

Proposition 2. *Let $\mathcal{M}$ be a II_∞ factor. Then*

$$\mathcal{M}_s = \mathcal{M}_a = \mathcal{M}^{\otimes n} \times_\alpha S_n. \tag{9}$$

Proof. The inclusion $\mathcal{M}_s \subset \mathcal{M}^{\otimes n} \times_\alpha S_n$ is obvious. For the inverse inclusion it suffices to show that the operators $P_{ij} = \frac{1}{2}(1 + U_{ij})$ $i, j = 1, \ldots, n$ belongs to $\mathcal{M}_s$. Since $\mathcal{M} \simeq \mathcal{M}_p \otimes \mathcal{L}(l_2)$, the factor $\mathcal{M}$ contains an isometry V such that $(V^*)^m \to 0$, $m \to \infty$ strongly (for example $V = 1 \otimes W$ where W is unilateral shift in l_2: $We_k = e_{k+1}$ for a standard basis $\{e_k\}_{k=1}^\infty$ in l_2). Since $(V^*)^m V^m = 1$ we have strong convergence

$$\big(1 \otimes 1 \otimes (V^*)^m \otimes \cdots \otimes (V^*)^m\big) P_s \big(1 \otimes 1 \otimes V^m \otimes \cdots \otimes V^m\big) \to \frac{2}{n!} P_{12},$$

$m \to \infty$. Thus $P_{12} \in \mathcal{M}_s$. Similar arguments show that $P_{ij} \in \mathcal{M}_s$ for any $i, j = 1, \ldots, n$.

The case of $\mathcal{M}_a$ can be treated in a completely similar way. ∎

In what follows we denote by $\mathrm{Tr}_\mathcal{N}$ the faithful normal semifinite trace on a II_∞ factor $\mathcal{N}$.

Corollary 1. *For any $A \in \mathcal{M}$ we have*

$$\mathrm{Tr}_{\mathcal{M}_s}(A^{\otimes n} P_s) = \mathrm{Tr}_{\mathcal{M}_a}(A^{\otimes n} P_a) = \frac{(\mathrm{Tr}_\mathcal{M} A)^n}{n!}. \tag{10}$$

Proof. According to the Proposition 2 we will use the trace on a factor $\mathcal{M}^{\otimes n} \times_\alpha S_n$. It is obvious that $\alpha_g(A^{\otimes n}) = A^{\otimes n}$ for any $g \in S_n$. Therefore for any $g \in S_n$ we have $\mathrm{Tr}_{\mathcal{M}^{\otimes n} \times_\alpha S_n}(A^{\otimes n} U_g) = \delta_{e,g} \mathrm{Tr}_{\mathcal{M}^{\otimes n}}(A^{\otimes n})$ (here $\delta_{g,h}$ is the Kronecker symbol). Then (10) follows from (6) and (7). ∎

3 L^2-Betti numbers of N-point configuration spaces

In this section, we apply the results described above to computation of L^2-Betti numbers of the spaces of configurations of N points in the universal covering of a compact manifold with infinite fundamental group. We start with the discussion of the structure of the spaces of square-integrable differential forms over configuration spaces.

Let X be a smooth connected Riemannian manifold. Consider the N-point configuration space

$$X^{(N)} := \{\{x_1, \dots, x_N\} \subset X\}, \tag{11}$$

the set of all N-point subsets of X. Clearly,

$$X^{(N)} = X \widetilde{\times \cdots \times} X / S_N, \tag{12}$$

where $X \widetilde{\times \cdots \times} X$ is the Cartesian product of N copies of X without coinciding components. $X^{(N)}$ is a Riemannian manifold equipped with the Riemannian structure induced from X.

For a Riemannian manifold R, we denote by $L^2\Omega^p(R)$ the space of square-integrable (w.r.t. the Riemannian volume) p-forms on R. We let Δ_R^p be the Hodge–deRham Laplacian in $L^2\Omega^p(R)$, and $\mathcal{H}^p(R) := \operatorname{Ker} \Delta_R^p$, the space of square-integrable harmonic p-forms on R.

For a Hilbert space $\mathcal{K}$, we use the notation

$$\mathcal{K}^{\overset{k}{\diamond} s} = \begin{cases} \mathcal{K}^{\widehat{\otimes} s}, & k \text{ is even}, \\ \mathcal{K}^{\wedge s}, & k \text{ is odd}. \end{cases} \tag{13}$$

The following result is a symmetrized version of the Künneth formula.

Theorem 3.

$$\mathcal{H}^p\big(X^{(N)}\big) = \bigoplus_{\substack{s_1, \dots, s_d = 0,1,2,\dots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \big(\mathcal{H}^1(X)\big)^{\overset{1}{\diamond} s_1} \otimes \cdots \otimes \big(\mathcal{H}^d(X)\big)^{\overset{d}{\diamond} s_d} \tag{14}$$

where $d = \dim X - 1$.

Proof. Let us first remark that the space $L^2\Omega^p\big(X^{(N)}\big)$ is unitarily isomorphic to $L^2\Omega^p_{\mathrm{sym}}\big(X^N\big)$, the latter being the space of square-integrable p-forms on $X^N := \overbrace{X \times \cdots \times X}^{N}$ which are symmetric w.r.t. the permutations of variables. It is easy to see that there exists a natural unitary isomorphism

$$L^2\Omega^p_{\mathrm{sym}}\big(X^{(N)}\big) = \bigoplus_{\substack{s_0, \dots, s_{d+1} = 0,1,2,\dots \\ s_0 + \cdots + s_{d+1} = N \\ s_1 + 2s_2 + \cdots + (d+1)s_{d+1} = p}} \big(L^2\Omega^0(X)\big)^{\overset{0}{\diamond} s_0} \otimes \cdots \otimes \big(L^2\Omega^{d+1}(X)\big)^{\overset{d+1}{\diamond} s_{d+1}}. \tag{15}$$

It has been proved in [1] that the restriction $(\Delta^p_{X^N})_{\mathrm{sym}}$ of $\Delta^p_{X^N}$ onto $L^2\Omega^p_{\mathrm{sym}}\big(X^N\big)$ is essentially self-adjoint on the space of smooth forms with compact support. Thus $(\Delta^p_{X^N})_{\mathrm{sym}}$ coincides with the Hodge–deRham Laplacian on $X^{(N)}$, and we have

$$\mathcal{H}^p\big(X^{(N)}\big) = \operatorname{Ker}\big(\Delta^p_{X^N}\big)_{\mathrm{sym}} = \operatorname{Ker}\big(\Delta^p_{X^N}\big) \cap L^2\Omega^p_{\mathrm{sym}}\big(X^N\big). \tag{16}$$

By the Künneth formula,

$$\operatorname{Ker}(\Delta^p_{X^N}) = \bigoplus_{\substack{1\le k_1,\dots,k_N\le d\\ k_1+\cdots+k_N=p}} (\mathcal{H}^{k_1}(X))\otimes\cdots\otimes(\mathcal{H}^{k_N}(X)), \tag{17}$$

(remark that $\mathcal{H}^0(X)=\mathcal{H}^{d+1}(X)=0$), which together with (15) implies the result. ■

Corollary 2. *If all spaces $\mathcal{H}^k(X)$ are finite-dimensional, then all spaces $\mathcal{H}^p\big(X^{(N)}\big)$ are so. Their dimensions are given by the following formula:*

$$\dim\mathcal{H}^p\big(X^{(N)}\big) = \sum_{\substack{s_1,\dots,s_d=0,1,2,\dots\\ s_1+s_2+\cdots+s_d=N\\ s_1+2s_2+\cdots+ds_d=p}} \beta_1^{(s_1)}\cdots\beta_d^{(s_d)}, \tag{18}$$

where

$$\beta_k^{(s)} := \begin{cases} \begin{pmatrix}\beta_k\\ s\end{pmatrix}, & k=1,3,\dots,\\ \begin{pmatrix}\beta_k+s-1\\ s\end{pmatrix}, & k=2,4,\dots,\end{cases} \tag{19}$$

$s\neq 0$, and $\beta_k^{(s)}=1$ for $s=0$. Here $\beta_k := \dim\mathcal{H}^k(X)$, $k=1,\dots,d$. This case occurs for instance when X is compact or has finite number of ends.

An important example of a manifold X with infinite dimensional spaces $\mathcal{H}^{(p)}(X)$ is given by an infinite covering of a compact Riemannian manifold (say M). In this case, the fundamental group $G=\pi_1(M)$ acts by isometries on X and consequently on all spaces $L^2\Omega^p(X)$. The orthogonal projection

$$\mathcal{P}_p : L^2\Omega^p(X)\to\mathcal{H}^{(p)}(X) \tag{20}$$

commutes with the action of G and thus belongs to the commutant $\mathcal{A}_p$ of this action which is a semifinite von Neumann algebra. The corresponding von Neumann trace $b_p := \operatorname{Tr}_{\mathcal{A}}\mathcal{P}_p$ gives a regularized dimension of the space $\mathcal{H}^{(p)}(X)$ and is called the L^2-Betti number of X (or M). L^2-Betti numbers were introduced in [3] studied by many authors (see e.g. [7] and references given there). It is known [3] that (because of the elliptic regularity of Δ^p_X) $b_p<\infty$.

It is natural to ask whether the notion of L^2-Betti numbers can be extended to configuration spaces over infinite coverings. It particular, is formula (19) valid in this case (with β_k replaced by b_k)? In what follows, we use the results of the first section for constructing a von Neumann algebra containing the projection

$$\mathbf{P}_p : L^2\Omega^p\big(X^{(N)}\big)\to\mathcal{H}^p\big(X^{(N)}\big), \tag{21}$$

and computing its von Neumann trace.

In what follows, we assume that G is an ICC group (that is, all non-trivial classes of conjugate elements are infinite). Under this condition we have that the von Neumann algebra $\mathcal{A}_p$ is a II_∞ factor.

Let us define the operator

$$\mathcal{P}_p^{(n)} := \begin{cases} \mathcal{P}_p^{\otimes n}P_s, & p \text{ is even},\\ \mathcal{P}_p^{\otimes n}P_a, & p \text{ is odd}\end{cases} \tag{22}$$

and the von Neumann algebra

$$\mathcal{A}_p^{(n)} := \begin{cases} \{\mathcal{A}_p^{\otimes n}, P_s\}'', & p \text{ is even}, \\ \{\mathcal{A}_p^{\otimes n}, P_a\}'', & p \text{ is odd}, \end{cases} \tag{23}$$

generated by $\mathcal{A}_p^{\otimes n}$ and projections P_s and P_a respectively. Thus, $\mathcal{P}_p^{(n)}$ is the orthogonal projection $\left(L^2\Omega^p(X)\right)^{\otimes n} \to (\mathcal{H}^p(X))^{\overset{p}{\diamond} n}$, $n = 1, 2, \ldots$. Obviously, $\mathcal{P}_p^{(n)} \in \mathcal{A}_p^{(n)}$. It follows from the Proposition 2 that $\mathcal{A}_p^{(n)} = \mathcal{A}_p^{\otimes n} \times_\alpha S_n$. We will use the convention $\mathcal{A}_p^{(0)} = \mathbb{C}^1$.

Further, we introduce the von Neumann algebra

$$\mathbf{A}^{(p)} = \prod_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \mathcal{A}_1^{(s_1)} \otimes \cdots \otimes \mathcal{A}_d^{(s_d)}, \tag{24}$$

$d = \dim X - 1$. Since all algebras $\mathcal{A}_k^{(s_k)}$ are II_∞-factors, so is $\mathbf{A}^{(p)}$, with the trace given by the product of the traces in $\mathcal{A}_k^{(s_k)}$.

Theorem 4. *We have* $\mathbf{P}_p \in \mathbf{A}^{(p)}$ *and*

$$\mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_d}}{s_d!}, \tag{25}$$

where b_k *are the* L^2*-Betti numbers of* X, $k = 1, \ldots, d$, *and* $b_0 = 1$.

Proof. It follows from Theorem 3 that

$$\mathbf{P}_p = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \mathcal{P}_1{}^{(s_1)} \cdots \mathcal{P}_d{}^{(s_d)}, \tag{26}$$

with the convention $\mathcal{P}_k^{(0)} = id$. The result follows now from Corollary 1. ■

We will use the notation $\mathbf{b}_p = \mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p$ and call $\mathbf{b}_p$ the p-th L^2-Betti number of Γ_X.

Remark 1. It is easy to see that formula (25) can be rewritten in the form

$$\mathbf{b}_p = \frac{1}{N!} \sum_{\substack{1 \le k_1, \ldots, k_N \le d \\ k_1 + \cdots + k_N = p}} b_{k_1} \cdots b_{k_N}, \tag{27}$$

or, according to the Künneth formula (17),

$$\mathbf{b}_p = \frac{1}{N!} \mathrm{Tr}_{\mathcal{A}^{\otimes N}} P,$$

where P is the orthogonal projection $L^2\Omega^p\left(X^N\right) \to \mathcal{H}^p\left(X^N\right)$.

Example 1. Let $X = \mathbb{H}^d$, the hyperbolic space of dimension d. It is known that the only non-zero L^2 Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided d is even). Then

$$\mathbf{b}_p = \begin{cases} \dfrac{\left(b_{d/2}\right)^k}{k!}, & p = \dfrac{kd}{2}, \\ 0, & p \ne \dfrac{kd}{2}. \end{cases} \tag{28}$$

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[1] Albeverio S., Daletskii A. and Lytvynov E., De Rham cohomology of configuration spaces with Poisson measure, *J. Funct. Anal.*, 2001, V.185, 240–273.

[2] Albeverio S. and Daletskii A., L^2-Betti numbers of Poisson configuration spaces, Preprint, Bonn University, 2003.

[3] Atiyah M.F., Elliptic operators, discrete groups and von Neumann algebras, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (1974, Orsay), 43–72; *Asterisque*, N 32–33, Soc. Math. France, Paris, 1976.

[4] Bratteli O. and Robinson D.W., Operator algebras and quantum statistical mechanics I, New York, Heidelberg, Berlin, Springer-Verlag, 1979.

[5] Daletskii A. and Samoilenko Yu., Von Neumann dimensions of symmetric and antisymmetric tensor products, *Methods of Functional Analysis and Topology*, 2003, V.9, N 2, 123–132.

[6] Jones V. and Sunder S., Introduction to subfactors, *London Math. Soc. Lecture Note Series*, Vol. 234, Cambridge Univ. Press, 1997.

[7] Mathai V., L^2 invariants of covering spaces, in "Geometric analysis and Lie theory in mathematics and physics", *Austral. Math. Soc. Lect. Ser. 11*, Cambridge, Cambridge Univ. Press, 1998, 209–242.

[8] von Neumann J., On rings of operators. III, *Ann. Math.*, 1940, V.41, 94–161.

[9] Takesaki M., Theory of operator algebras, New York, Springer-Verlag, 1979.