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Matrix Impulsive System of Differential Equations with Bilinear Main Part

Vladimir A. CHIRICALOV

1-7 Pechenigivska Str., ap. 112, 04107 Kyiv, Ukraine
E-mail: *cva@skif.kiev.ua*

The main purpose of the paper is to summarize some fundamental results concerning the matrix impulsive differential equations with impulses at fixed points. The theory of impulsive matrix differential equations is interesting in itself and it will assume greater importance in the near future, since the application of the theory to various fields of science is also increasing, especially in the theory of polyphase transmission line and surge phenomena.

1 Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbation whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulsive effects, appear as a natural description of observed evolution phenomena of several real world problems. In the theory of impulsive systems, there are some problems similar to the ones considered in the theory of ordinary differential equations but there are also problems that are specific to the theory of impulsive systems.

We will be considering the matrix differential equations. It means that a phase space of evolution process be a matrix space $R^{n\times m}$. The point that describes the state of the process at the time t we denote $X(t) \in R^{n\times m}$. The topological product of the phase space $R^{n\times m}$ and real axis R, i.e. $R\times R^{n\times m}$ will be called the extended phase space of the considered evolution process. The process itself goes as follows: the point $P_t = (t, X(t)), X(t) \in R^{n\times m}$, begins its motion from the initial point $P_{t_0} = (t_0, X_0)$ and moves along the curve $\{(t, X(t)); t > t_0\}$ until the time $t_1 > t_0$ at which the point P_t meets the set Γ_1. At $t = t_1$ the operator $\mathcal{M}_1$ transfers the point $X(t_1) \in R^{n\times m}$ into the point $X(t_1 + 0) = X^+(t_1) \in R^{n\times m}$ then the point P_t continues to move further along the curve $\{(t, X(t)); t > t_1\}$ with initial value $(t_1, X^+(t_1))$, $X^+(t_1) = \mathcal{M}_1(X(t_1))$, where $X(t) = X(t, t_1, X_1^+) = X_t(t_1, X_1^+)$ is the solution of equation

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t, X), \tag{1}$$

where $F : R^+ \times R^{n\times m} \to R^{n\times m}$, R^+ is the nonnegative real line. The point P_t moves along the curve until the moment $t_2 > t_1$ at which the point P_t again meet the new set Γ_2. The moment t_i at which the point P_t hits the set Γ_i $(i = 1, 2, \ldots)$ are called moment of impulsive effect. The sets Γ_i in extended phase space $R^+ \times R^{n\times m}$ may be determined by equation $\mathcal{F}_i(t, X) = 0$; $\mathcal{F}_i : R^+ \times R^{n\times m} \to R^+$. When the point P_t hits the set Γ_i at the moment t_i this equation becomes identity: $\mathcal{F}_i(t_i, X(t_i)) \equiv 0$. If this equation is solved with respect to t, we have explicit form of the set $\Gamma_i : t = \tau_i(X)$, $\tau_i : R^{n\times m} \to R^+$. As a special case we are considering the sets Γ_i as a sequence of hyperplane $t = t_i = \text{const}$, where $\{t_i\}$ is a given sequence of times (finite or infinite). A solution of an impulsive differential system may be a piecewise continuous function having a countable number of discontinuities of the first kind if the integral curve encounters the sets Γ_i at a countable number of points that are not the fixed point of operator $\mathcal{M}_i$. The matrix differential equations without impulses have been considered in the papers [1–6].

2 Matrix differential system with impulses at fixed times

If in a real process described by system (1) the impulses occur at fixed times the mathematical model of this process will be given by the following matrix impulsive system

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t,X), \qquad t \neq t_i, \tag{2}$$

$$\triangle X(t) = D_i X(t)\widetilde{D}_i + \widetilde{F}_i(X(t)), \qquad t = t_i, \tag{3}$$

where $F \in C[R^+ \times \Re, R^{n\times m}]$ is continuous map of the direct product $R^+ \times \Re$ into space of matrices $R^{n\times m}$, $\Re \subset R^{n\times m}$ being an open set, $R^{n\times m}$ is the space of matrices with real entries, $\tilde{F} : R^{n\times m} \to R^{n\times m}$ is nonlinear matrix-valued function, $\triangle X(t_i) = X(t_i+0) - X(t_i)$, $X(t_i+0) = \lim_{h\to 0+} X(t_i+h)$, $D \in R^{n\times n}$, $\widetilde{D} \in R^{m\times m}$. If the matrix-valued functions $\widetilde{F}_i(X)$ and $F(t,X)$ are independent from X we have matrix nonhomogeneous bilinear impulsive system

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t), \qquad t \neq t_i, \tag{4}$$

$$\triangle X(t) = D_i X(t)\widetilde{D}_i + \widetilde{F}_i, \qquad t = t_i, \tag{5}$$

where $A(t)$, $B(t)$, $F(t)$ are Lebesgue summable matrix-valued function of real variable t on the interval $[a,b]$.

The space of matrices $R^{n\times m}$ is topologically isomorphic to the Euclidean space R^{nm} and therefore the space of solutions M of the system (2), (3) is topologically isomorphic to the space of solutions $\widetilde{M}$ of the system of vector differential equations

$$\frac{dx}{dt} = (A(t)\otimes I_m - I_n \otimes B^T(t))x + f(t), \qquad t \neq t_i, \tag{6}$$

$$\triangle x(t) = (D_i \otimes \widetilde{D}_i^T)x(t) + \widetilde{f}_i, \ t = t_i. \tag{7}$$

Here the superscript T denotes the transpose, $\otimes$ is the sign of tensor product of matrices, $x^T = (x_{1,*}, x_{2,*}, \ldots, x_{n,*})$, $f^T = (f_{1,*}, f_{2,*}, \ldots, f_{n,*})$, $\widetilde{f_i}^T = (\widetilde{f^i}_{1,*}, \widetilde{f^i}_{2,*}, \ldots, \widetilde{f^i}_{n,*})$ $(i = 1, 2, \ldots)$ are vectors, $x_{i,*}$ is the i-th row of the matrix X, $f_{i,*}$ is the i-th row of the matrix F. Without lost of generality we take $t_0 = 0$. If $f = 0$, $\widetilde{f} = 0$ we have the homogeneous vector bilinear system of differential equations. It is well known [7] that a solution of the homogeneous vector system of differential equations is $x_t = U_\tau^t x_\tau$, where $x_t = x(t)$, U_τ^t is evolution matrix operator. The structure of the evolution operator for impulsive differential system of equations has been considered in [8, 9]. It should be remarked that U_τ^t for impulsive differential system in the Banach space has been considered in [9] and it was call there *multiplicative Stiltjes integral* $U_\tau^t = \widehat{\int}{}_\tau^t \exp(dF(s))$ which is the solution of equivalent integral equation [9]

$$U_\tau^t = I + \int_\tau^t (dF(s))U_\tau^s, \tag{8}$$

where I is identity operator, operator-valued function $F(t)$ has bounded variation, and $F(t)$ is not continuously differentiable on $[\tau, t]$, it has finite number of discontinuities of first kind at some moment of time t_i $(i = 1, 2, \ldots)$. As was underlined in monograph [7] larger part of result, concerning the properties of differential system of equations in Banach spaces without impulses, may be generalized into impulsive differential system of equations. We denote $[U_\tau^t] : R^{n\times m} \to R^{n\times m}$ the evolution operator of homogeneous matrix impulsive differential system of equation (2), (3) which corresponding to evolution operator (fundamental matrix) U_τ^t associated homogeneous vector system (6), (7). The vector function $x(t) = U_0^t x_0$ is a solution of homogeneous system (6), (7) for any constant vector x_0 and if x_0 ranges over the whole space R^{nm}, then the family of function $x(t)$ forms a space of solutions $\widetilde{M}$ of the system (6), (7). The matrix function $[U_0^t]X_0$, accordingly, is a solution of the homogeneous system, associated to the system (2), (3), for any constant matrix $X_0 \in R^{n\times m}$ and if X_0 ranges over the whole matrix space $R^{n\times m}$ then the family of matrix functions $X_t = [U_0^t]X_0$ form a space of solutions M of the system (2), (3). The operator $[U_0^t]$ we call *fundamental operator* or *evolution operator* (*transfer operator*) [9]. Some problems of an impulsive matrix differential equations have been considered in [12–16].

3 Homogeneous matrix bilinear impulsive differential system

The solution of the homogeneous system (2), (3) can be written in the form $X_t(X_\tau) = [U_\tau^t]X_\tau$, where $[U_\tau^t]$ is nonsingular solution of homogeneous bilinear matrix impulsive differential system (2), (3), which satisfies the condition $[U_\tau^\tau] = [I]$, $[I]$ is used to denote the unit operator in the space of matrix $R^{n\times m}$. The evolution operator of the equation (2), (3) on the interval of continuity we denote $[\Omega_\tau^t]$: $R^{n\times m} \to R^{n\times m}$, where $[\Omega_\tau^t]Z = \underset{A}{\Omega}{}_\tau^t Z \underset{B}{\Omega}{}_t^\tau$, $Z \in R^{n\times m}$; $\underset{A}{\Omega}{}_\tau^t$, $\underset{B}{\Omega}{}_\tau^t$ are matriciants associated with the matrices $A(t)$ and $B(t)$ respectively. $\underset{A}{\Omega}{}_\tau^\tau = I_n$, $\underset{B}{\Omega}{}_\tau^\tau = I_m$; I_n, I_m are unit matrices in the Euclidean spaces R^n and R^m. Let an interval $[\tau, \tau+h]$ contain a finite number of points of discontinuity t_{j+i}, $(i = \overline{s+1, k})$, $s < k$, if $\tau < t_{j+s+1}$ and $t > t_{j+k}$, then evolution operator of system (2), (3) can be represented as

$$[U_\tau^t] = [\Omega_{t_{j+k}}^t] \prod_\nu \left(([I] + [D_\nu])[\Omega_{t_{\nu-1}}^{t_\nu}] \right), \tag{9}$$

where $\nu = j+k, j+k-1, \ldots, j+s+1$, $(k > s)$, we denote $\tau = t_{j+s}$ and it should be noted that point of time τ is not the point of discontinuity. The evolution operator U_τ^t of the system (6), (7), which corresponding to the operator $[U_\tau^t]$ can be written as

$$U_\tau^t = (\underset{A}{\Omega}{}_{t_{j+k}}^t \otimes \underset{B}{\widetilde{\Omega}}{}_t^{t_{j+k}}) \prod_\nu \left((I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T)(\underset{A}{\Omega}{}_{t_{\nu-1}}^{t_\nu} \otimes \underset{B}{\widetilde{\Omega}}{}_{t_\nu}^{t_{\nu-1}}) \right), \tag{10}$$

where $\nu = j+k, j+k-1, \ldots, j+s+1$, $(k > s)$, $I_{nm} = I_n \otimes I_m$, $\widetilde{\Omega} = \Omega^T$, U_τ^t is a solution of the Cauchy problem for the homogeneous system associated with the system (6), (7), with initial conditions $U_\tau^\tau = I_{nm}$, I_{nm} is unit operator in the Euclidean space R^{nm}. Any matrix solution $Z(t)$ of the vector system (6), (7) can be represented as $Z(t) = U_\tau^t Z(\tau)$. By using the Jacobi formula we get from (10), $\det Z_t = \det U_\tau^t \det Z_\tau$. We use the properties: $\det(A_n \otimes B_m) = (\det A)^n (\det B)^m$, $\det \underset{A}{\Omega}{}_{t_{\nu-1}}^{t_\nu} = \exp \int_{t_{\nu-1}}^{t_\nu} \operatorname{Tr} A(s) ds$, $\det \underset{B}{\Omega}{}_{t_\nu}^{t_{\nu-1}} = \exp(-\int_{t_{\nu-1}}^{t_\nu} \operatorname{Tr} B(s) ds)$. Then we have

$$\det U_\tau^t = \exp\left(\int_\tau^t (n \operatorname{Tr} A(s) - m \operatorname{Tr} B(s)) ds \right) \times \prod_\nu \det(I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T), \tag{11}$$

where $\nu = \overline{j+s+1, j+k}$, $(k > s)$, $\operatorname{Tr} A(s) = \sum_{i=1}^n a_{ii}(s)$, $\operatorname{Tr} B(s) = \sum_{i=1}^m b_{ii}(s)$. Because the matrices $(I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T)$ are nonsingular for all ν, it follows from (11) that the matrix $Z(t)$

is nonsingular if $Z(\tau)$ is such. The operator $U_\tau^t : R^{nm} \to R^{nm}$ is topologically equivalent the operator $[U_\tau^t] : R^{n\times m} \to R^{n\times m}$ therefore the operator $[U_\tau^t]$ is nonsingular in the space $R^{n\times m}$. For a nonsingular operator $[U_\tau^t]$, the inverse operator $[U_\tau^t]^{-1} = [U_t^\tau]$ is given by the following

$$[U_t^\tau] = \left(\prod_\nu [\Omega_{t_\nu}^{t_{\nu-1}}]([I] + [D_\nu])^{-1} \right) [\Omega_t^{t_{j+k}}], \tag{12}$$

where $\nu = \overline{j+s+1, j+k}$, $(k > s)$, $t_{j+s} = \tau$, $\tau < t_{j+s+1}$, $t > t_{j+k}$, $[\Omega_\tau^t]^{-1} = [\Omega_t^\tau]$.

4 Nonhomogeneous bilinear matrix impulsive differential system

We call the system (2), (3) with $F(t) \neq 0$, $\widetilde{F}_i \neq 0$ $(i = 1, 2, \ldots)$ nonhomogeneous bilinear matrix equation

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t,), \qquad t \neq t_i, \tag{13}$$

$$\triangle X(t) = D_i X(t) \widetilde{D}_i + \widetilde{F}_i, \qquad t = t_i, \quad i = 1, 2, \ldots, \tag{14}$$

where $A(t)$, $B(t)$, D_i, $\widetilde{D}_i$ are the same as in system (2), (3), $F : [a, b] \to R^{n\times m}$ is a matrix-valued function, continuous (piecewise continuous) on the interval $[a, b]$, $\widetilde{F}_i \in R^{n\times m}$ are constant matrices, $a < t_i < b$, $a = t_0$. We use change of variables $X(t) = [U_{t_0}^t]Y(t)$, which is called "variation of parameter". Taking into account the properties of evolutionary operator $[U_{t_0}^t]$ we get

$$dY/dt = [U_t^{t_0}]F(t), \qquad t \neq t_i, \qquad \triangle Y(t_i) = [U_{t_i+0}^{t_0}]\widetilde{F}_i, \qquad t = t_i, \quad i = 1, 2, \ldots \tag{15}$$

the jump condition can also be written in the form $\triangle Y(t_i) = [U_{t_i}^{t_0}]\widehat{F}_i$, where $\widehat{F}_i = ([I]+[D_i])^{-1}\widetilde{F}_i$. Because the operators $([I] + [D_i]) : R^{n\times m} \to R^{n\times m}$ are nonsingular for all i, $\widehat{F}_i$ is the unique solution of matrix algebraic equation

$$\widehat{F}_i + D_i \widehat{F}_i \widetilde{D}_i = \widetilde{F}_i. \tag{16}$$

The solution of the equation (16) can be written in the form [7]

$$\widehat{F}_i = \sum_{j_1, j_2, r_1=0, r_2=0}^{n, m, n_{j_1}, m_{j_2}} \acute{P}_{j_1} \acute{Q}_{j_1}^{r_1} \widetilde{F}_i \grave{P}_{j_2} \grave{Q}_{j_2}^{r_2} \frac{\partial^r \varphi(\lambda_{j_1}, \mu_{j_2})}{r_{12}\partial\lambda_{j_1}^{r_1} \partial\mu_{j_2}^{r_2}}, \tag{17}$$

where $\varphi(\lambda, \mu) = (1 + \lambda\mu)^{-1}$, $r = r_1 + r_2$, $r_{12} = r_1!r_2!$. The matrices $\acute{Q}_{j_1}$, $\grave{Q}_{j_2}$ are defined by the spectral expansion of the matrices D_j and $\widetilde{D}_j$

$$D_j = \sum_{j_1=1}^{n} (\lambda_{j_1} \acute{P}_{j_1} + \acute{Q}_{j_1}), \qquad \widetilde{D}_j = \sum_{j_2=1}^{m} (\mu_{j_2} \grave{P}_{j_2} + \grave{Q}_{j_2}). \tag{18}$$

Taking into account (16) we get for $t > t_0$

$$Y(t) = Y_0 + \int_{t_0}^{t} [U_\tau^{t_0}] F_1(\tau) d\tau. \tag{19}$$

where $F_1(\tau) = F(\tau) + \sum_i [U_{t_i}^{t_0}]\widehat{F}_i \delta(t - t_i)$.

Corollary 1. *Let $[U^t_\tau]$ be a fundamental operator homogeneous matrix system (2), (3), with nonsingular operators $([I]+[D_i])$, $(i=1,2,\ldots)$. Then every solution of nonhomogeneous matrix bilinear impulsive differential system (2), (3), for $t>t_0$ is given by the formula*

$$X(t) = [U^t_{t_0}]X_0 + \int_{t_0}^{t}[U^t_\tau]F(\tau)d\tau + \sum_i [U^t_{t_i}]\widehat{F}_i, \tag{20}$$

where i is defined by the inequalities $t_0 < t_i < t$.

This formula can be written by using Stieltjes integral of operator-valued function $[U^t_\tau]$ by the matrix-valued function $L(t)$, where $L(t) = \int_{t_0}^{t}(F(\tau)+\sum_i \widehat{F}_i\delta(\tau-t_i))d\tau = \int_{t_0}^{t}F(\tau)d\tau + \sum_i \widehat{F}_i\eta(\tau-t_i)$ is matrix-valued function of bounded variation on $[t_0,t)$. Here $\delta(t-t_i)$ is the Dirac delta function, $\eta(t-t_i)$ is the Heaviside step-function or jump-function. Both of these functions are generalized functions or distributions

$$X(t) = [U^t_{t_0}]X_0 + \int_{t_0}^{t}[U^t_\tau]dL(t), \tag{21}$$

where $[U^t_\tau]$ is the multiplicative Stiltjes integral [7, 10] or evolution operator of homogeneous bilinear matrix impulsive equation corresponding to (2), (3).

5 Bilinear matrix impulsive system of differential equation with constant coefficients

Let the matrices $A(t)$, $B(t)$, D_i, $\widetilde{D}_i$ in system (2), (3) be constant. Then we have a bilinear matrix impulsive system of differential equations with constant coefficients

$$\frac{dX}{dt} = AX - XB + F(t), \qquad t \neq t_i, \tag{22}$$

$$\triangle X(t) = DX(t)\widetilde{D} + \widetilde{F}_i, \qquad t = t_i, \quad i=1,2,\ldots. \tag{23}$$

At first we consider homogeneous system, i.e. $F(t)\equiv 0$, $\widetilde{F}_i \equiv 0$. Suppose that the times t_i are indexed by the set of natural numbers such that $t_i \to \infty$ for $i\to\infty$. Without loss of generality, we can assume that $t_1 > t_0$. It is easy to see that for any (t_0, X_0), there exists a unique solution $X^t_{t_0}$ of (22), (23) for $t \geq t_0$, it can be written as $X_t(t_0,X_0) = [U^t_{t_0}]X_0$, where

$$[U^t_{t_0}] = [\Omega^t_{t_k}]\prod_{\nu=k}^{1}\Big(([I]+[D])[\Omega^{t_\nu}_{t_{\nu-1}}]\Big) \tag{24}$$

and $[\Omega^t_s]Z = e^{A(t-s)}Ze^{-B(t-s)}$. Solutions of the homogeneous matrix system are not invariant with respect to shifts because, due to the times of an impulsive effect $t=t_i$, homogeneous matrix system is not autonomous. However, in some cases expression (24) can be simplified. For example, if the matrices A and D, B and $\widetilde{D}$ commute, then matrix exponent e^{At} commutes with the matrix D and e^{-Bt} commutes with the matrix $\widetilde{D}$ and equality (24) can be written as

$$X_t(t_0,X_0) = ([I]+[D])^{p(t,t_0)}e^{A(t-t_0)}X_0e^{-B(t-t_0)}, \tag{25}$$

where $p(t,t_0)$ is the number of points t_i which belong to the segment $[t,t_0]$, i.e. $p(t,t_0)=k$ if $t_k < t < t_{k-1}$, $t_0 < t_1$. In particular, if the times t_i are equidistant, $t_i = t_1 + (i-1)\theta$ and operator $([I]+[D])$ is nonsingular, then from (25) we get

$$X_t(t_0,X_0) = e^{Ln([I]+[D])p(t,t_0)}e^{A(t-t_0)}X_0e^{-B(t-t_0)},$$

where $p(t,t_0) = (t-t_1)/\theta - \{(t-t_1)/\theta\})$, curly brackets $\{\cdot\}$ means fractional part of the real value $(t-t_1)/\theta$. For the equivalent vector equation (6), (7), we have

$$x_t(x_0) = \Theta(t)e^{S(t-t_1)},$$

where

$$S = \widehat{D} + \frac{1}{\theta}Ln(I_{nm} + D \otimes \widetilde{D}), \widehat{D} = A \otimes I_m - I_n \otimes B^T,$$
$$\Theta(t) = e^{\widehat{D}(t_1-t_0)}e^{-Ln(I_{nm}+D\otimes\widetilde{D})(\{(t-t_1)/\theta\}+1)}.$$

From this, it can be seen that behavior of the solution of an impulsive system is defined by eigenvalues of the matrix S because matrix $\Theta(t)$ is bounded. Consequently, if the real parts of all eigenvalues of the matrix S are negative, then all solutions of homogeneous system corresponding to the system (22), (23) tend to zero for $t \to \infty$.

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