Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 666–670

Quantization of Nonlinear Fields on the Classical Background by Means of Bogoliubov Group Variables

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The scheme of quantization of gravitational field in the neighborhood of the classical component by means of the Bogoliubov group variables is presented. Nontrivial exact solutions of Einstein equation are chosen to be the classical background. The system is assumed to be invariant with respect to arbitrary group transformation of the variable x.

1 Introduction

In the present article the theory of quantization of nonlinear boson fields in a vicinity of a nontrivial classical component by means of the Bogoliubov group variables is presented. It is possible to show that quantization of any nonlinear physical system should include quantization of a boson field in a vicinity of a classical background. If it is impossible to consider a classical field from the very beginning to be small (gravitation and extended particles), there is a problem of the description of properties of the physical system, in which main effect is the separation of a classical boson field. The known theories of quantization on a classical background meet two basic difficulties. The first and the main one is a problem of the conservation laws, and the second is the zero-mode problem. To bypass these problems we propose to use an idea of Bogoliubov. In this work we propose new quantizations that will allow to use the Bogoliubov group variables for systems having arbitrary symmetry group (provided that we know representation of this group), including ones for non-stationary systems.

2 The basic ideas of a new method of quantization

In this section we present schematically the basic ideas of quantization of boson fields by means of method of the Bogoliubov group variables. In subsequent sections we shall show how this approach works in the case of gravitational field.

Development of a scheme of quantization is done in three basic stages.

I. At the first stage we pass to new variables, i.e., Bogoliubov group variables.

Let variable x' be related to x by the group transformation:

$$x' = F(x,a), \qquad x'' = F\left(F(x,a),b\right) = F(x,c), \qquad c = \varphi\left(a,b\right).$$

Variations of coordinates at a variation of parameters of group a are:

$$\left(\delta x'\right)^i = \xi^i_s(x')B^s_p(a)\delta a^p,$$

where $i = 0,1,2,3$ is the number of the coordinate, $p = 1,\ldots,r$, and k is the number of generators of group. The group properties of the transformation are defined by the tensor $B^s_p(a)$.

Let us define Bogoliubov transformation as follows:

$$f(x) = gv(x') + u(x'),$$

the dimensionless parameter g is supposed to be large, and $(u(x'), a)$ are independent new variables (Bogoliubov group variables). The explicitly separated large component depends on x', as well as a quantum part. Thus, we restore invariance with respect to the transformation group, which was broken by explicit separation of classical component, as it was pointed out in Introduction. However, consideration of the variable τ as independent leads to the fact that the right-hand side of our equation contains now k variables more than the left one. In order to equalize the number of variables we shall impose some invariant conditions on $u(x')$.

Let us make it as follows: we choose some functions $N^a(x')$ (the number of functions is equal to the number of group parameters) and we shall impose the following conditions on the function $u(x)$:

$$\omega(N^a, u) = \int_\Sigma d\sigma\big(N^a_n u - N^a u_n\big) = 0.$$

This form is invariant with respect to group transformations.

Use of this condition allows to obtain the equations, which determine group variables as functionals of initial $f(x)$ and $f_n(x)$ on D in the differential form:

$$\frac{\delta\tau^a}{\delta f(x)} = -\frac{1}{g}Q^a_b\tilde{N}^b_n(x'), \qquad \frac{\delta\tau^a}{\delta f_n(x)} = \frac{1}{g}Q^a_b\tilde{N}^b(x'),$$

where Q^a_b is a solution of the equation $Q^a_b = \delta^a_b - \frac{1}{g}R^a_c Q^c_b$. $\tilde{N}^a$ is a linear combination of N^a, such that parities $\omega(\tilde{N}^a, v_b) = 0$; and R^a_c is a complex number obtained with the help of $v(x')$ and $u(x')$.

II. At the second stage we develop the perturbation theory. All integrals of motions are expressed in new variables. Analyzing the obtained expressions, we come to conditions at which application of perturbation theory is correct, namely: the classical part has to satisfy some equations. These equations appear to coincide with Euler–Lagrange equations. However, this result is not trivial: it was received not as a consequence of variational principle, but as a condition for development of a perturbative scheme for our system in terms of new variables, while for obtaining of these equations we did not need to know the Hamiltonian structure as the generator of translations in time.

So, at the second stage we obtain equations of motion for classical components.

It is done as follows: Having in explicit form expressions for new variable as functionals from old variable on D it is possible to define the operators $\hat{q}(x)$ and $\hat{p}(x)$:

$$\hat{p}(x) = \frac{1}{\sqrt{2}}\left(f_n(x) + i\frac{\delta}{\delta f(x)}\right), \qquad \hat{q}(x) = \frac{1}{\sqrt{2}}\left(f(x) - i\frac{\delta}{\delta f_n(x)}\right).$$

They are Hermitian in space, where the scalar product looks like:

$$\langle F_1 \,|\, F_2\rangle = \int DfDf_nF_1[f, f_n]F_2[f, f_n],$$

and also satisfy to formal commutation relations:

$$[\hat{q}(x), \hat{p}(x')] = i\delta(x - x').$$

Therefore, we can treat $\hat{q}(x)$ and $\hat{p}(x)$ as operators of coordinate and momentum of field oscillators, and to develop a scheme of secondary quantization. However, there is another pair of self-adjoint operators, constructed from $f_n(x)$, $f(x)$, $\frac{\delta}{\delta f(x)}$ and $\frac{\delta}{\delta f_n(x)}$, which satisfy the same commutation relation. Thus, the number of possible field states appears to be doubled and requires a reduction.

Therefore, we shall act as follows: We shall develop a scheme of the perturbation theory in terms of Bogoliubov variables despite appearance of extra states. After that we make a reduction of states number, and this reduction will be, as a matter of fact, a dynamical scheme.

Now we have quantum operators of coordinate and momentum (and they are are expressed in terms of new variable). Now we can perform quantization of the system by the following substitution:

$$f(x') \longrightarrow \hat{q}(x), \qquad f_n(x') \longrightarrow \hat{p}(x).$$

In terms of the new variable $\hat{q}(x)$ and $\hat{p}(x)$ are series in inverted powers of the coupling constant. Hence, and the integrals of a motion of system are series:

$$O = g^2 O_{-2} + g O_{-1} + O_0 + g^{-1} O_1 + \cdots .$$

Here O_{-2} are complex numbers, the operators O_{-1} are linear with respect to $u(x')$, $u_n(x')$, $\frac{\partial}{\partial u(x')}$, $\frac{\partial}{\partial u_n(x')}$. Since there are no normalizable eigenvectors of such operators, for correct development of perturbation theory, it is necessary that operators are identically equal to zero. This requirement is satisfied if the classical component satisfies equations of motion on D (at certain boundary conditions). Then the operators O_{-1} are equal to zero.

Therefore, further we shall consider the classical part to be the solution of Cauchy problem with boundary conditions on D.

III. The third stage is construction of the system state space.

Above we have mentioned that the number of possible field states is double in comparison with a real situation, and consequently reduction of number of states is necessary.

First of all, we shall analyze the number of independent variables. The initial number of independent variables was ∞. After definitions of the Bogoliubov group variables (they, we remind, are considered as independent) this number became $\infty + k$. This number was doubled because of the way of determination of the creation-annihilation operators: $(\infty + k) * 2 = 2 * \infty + 2 * k$. Constraints, which are introduced together with definition of Bogoliubov transformation, reduce number of independent variable by k (the number of group parameters). And at the present moment the number of possible field states is $2 * \infty + k$.

Let us separate from field variables $u(x')$ k some variables r_a, which have no physical sense and are related to the method of realization of the perturbation theory. Then the number of states is $2 * \infty$, and it is possible to describe the field by means of the variable $w(x')$, that is defined as follows:

$$u(x') = w(x') + \tilde{N}^a(x') r_a, \qquad u_n(x') = w_n(x') + \tilde{N}^a_n(x') r_a.$$

The necessary reduction of the state number can be made as follows: let us assume that a state of a field is defined by functionals $w(x')$ and $w_n(x')$, on which $\frac{\delta}{\delta w(x')}$ and $\frac{\delta}{\delta w_n(x')}$ act by a following way:

$$\frac{\delta}{\delta w(x')} \longrightarrow \frac{\delta}{\delta w(x')} - i w_n(x'), \qquad \frac{\delta}{\delta w_n(x')} \longrightarrow -i w(x').$$

(It can be achieved, for example, by introduction of holomorphic representation in a certain isomorphically connected space.) After reductions we have the following independent variables:

$$(k \text{ of group parameters}) + (k \text{ variable } r_a) + (\infty - k\text{-dimensional space of functions } w).$$

Variables r_a have no physical sense. They have appeared as the remainder of reduction of states space in the terms of Bogoliubov group variables. Separation of these variable is related to structure of integrals of a motion in zero order and also dynamical by nature.

At this stage of construction of our scheme we obtained the correct number of degrees of freedom that will be equal to their real number. During realization of the reduction we obtain equations of motion for quantum correction for our field (the equations appear to be nonlinear). Also at this stage the explicit expressions for creation-annihilation operators are obtained, as well as the spectrum of frequencies, that is actually the energy spectrum. Besides, we obtain expressions for integrals of motion as generators of corresponding symmetry transformations:

$$O_0 = i\Im^{\alpha}\frac{\partial}{\partial\tau^{\alpha}}.$$

In particular, the Hamiltonian is the generator of shift on time. Let us note that equations of motion of a classical part and explicit form of the Hamiltonian are received independently, that is to overcome basic difficulty of the description of non-stationary systems, which was described in Introduction.

Also in the zero-th order we shall obtain Heizenberg equation in terms of new variables.

The summary of the received results:

– equation of motion of a classical part;
– equation of motion of the quantum addendum;
– energy spectrum;
– expressions for integrals of motion;
– Heisenberg equations;
– explicit form of the field operator in the terms of new variables,

allows to assert that our theory gives the complete description of a system of boson fields, quantized on classical background. At the same time, the theory guarantees exact fulfillment of the conservation laws in any order of perturbation theory and also allows to avoid a zero-mode problem.

3 Gravity quantization as example of application of Bogoliubov group variables

Now we apply an approach presented above to description of a gravitational field from the point of view of quantum field theory. We consider quantization in vicinity of a solution of Einstein equation and use the method of Bogoliubov transformations for this purpose.

Let the variables x' be connected with x by an arbitrary group transformation:

$$x' = f(x,a), \qquad x'' = f\left(f(x,a),b\right) = f(x,c), \qquad c = \varphi\left(a,b\right).$$

Let us consider pairs of functions $f_{st}(x)$, $f_n^{st}(x)$ and define the Bogoliubov transformation as follows:

$$f_{st}(x) = gv_{st}(x') + u_{st}(x'), \qquad f_n^{st}(x) = gv_n^{st}(x') + u_n^{st}(x'),$$

the dimensionless parameter g is assumed to be large, and group parameters a^p are new independent variables.

For quantization we use operators $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x')$:

$$u_{st}(x') \mapsto \hat{q}^{st}(x), \qquad u_n^{st}(x') \mapsto \hat{p}^{st}(x).$$

In the expansion of action into series with respect to inverted powers of coupling constant operators S_0 are complex numbers. There are no normalizable eigenvectors of S_1, so it is necessary to set them to zero for perturbation theory construction. It is possible if the following equations hold true on Σ:

$$F_{stn} = \frac{2a}{\sqrt{F}}\left(F_{stn} - \frac{1}{2}F_n F_{st}\right),$$

$$
\begin{aligned}
F^{st}_{nn} &= \frac{a}{2\sqrt{F}}\left(F_{lkn}F^{lk}_{n} - \frac{1}{2}F^2_n\right)F^{st} - \frac{2a}{\sqrt{F}}\left(F^{st}_nF^{kl}_nF_{stn} - \frac{1}{2}F_nF^{st}_n\right) \\
&\quad - a\sqrt{F}\left(R^{st} - \frac{1}{2}F^{st}R\right) - \sqrt{F}\left(F^{sl}c^t_{;l} - F^{st}c^l_{;l}\right).
\end{aligned}
$$

here $F_{st}(x')$ is linearly connected with $v(x')$. These equations could be treated as evolution equations.

Hereinafter, we assume $F_{st}(x)$ to be a solution of those equations, and $F_{st}(x')$ and $F^{st}_n(x')$ are the solution of the Cauchy problem on Σ, so we can state that on the 3D manifold the evolution equation holds true. Also it is possible to obtain conditions on the choice of Σ surface. These equations could be treated as constraint equations

$$
\frac{1}{\sqrt{F}}\left(F^{st}_nF_{stn} - \frac{1}{2}F^2_n\right) - \sqrt{F}R(F) = 0, \qquad F^{sl}_{;l} = 0.
$$

According to (3+1) ADM theory if evolution equations and constraint equations are satisfied in 3D space it means that in 4D manifold Einstein equations are fulfilled.

So we can state that Einstein equations is a necessary condition for the perturbation theory to be applicable. We would like to underline that Einstein equations were obtained in the process of perturbation theory construction as a condition of validity, not as a consequence of the variational principle.

Quantum addendum to the classical gravitational field is obtained in the process of construction of the system state space. It should be a solution of the equations

$$
\begin{aligned}
&\phi_{stm}(\vec{x}) = \frac{2a}{\sqrt{F}}\left(\phi_{stm}(\vec{x}) - \frac{1}{2}F_{st}\phi_m(\vec{x})\right), \\
&\omega^2_m\phi^{st}_m(\vec{x}) = a\sqrt{F}\left(R\phi^{st}_m(\vec{x}) - R^{st}(\phi_m(\vec{x}))\right).
\end{aligned}
$$

Creation-annihilation operators also may be constructed in the process of system state space construction, as well as Hamiltonian as generator of time transformation:

$$
H = -iB^{-1}{}^p_0(a)\frac{\partial}{\partial a^p}.
$$

We applied Bogoliubov transformation to the quantization of gravitational field in the neighborhood of a nontrivial classical component that permitted us to avoid zero-mode problem.

Einstein equations for the classical component were obtained as a necessary condition for the perturbation theory to be applicable, not as a consequence of the variational principle.

We obtained expressions for quantum corrections of the field operator and explicit form of state vector that allows us to calculate quantum corrections to the observables such as effective mass, energy spectrum and so on.

Such calculations for the physically interesting cases like Kerr, Schwartzshild and other exact solutions of Einstein equation are planned for research in the nearest future.

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