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# A Discussion on the Different Notions of Symmetry of Differential Equations

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A discussion is presented, within a simple unifying scheme, about different types of symmetry of PDE's, with the introduction and a precise characterization of the notions of "standard" and "weak" conditional symmetries, together with their relationship with exact and partial symmetries. An extensive use of "symmetry-adapted" variables is made; some clarifying examples are also provided.

## 1 Introduction

This paper is essentially a presentation of a unifying and comprehensive scheme, where several different notions of symmetry for differential problems may be considered and compared. In particular, this approach will permit the introduction of "subtler" notions of conditional symmetries (or "nonclassical symmetries ") [1–5], with a clear distinction and characterization of these symmetries and of other related concepts, including the more recently introduced notions of "partial symmetries" [6] (see also [7, 8]), and of "hidden symmetries" (see e.g. [9, 10]).

For simplicity, we will consider here only the case of partial differential equations (PDE)

$$\Delta \equiv \Delta_a\big(x, u^{(m)}\big) = 0 \qquad (a = 1, \ldots, \nu) \tag{1}$$

for the $q$ functions $u_\alpha = u_\alpha(x)$ of the $p$ variables $x_i$ (as usual, $u^{(m)}$ denotes the functions $u_\alpha$ together with their $x$ derivatives up to the order $m$), and only "geometrical" or Lie-point symmetries, i.e. symmetries generated by vector fields of the form (sum over repeated indices)

$$X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \phi_\alpha(x, u)\frac{\partial}{\partial u_\alpha} \tag{2}$$

although the relevant results could be suitably extended also to other types of symmetries, as generalized or Bäcklund, potential or nonlocal symmetries, whose importance is well known and also recently further emphasized (cf. e.g. [11–14]).

## 2 Exact symmetries

Let us start with the basic and standard definition, with the usual nondegeneracy and regularity assumptions tacitly understood (see [15–19]):

**Definition 1.** A system of PDE $\Delta_a\big(x, u^{(m)}\big) = 0$ is said to admit the Lie-point symmetry generated by the vector field $X$ (or to be *symmetric* under $X$) if the following condition

$$X^*(\Delta)|_{\Delta=0} = 0 \tag{3}$$

is satisfied, or – equivalently (at least under mild hypotheses) – if there are functions $G = G_{ab}(x, u^{(m)})$ such that

$$(X^*(\Delta))_a = G_{ab}\,\Delta_b. \tag{4}$$

We simply denote by $X^*$ the "appropriate" prolongation of $X$ for the equation at hand, or – alternatively – its infinite prolongation (indeed, only a finite number of terms will appear in calculations).

Let us also give this other definition:

**Definition 2.** A system of PDE as before is said to be *invariant* under $X$ if

$$X^*(\Delta) = 0. \tag{5}$$

For instance, the Laplace equation $u_{xx} + u_{yy} = 0$ is *invariant* under the rotation symmetry $X = y\partial/\partial x - x\partial/\partial y$; the heat equation $u_t = u_{xx}$ is *symmetric* but *not* invariant under

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$$

indeed one has $X^*(u_t - u_{xx}) = -x(u_t - u_{xx})$.

We then have:

**Theorem 1.** *Let $\Delta = 0$ be a nondegenerate system of PDE's, symmetric under a projectable vector field $X$, according to Definition 1. Then, there are new $p + q$ variables $s$, $z$ and $v$, with $s \in \mathbb{R}$, $z \in \mathbb{R}^{p-1}$ and $v \equiv (v_1(s, z), \ldots, v_q(s, z))$, and a new system of PDE's, say $K = 0$, with $K_a = S_{ab}(s, z, v^{(m)})\,\widetilde{\Delta}_b(s, z, v^{(m)})$ (where $v^{(m)}$ stands for $v(s, z)$ and its derivatives with respect to $s$ and $z$, and $\widetilde{\Delta} = \widetilde{\Delta}(s, z, v^{(m)})$ is $\Delta$ when expressed in terms of the new variables $s, z, v$), which is locally* equivalent *to the initial system and is* invariant *(as in Definition 2) under the symmetry $X = \partial/\partial s$, i.e. $K_a = K_a(z, v^{(m)})$.*

**Proof.** (a sketch) Given the symmetry $X$, one has to introduce "canonical variables" (or symmetry-adapted variables) $s, z \equiv (z_1, \ldots, z_{p-1})$, which are defined by

$$Xs \equiv \xi_i\frac{\partial s}{\partial x_i} + \phi_\alpha\frac{\partial s}{\partial u_\alpha} = 1, \qquad Xz_k = 0 \qquad (k = 1, \ldots, p-1).$$

Using the method of characteristics, one also finds the $q$ dependent variables $v = v_\alpha(s, z)$; once written in these coordinates, the symmetry field and all its prolongations are simply given by

$$\widetilde{X} = \widetilde{X}^* = \frac{\partial}{\partial s} \tag{6}$$

whereas the symmetry condition (Definition 1) becomes $\left.\frac{\partial\widetilde{\Delta}}{\partial s}\right|_{\widetilde{\Delta}=0} = 0$, or

$$\frac{\partial}{\partial s}\widetilde{\Delta}_a = G_{ab}\widetilde{\Delta}_b. \tag{7}$$

It is not difficult to show (cf. [20]) that for any $\widetilde{\Delta}_a$ satisfying the system (7) there are smooth locally invertible functions $S_{ab}$ such that the combinations $K_a := S_{ab}\widetilde{\Delta}_b$ are independent of $s$, as claimed. We have assumed here for convenience that the vector fields $X$ are "projectable", or – more explicitly – that the functions $\xi$ in (2) do not depend on $u$ (as happens in most cases in the study of PDE's) in order to simplify calculations in the introduction of the canonical coordinates, and to get a more direct relationship between symmetries and symmetry-invariant solutions (for a discussion on this point, cf. [21]). ■

See [22] for a result in an analogous problem, although with different aim (i.e., constructing equations invariant under a given Lie algebra). It should be also emphasized that the result in Theorem 1 is not the same as (but is related to and includes in particular) the well known result concerning the reduction of the given equations to $X$-invariant equations for the invariant variables $w(z)$: indeed, introducing the new "symmetry-adapted" variables $s$, $z$ and $v(s,z)$, we have transformed the equations into *equivalent* equations for $v(s,z)$. If one now *further* assumes that $\partial v/\partial s = 0$, i.e. if one looks for the $X-$invariant solutions where $v = w(z)$, then the equations $K_a = 0$ become a system of equations

$$K_a^{(0)}\big(z, w^{(m)}\big) = 0 \tag{8}$$

involving only the variables $z$ and functions depending only on $z$ (see [23] for a general discussion on the reduction procedure).

# 3 "Standard" and "weak" conditional symmetries, and related notions

Let us now consider the case of non-exact symmetries. A fundamental and largely comprehensive notion has been introduced by Fushchych [24]: let us say that $X$ is a *conditional symmetry of the equation* $\Delta = 0$ *in the sense of Fushchych* if there is a supplementary equation $E = 0$ such that $X$ is an exact symmetry of the system $\Delta = E = 0$.

The simplest and more common case is obtained choosing as supplementary equation the "side condition" or "invariant surface condition"

$$X_Q u \equiv \xi_i \frac{\partial u}{\partial x_i} - \phi = 0, \tag{9}$$

where $X_Q$ is the symmetry written in "evolutionary form" [16]: this corresponds to the usual (properly called) conditional symmetry (CS) (also called $Q$-conditional symmetry), and the above condition indicates that we are looking precisely for solutions which are *invariant* under $X$.

To avoid unessential complications with notations, we will consider from now on only the case of a single PDE $\Delta = 0$ for a single unknown function $u(x)$. The extension to more general cases is in principle completely straightforward.

It is known that the above definition of CS suffers from some intrinsic difficulties, essentially due to the necessity of introducing and dealing with the differential consequences of (9) (for a discussion of this point, see e.g. [16, 25, 26], and [27–29] for a more complete definition). Related to these difficulties is the quite embarrassing sentence by Olver and Rosenau [25] (see also [26]), which says – essentially – that, given any differential equation, *any* vector field $X$ is a CS, and *any* solution of the equation is an invariant solution under some $X$.

To clarify this point, we will introduce a subtler definition of CS. This will be made resorting once again to the canonical coordinates $s$, $z$, $v = v(s,z)$, introduced in the proof of Theorem 1. First of all, in these coordinates the invariance condition $X_Q v = 0$ becomes

$$\frac{\partial v}{\partial s} = 0 \tag{10}$$

and the condition of CS takes the simple form (let us now retain for simplicity the same notation $\Delta$, instead of $\widetilde{\Delta}$, also in the new coordinates)

$$\frac{\partial \Delta}{\partial s}\Big|_{\Sigma} = 0 \tag{11}$$

here $\Sigma$ stands for the set of the simultaneous solutions of $\Delta = 0$ and $v_s = \partial v/\partial s = 0$, together with the derivatives of $v_s$ with respect to all the variables $s$ and $z_k$. Introducing the global notation $v_s^{(\ell)}$ to indicate $v_s$, $v_{ss}$, $v_{sz_k}$ etc., we shall say that $X = \partial/\partial s$ is a CS in *standard sense* if the equation takes the form

$$\Delta = R\big(s, z, v^{(m)}\big) K\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0, \tag{12}$$

where the point to be emphasized is that $K$ does not depend *explicitly* on $s$, and $R$, $K$ do not contain $v_s^{(\ell)}$. It is then clear that, if one now looks for solutions of $\Delta = 0$ which are independent on $s$, i.e. such that $v_s^{(\ell)} = 0$, or of the form $v = w(z)$, then equation (12) becomes a "reduced" equation $K^{(0)}(z, w^{(m)}) = 0$, just as in the exact symmetry case.

But this is clearly only a special case. Indeed, the equation $\Delta = 0$ may also take the form

$$\Delta = \sum_{r=1}^{\sigma} s^{r-1} K_r\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0, \tag{13}$$

where the part not containing $v_s^{(\ell)}$ is a polynomial in the variable $s$, with coefficients $K_r$ not depending explicitly on $s$, or also – more in general (with some different regrouping of the terms containing $s$ into linearly and functionally independent terms $R_r$)

$$\Delta = \sum_{r=1}^{\sigma} R_r\big(s, z, v^{(m)}\big)\, K_r\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0. \tag{14}$$

In this case, if one looks for $X$-invariant solutions $w(z)$ of $\Delta = 0$, one is faced with the system of reduced equations (not containing $s$ nor functions of $s$)

$$K_r^{(0)}\big(z, w^{(m)}\big) = 0, \qquad r = 1, \ldots, \sigma. \tag{15}$$

Assume that this system admits some solution (it is known that the existence of invariant solutions is by no means guaranteed in general, neither for "standard" CS, nor for "exact" Lie symmetries), we will say that $X$ is *weak CS of order $\sigma$*.

We now see that the set of the solutions of the above system can be characterized equivalently as the set of the solutions of the system

$$\Delta = 0, \quad \frac{\partial \Delta}{\partial s} = 0, \quad \ldots, \quad \frac{\partial^{\sigma-1}\Delta}{\partial s^{\sigma-1}} = 0, \quad v_s^{(\ell)} = 0. \tag{16}$$

Coming back to the original coordinates $x$, $u$, the set of conditions (16) becomes

$$\Delta = \Delta^{(1)} = \cdots = \Delta^{(\sigma-1)} = 0, \qquad X_Q u = 0, \tag{17}$$

where

$$\Delta^{(1)} := X^*(\Delta), \qquad \Delta^{(2)} := X^*\big(\Delta^{(1)}\big), \qquad \ldots \tag{18}$$

(as already pointed out, also the differential consequences of $X_Q u = 0$ must be taken into account), and a CS of order $\sigma$ can be characterized by the condition

$$X^*(\Delta)|_{\Sigma_\sigma} = 0, \tag{19}$$

where $\Sigma_\sigma$ is the set (if not empty, of course) of the solutions of the system (17).

We can summarize our discussion in the following form.

**Proposition 1.** *Given a PDE $\Delta = 0$, a projectable vector field $X$ is a "standard" conditional symmetry for the equation if it is a symmetry for the system*

$$\Delta = 0, \qquad X_Q u = 0$$

*and this corresponds to the existence of a reduced equation in $p-1$ independent variables, which – if admits solutions – gives $X$-invariant solutions of $\Delta = 0$. A vector field $X$ is a "weak" CS (of order $\sigma$) if it is a symmetry of the system*

$$\Delta = 0, \quad \Delta^{(1)} := X^*(\Delta) = 0, \quad \Delta^{(2)} := X^*\big(\Delta^{(1)}\big) = 0, \quad \ldots, \quad \Delta^{(\sigma-1)} = 0, \quad X_Q u = 0$$

*and this corresponds to the existence of a system of $\sigma$ reduced equations, which – if admits solutions – gives $X-$invariant solutions of $\Delta = 0$. Introducing $X-$adapted variables $s$, $z$, such that $Xs = 1$, $Xz = 0$, the PDE has the form* (12) *in the case of standard CS, or* (14) *in the case of weak CS.*

If one neglects the invariance condition $X_Q u = 0$, one is actually dealing with the case of *partial* symmetries. Indeed (see [6–8]), $X$ is precisely a partial symmetry of order $\sigma$ if $X$ is a symmetry of the system

$$\Delta = \Delta^{(1)} = \cdots = \Delta^{(\sigma-1)} = 0. \tag{20}$$

If this is the case, $X$ maps one into another the solutions of the system (20), which is then a "symmetric set of solutions of $\Delta = 0$" [7]. In particular, if in this set there are some solutions which are left fixed by $X$, then $X$ is also a CS (either standard or weak) of $\Delta = 0$.

We can then rephrase the Olver–Rosenau statement [25] in the form:

**Proposition 2.** *Any vector field $X$ is either an exact, or a standard CS, or a weak CS. Similarly, any $X$ is either an exact or a partial symmetry.*

It is well known that the set of the solutions which can be obtained in this way may be empty or contain only trivial solutions (e.g., $u = \text{const}$): it is clear that the choice of good candidates for these "non-exact" symmetry generators should be guided by some reasonable criterion and motivated guess. It is also clear that all the notions of non-exact symmetries considered above can be viewed as special cases of CS in the sense of Fushchych.

In all the above discussion, we have considered the case of a single vector field $X$; clearly, the situation becomes richer and richer if more than one vector field is taken into consideration. First of all, the reduction procedure itself must be adapted and refined when the given equation admits an algebra of symmetries of dimension larger than 1 (possibly infinite): for a recent discussion see [30]. Secondly, for instance, it can happen that the reduced equations (8) or (15) may admit some new symmetry $Y$ not shared by the original equation $\Delta = 0$: this is (essentially) the case of "hidden symmetries" [9,10]. Different reduction procedures have been also proposed, based on the introduction of multiple suitable differential constraints: see, e.g., [11,31–33], and also [17].

## 4 Examples

We will give here some simple examples, to illustrate the properties of the different types of symmetries introduced above, and the different solutions that can be obtained accordingly.

**Example 1.** Consider the equation, proposed by Popovych [29]

$$u_t + u_{xx} - u + t(u_x - u) = 0, \qquad u = u(x,t).$$

The vector field $X = \partial/\partial t$ is not an exact nor a standard CS, but is a weak CS, indeed the system of equations (15) (here $s = t$) becomes $u_{xx} = u$, $u_x = u$, with solution $u = c\exp(x)$. The same vector field $X = \partial/\partial t$ is a weak CS also for this variation of the above equation:

$$u_t - u_{tt} + u_{xx} - u + t(u_x - u) = 0$$

with the same solution as above. But $X$ is now also a partial symmetry: indeed, the equation $\Delta^{(1)} = 0$ is now $u_x - u = 0$, and combining it with $\Delta = 0$ we find the more general solution $u = c\exp(x) + c_1\exp(x+t)$. Considering this other variation of the Popovych example

$$t^2(u_t - u) + u_{xx} - u + t(u_x - u) = 0$$

here $X = \partial/\partial t$ is only a partial symmetry, leading to the solution $u = c\exp(x+t)$ (strictly speaking, it is also a weak CS, but producing only the trivial solution $u = 0$!).

**Example 2.** It is well known that the Korteweg-de Vries equation

$$u_t + u_{xxx} + uu_x = 0, \qquad u = u(x,t)$$

does not admit (standard) CS, apart from its exact symmetries. There are however weak CS; e.g. the scaling

$$X = 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$$

is an exact symmetry for the system $\Delta = 0$, $\Delta^{(1)} := X^*(\Delta) = 0$, $X_Q u = 0$, which means that this is a weak CS, giving the scaling-invariant solution $u = x/t$. But also, if we neglect the invariance condition $X_Q u = 0$, we obtain the (clearly larger) symmetric set of solutions $u = (x + c_1)/(t + c_2)$, showing that the above $X$ is also a partial symmetry.

**Example 3.** The symmetry properties of the Boussinesq equation

$$u_{tt} + u_{xxxx} + uu_{xx} + u_x^2 = 0, \qquad u = u(x,t) \tag{21}$$

have been the object of several papers (see e.g. [11, 34, 35]). For what concerns standard CS, writing the general vector field in the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{22}$$

a complete list of CS has been given both for the case $\tau \neq 0$ (and therefore, without any restriction, $\tau = 1$) [34] and for the case $\tau = 0$ [35, 36]; it has been also shown that the invariant solutions under these CS are precisely those found by means of the "direct method", which is not based on (but clearly related to) symmetry properties [34, 35, 37].

To complete the analysis, one can also look for symmetries with $\xi = 0$. It is not difficult to verify that no standard CS of this form is admitted. There are however weak CS: an example is

$$X = \frac{\partial}{\partial t} + \left(\frac{1}{t^2} - \frac{2u}{t}\right)\frac{\partial}{\partial u} \tag{23}$$

one obtains from this: $s = t$, $z = x$ and $u(x,t) = t^{-1} + t^{-2}v(x,t)$, giving

$$vv_{xx} + v_x^2 + 6v + t(v_{xx} + 2) + t^2 v_{xxxx} - 4tv_t + t^2 v_{tt} = 0 \tag{24}$$

which is precisely of the form (14) (the role of $s$ is played here by $t$). Looking indeed for solutions with $v = w(x)$, one gets a system of three ODE's

$$ww_{xx} + w_x^2 + 6w = 0, \qquad w_{xx} + 2 = 0, \qquad w_{xxxx} = 0$$

(cf. (16)), admitting the common solution $w = -x^2$ and giving the (quite elementary) solution $u = 1/t - x^2/t^2$ of the Boussinesq equation.

Another example of weak CS for the Boussinesq equation is the following

$$X = t^2\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \left(2x + \frac{10}{3}t^3\right)\frac{\partial}{\partial u} \tag{25}$$

now $s = t$, $z = x - t^3/3$ and $u = -2sz - s^4 + v(s,z)$. The additional equations $\Delta^{(1)} := X^*(\Delta) = 0$ etc.: now become

$$\Delta^{(1)} = -10t - 3u_x - 2tu_{xt} - \frac{5}{3}t^3 u_{xx} - xu_{xx} = 0, \qquad \Delta^{(2)} = 2 + u_{xt} + t^2 u_{xx} = 0 \tag{26}$$

and taking into account also the invariance condition $X_Q u = 0$, we easily conclude that this is a weak CS of order $\sigma = 3$ and obtain the solution

$$u(x,t) = -\frac{t^4}{3} - 2tx - \frac{12}{(x - t^3/3)^2} \tag{27}$$

If instead we do *not* impose the invariance condition $X_Q u = 0$ and solve the three equations (21), (26), we find, in addition to the invariant solution (27), also the following family of solutions $u(x,t) = -t^4/3 + c_1 t - 2tx + c_2$, showing that the above symmetry is also a partial symmetry.


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