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Bogoyavlenskij Symmetries of Isotropic and Anisotropic MHD Equilibria as Lie Point Transformations

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Isotropic Magnetohydrodynamic (MHD) Equilibrium Equations and generalized Anisotropic Magnetohydrodynamic (Chew–Goldberger–Low, CGL) Equilibrium Equations possess infinite-dimensional groups of intrinsic symmetries. We show that certain non-trivial Lie point transformations (that can be obtained by direct application of the general Lie group analysis method) are equivalent to Bogoyavlenskij symmetries in both isotropic and anisotropic cases.

1 Introduction

Among all descriptions of plasma as a continuous medium, two single-fluid models are used most frequently. They are the isotropic magnetohydrodynamics (MHD) equations [1] and the anisotropic CGL (Chew–Goldberger–Low) magnetohydrodynamics equations [2]. Both of them are generalizations of Navier–Stokes system onto the case of conducting fluids; they are derived from Boltzmann and Maxwell equations under different isotropy assumptions.

The isotropic MHD approximation employs scalar pressure p and is valid when the mean free path of plasma particles is much smaller than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions.

On the other hand, when the mean free path for particle collisions is long compared to Larmor radius, for instance, in strongly magnetized or rarified plasmas, the CGL approximation should be used. In this model, the pressure is a tensor with two different components: the pressure along the magnetic field $p_\parallel$ and in the transverse direction $p_\perp$. In the limit $p_\perp = p_\parallel = p$, CGL and MHD models coincide.

The applications of these models include but are not limited to the problem of controlled thermonuclear fusion, astrophysical applications (star formation, solar activity, astrophysical jets) and terrestrial applications (laboratory and industrial plasmas, ball lightning models). The relevant references are [1, 3–9].

Similarly to the Navier–Stokes gas dynamics equations, both MHD and CGL models are essentially nonlinear, and there are no methods so far for solving corresponding general initial/boundary value problems. There also is a lack of physically relevant particular analytical solutions that could model specific phenomena.

In recent papers [10, 11] Bogoyavlenskij introduced new symmetry transforms of the ideal MHD equilibrium equations. In certain classes of plasma configurations, Bogoyavlenskij symmetries break geometrical symmetry, thus giving rise to important classes of non-symmetric MHD equilibrium solutions.

In [12], we have shown that anisotropic CGL plasma equilibria possess similar topology-dependent infinite-dimensional symmetries, which generalize Bogoyavlenskij symmetries.

It is shown that Bogoyavlenskij symmetries for isotropic plasmas and generalized Bogoyavlenskij symmetries for anisotropic plasmas are equivalent to in particular Lie groups of point transformations, which are found independently using the classical Lie group analysis method.

The Lie symmetry method [13] used in this work finds Lie groups of point symmetries of partial differential equations. Lie transformations are used to build particular solutions of the system under consideration, to reduce the order and to obtain invariants. Self-similar solutions constructed from Lie symmetries often have direct physical significance. Many appropriate examples can be found in [14].

We remark, however, that not all symmetries of a given system can be found by the Lie method, but only continuous symmetries that have one-parametric Lie group structure.

In Section 2 of this paper, the isotropic and anisotropic plasma equilibrium equations are described, together with their most important properties, including their Bogoyavlenskij symmetries.

In Sections 3 and 4, the general Lie point symmetry method is described, and an equivalence correspondence is established between Bogoyavlenskij symmetries for isotropic and anisotropic plasmas and certain infinite-dimensional Lie point symmetry groups.

2 MHD and CGL equilibrium equations

The equilibrium states of isotropic moving plasmas are described by the system of MHD equilibrium equations, which under the assumptions of infinite conductivity and negligible viscosity has the form [1]

$$\rho \boldsymbol{V} \times \operatorname{curl} \boldsymbol{V} - \frac{1}{\mu} \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\boldsymbol{V}^2}{2} = 0, \tag{1}$$

$$\operatorname{div} \rho \boldsymbol{V} = 0, \qquad \operatorname{curl} (\boldsymbol{V} \times \boldsymbol{B}) = 0, \qquad \operatorname{div} \boldsymbol{B} = 0. \tag{2}$$

Here $\boldsymbol{V}$ is plasma velocity; $\boldsymbol{B}$ is the vector of the magnetic field induction; ρ, plasma density; P, plasma pressure; and μ, magnetic permeability coefficient.

In the case of incompressible plasma, the equation

$$\operatorname{div} \boldsymbol{V} = 0 \tag{3}$$

is added to the above system. In this paper we restrict our consideration to incompressible plasmas.

It is known [10, 11, 15, 16] that all compact incompressible MHD equilibrium configurations, except the Beltrami case $\operatorname{curl} \boldsymbol{B} = \alpha \boldsymbol{B}$, $\alpha = \text{const}$, have two-dimensional magnetic surfaces – the vector fields $\boldsymbol{B}$ and $\boldsymbol{V}$ are in every point tangent to magnetic surfaces. The magnetic surfaces may not exist for unbounded incompressible MHD equilibrium configurations with $\boldsymbol{V} \,||\, \boldsymbol{B}$.

In the case when plasma Larmor radius is small compared to characteristic dimensions of the system, the set of equations was found by Chew, Goldberger and Low [2]. The anisotropic equilibrium equations are:

$$\rho \boldsymbol{V} \times \operatorname{curl} \boldsymbol{V} - \frac{1}{\mu} \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} = \operatorname{div} \mathbb{P} + \rho \operatorname{grad} \frac{\boldsymbol{V}^2}{2}, \tag{4}$$

$$\operatorname{div} \rho \boldsymbol{V} = 0, \qquad \operatorname{curl}(\boldsymbol{V} \times \boldsymbol{B}) = 0, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{5}$$

where $\mathbb{P}$ is the pressure tensor with two independent components:

$$\mathbb{P} = \mathbb{I} p_\perp + \frac{p_\parallel - p_\perp}{\boldsymbol{B}^2} (\boldsymbol{B}\boldsymbol{B}). \tag{6}$$

Here $\mathbb{I}$ is a unit tensor.

For this system to be closed, one needs to add to it two equations of state. In this paper we will consider incompressible CGL plasmas: $\operatorname{div} \boldsymbol{V} = 0$.

Using vector calculus identities, the pressure tensor divergence may be rewritten in the form

$$\operatorname{div}\mathbb{P} = \operatorname{grad} p_\perp + \tau \operatorname{curl}\boldsymbol{B}\times\boldsymbol{B} + \tau \operatorname{grad}\frac{\boldsymbol{B}^2}{2} + \boldsymbol{B}(\boldsymbol{B}\cdot\operatorname{grad}\tau), \tag{7}$$

$$\tau = \frac{p_\parallel - p_\perp}{\boldsymbol{B}^2}. \tag{8}$$

Hence the system (4), (5) rewrites as

$$\rho\boldsymbol{V}\times\operatorname{curl}\boldsymbol{V} - \left(\frac{1}{\mu}-\tau\right)\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}$$
$$= \operatorname{grad} p_\perp + \rho \operatorname{grad}\frac{\boldsymbol{V}^2}{2} + \tau \operatorname{grad}\frac{\boldsymbol{B}^2}{2} + \boldsymbol{B}(\boldsymbol{B}\cdot\operatorname{grad}\tau), \tag{9}$$

$$\operatorname{div}\boldsymbol{V} = 0, \qquad \operatorname{div}\boldsymbol{B} = 0, \qquad \operatorname{curl}(\boldsymbol{V}\times\boldsymbol{B}) = 0. \tag{10}$$

Recently Bogoyavlenskij [10, 11] found that the ideal MHD equilibrium equations (1)–(3) possess the following symmetries.

Let $\{\boldsymbol{V}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r}), P(\boldsymbol{r}), \rho(\boldsymbol{r})\}$ be a solution of (1)–(3), where the density $\rho(\boldsymbol{r})$ is constant on both magnetic field lines and streamlines. Then $\{\boldsymbol{V}_1(\boldsymbol{r}), \boldsymbol{B}_1(\boldsymbol{r}), P_1(\boldsymbol{r}), \rho_1(\boldsymbol{r})\}$ is also a solution, where

$$\boldsymbol{V}_1 = \frac{b(\boldsymbol{r})}{m(\boldsymbol{r})\sqrt{\mu\rho}}\boldsymbol{B} + \frac{a(\boldsymbol{r})}{m(\boldsymbol{r})}\boldsymbol{V}, \qquad \boldsymbol{B}_1 = a(\boldsymbol{r})\boldsymbol{B} + b(\boldsymbol{r})\sqrt{\mu\rho}\,\boldsymbol{V}, \tag{11}$$
$$\rho_1 = m^2(\boldsymbol{r})\rho, \qquad P_1 = CP + \left(C\boldsymbol{B}^2 - \boldsymbol{B}_1^2\right)/(2\mu).$$

Here

$$a^2(\boldsymbol{r}) - b^2(\boldsymbol{r}) = C = \text{const},$$

and $a(\boldsymbol{r})$, $b(\boldsymbol{r})$, $c(\boldsymbol{r})$ are functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi = \text{const}$, when they exist).

These transformations form an infinite-dimensional Abelian group [11]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2, \tag{12}$$

where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic surfaces. The group G_m has eight connected components.

In [12] we have found that a similar group of transformations is present in the case of anisotropic (CGL) incompressible plasma equilibria: let $\{\boldsymbol{V}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r}), p_\perp(\boldsymbol{r}), p_\parallel(\boldsymbol{r}), \rho(\boldsymbol{r})\}$ be a solution of the CGL system (9), (10), where the density $\rho(\boldsymbol{r})$ and the anisotropy factor $\tau(\boldsymbol{r})$ (8) are constant on both magnetic field lines and streamlines. Then $\{\boldsymbol{V_1}(\boldsymbol{r}), \boldsymbol{B_1}(\boldsymbol{r}), p_{\perp 1}(\boldsymbol{r}), p_{\parallel 1}(\boldsymbol{r}), \rho_1(\boldsymbol{r})\}$ is also a solution, where

$$\rho_1 = m^2(\boldsymbol{r})\rho,$$
$$\boldsymbol{V}_1 = \frac{b(\boldsymbol{r})\sqrt{1/\mu-\tau}}{m(\boldsymbol{r})\sqrt{\rho}}\boldsymbol{B} + \frac{a(\boldsymbol{r})}{m(\boldsymbol{r})}\boldsymbol{V}, \qquad \boldsymbol{B}_1 = \frac{a(\boldsymbol{r})}{n(\boldsymbol{r})}\boldsymbol{B} + \frac{b(\boldsymbol{r})\sqrt{\rho}}{n(\boldsymbol{r})\sqrt{1/\mu-\tau}}\boldsymbol{V},$$
$$p_{\perp 1} = Cp_\perp + \frac{(C\boldsymbol{B}^2 - \boldsymbol{B}_1^2)}{2\mu},$$
$$p_{\parallel 1} = p_\parallel n^2(\boldsymbol{r})\frac{\boldsymbol{B}_1^2}{\boldsymbol{B}^2} + p_\perp\left(C - n^2(\boldsymbol{r})\frac{\boldsymbol{B}_1^2}{\boldsymbol{B}^2}\right) + \frac{(C\boldsymbol{B}^2 + {\boldsymbol{B}_1}^2(1-2n^2(\boldsymbol{r})))}{2\mu}. \tag{13}$$

Here

$$a^2(\boldsymbol{r}) - b^2(\boldsymbol{r}) = C = \text{const},$$

and $a(\boldsymbol{r})$, $b(\boldsymbol{r})$, $m(\boldsymbol{r})$, $n(\boldsymbol{r})$ are functions constant on both magnetic field lines and streamlines.

For these symmetries, the anisotropy factor $\tau(\boldsymbol{r})$ is transformed as follows:

$$\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{{\boldsymbol{B_1}}^2} = \frac{1}{\mu} - n^2(\boldsymbol{r})\left(\frac{1}{\mu} - \tau\right). \tag{14}$$

The transformations for anisotropic case form an Abelian Lie group $G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ with sixteen connected components.

3 Lie group formalism for the MHD equilibrium equations

A solution to a system of l first-order partial differential equations

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) = 0,\\
&\boldsymbol{E} = (E^1, \ldots, E^l), \qquad \boldsymbol{x} = (x^1, \ldots, x^n) \in X, \qquad \boldsymbol{u} = (u^1, \ldots, u^m) \in U,\\
&\underset{1}{\boldsymbol{u}} = \left(\frac{\partial u^j}{\partial x^i} \Big| \, i = 1, \ldots, n;\ j = 1, \ldots, m \right) \in U_1
\end{aligned} \tag{15}$$

represents a manifold Ω in $(m+n)$-dimensional space $X \times U$, which corresponds to a manifold Ω^1 in $(m + n + mn)$-dimensional prolonged (jet) space $X \times U \times U_1$ of dependent and independent variables together with partial derivatives [13].

Studying ideal MHD equilibria, one should take into account the generally the plasma domain is spanned by nested 2-dimensional *magnetic surfaces* – surfaces on which magnetic field lines and plasma streamlines lie [16].

The Lie method of seeking one-parametric groups of transformations that map solutions of (15) into solutions consists in finding the Lie algebra of vector fields tangent to the solution manifold Ω^1 in the jet space. These vector fields serve as infinitesimal generators for a Lie symmetry group with representation

$$\begin{aligned}
(x')^i &= f^i(\boldsymbol{x}, \boldsymbol{u}, a) \qquad (i = 1, \ldots, n),\\
(u')^j &= g^j(\boldsymbol{x}, \boldsymbol{u}, a) \qquad (j = 1, \ldots, m),
\end{aligned} \tag{16}$$

and have the form

$$\boldsymbol{v} = \sum_i \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^k} + \sum_{i,k} \xi_i^k(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) \frac{\partial}{\partial u_i^k}. \tag{17}$$

Components of these tangent vector fields are expressed through the group representation as follows:

$$\begin{aligned}
&\xi^i(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial f^i(\boldsymbol{x}, \boldsymbol{u}, a)}{\partial a}\Big|_{a=0}, \qquad \eta^j(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial g^j(\boldsymbol{x}, \boldsymbol{u}, a)}{\partial a}\Big|_{a=0},\\
&i = 1, \ldots, n, \qquad j = 1, \ldots, m.
\end{aligned} \tag{18}$$

The variables ξ_i^k in (17) are the coordinates of the prolonged tangent vector field corresponding to the derivatives u_i^k:

$$\xi_i^j(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) = D_i \eta^j - \sum_{k=1}^n u_k^j D_i \xi^k, \qquad D_i \equiv \frac{\partial}{\partial x^i} + \sum_{j=1}^m u_i^j \frac{\partial}{\partial u^j}. \tag{19}$$

The relation (19) defines an isomorphism between tangent vector fields (17) and infinitesimal operators

$$X = \sum_i \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^k}. \tag{20}$$

The explicit reconstruction of the transformations (16) from a generator (17) is done by solving the initial value problem

$$\begin{aligned} &\frac{\partial f^i(a)}{\partial a} = \xi^i(\boldsymbol{f}, \boldsymbol{g}), \qquad && \frac{\partial g^k(a)}{\partial a} = \eta^k(\boldsymbol{f}, \boldsymbol{g}), \\ &f^i(0) = x^i, && g^k(0) = u^k. \end{aligned} \tag{21}$$

To find all Lie group generators admissible by the original system (15), one needs to solve the determining equations

$$\boldsymbol{v}\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}})\Big|_{\boldsymbol{E}(\boldsymbol{x},\boldsymbol{u},\underset{1}{\boldsymbol{u}})=0} = 0. \tag{22}$$

All l determining equations (22) are linear partial differential equations with respect to $m+n$ unknown functions (18) of $m+n$ variables.

4 Connection between Bogoyavlenskij symmetries and Lie transformations

In this section the equivalence is established between Bogoyavlenskij symmetries (11), (13) for isotropic and anisotropic cases and certain infinite-dimensional Lie group transformations.

Theorem 1 shows that the application of the Lie group formalism to MHD equilibria (1), (2) yields certain Lie point transformations, some of which depend on arbitrary functions.

Theorem 2 proves that these Lie point transformations are equivalent to the groups G_m of Bogoyavlenskij symmetries (11).

Theorems 3 and 4 contain similar statements for Anisotropic (CGL) plasma equilibria.

Theorem 1. *Consider the incompressible MHD equilibrium system of equations* (1)–(3)*, where the density* $\rho(\boldsymbol{r})$ *is constant on both magnetic field lines and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\boldsymbol{r})\left(\sum_{k=1}^{3} \frac{B_k}{\mu\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu}(\boldsymbol{V}\cdot\boldsymbol{B})\frac{\partial}{\partial P}\right), \tag{23}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial P}, \tag{24}$$

$$X^{(3)} = N(\boldsymbol{r})\left(2\rho\frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k}\right), \tag{25}$$

$$X^{(4)} = \frac{\partial}{\partial P}. \tag{26}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$*. Here* $M(\boldsymbol{r})$*,* $N(\boldsymbol{r})$ *are arbitrary smooth functions constant on both magnetic field lines and streamlines.*

The proof of Theorem 1 is given in [17]. All infinitesimal operators are obtained by direct differentiation of Bogoyavlenskij symmetries with respect to a properly chosen parameter.

Remark 1. Let us explicitly write down the transformations contained in the infinitesimal operators (23)–(26). According to the reconstruction procedure (21), for the operator (23), we have

$$\rho_1 = \rho, \qquad \boldsymbol{x}_1 = \boldsymbol{x},$$

and solve the linear initial value problem

$$\begin{aligned} &\frac{\partial \boldsymbol{V}_1}{\partial \tau} = \boldsymbol{B}_1 \frac{M(\boldsymbol{r})}{\mu\rho}, \qquad && \frac{\partial \boldsymbol{B}_1}{\partial \tau} = \boldsymbol{V}_1 M(\boldsymbol{r}), \qquad && \frac{\partial P_1}{\partial \tau} = -\frac{M(\boldsymbol{r})}{\mu}(\boldsymbol{V}_1 \cdot \boldsymbol{B}_1), \\ &\boldsymbol{V}_1(\tau = 0) = \boldsymbol{V}, && \boldsymbol{B}_1(\tau = 0) = \boldsymbol{B}, && P_1(\tau = 0) = P. \end{aligned} \tag{27}$$

The solution is

$$\begin{aligned} &\boldsymbol{B}_1 = \cosh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\boldsymbol{B} + \sinh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\sqrt{\mu\rho}\,\boldsymbol{V}, \\ &\boldsymbol{V}_1 = \sinh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\frac{\boldsymbol{B}}{\sqrt{\mu\rho}} + \cosh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\boldsymbol{V}, \\ &P_1 = P + (\boldsymbol{B}^2 - \boldsymbol{B}_1^2)/(2\mu), \qquad \rho_1 = \rho. \end{aligned} \tag{28}$$

The infinitesimal operator (23) thus contains the possibility of "mixing" the components of the vector fields $\boldsymbol{B}$ and $\boldsymbol{V}$ of the original solution into a new solution.

The same way by solving a corresponding initial value problem (21) we find that transformations contained in the operator (24) are scalings

$$\rho_1 = \rho, \qquad \boldsymbol{B}_1 = \exp(\tau)\boldsymbol{B}, \qquad \boldsymbol{V}_1 = \exp(\tau)\boldsymbol{V}, \qquad P_1 = \exp(2\tau)P; \tag{29}$$

the operator (25) corresponds to infinite-dimensional scalings

$$\rho_1 = \exp(2N(\boldsymbol{r})\tau)\rho, \qquad \boldsymbol{B}_1 = \boldsymbol{B}, \qquad \boldsymbol{V}_1 = \exp(-N(\boldsymbol{r})\tau)\boldsymbol{V}, \qquad P_1 = P; \tag{30}$$

the operator (26) – to translations

$$\rho_1 = \rho, \qquad \boldsymbol{B}_1 = \boldsymbol{B}, \qquad \boldsymbol{V}_1 = \boldsymbol{V}, \qquad P_1 = P + \tau. \tag{31}$$

Theorem 2. *Lie point transformations* (28)–(30) *are equivalent to the group G_m of Bogoyavlenskij transformations* (11).

The proof of Theorem 2 can also be found in [17].

Theorems 3, 4 establish connection of the infinite-dimensional symmetries of anisotropic plasmas with Lie symmetry method.

Theorem 3. *Consider the incompressible anisotropic CGL equilibrium system of equations* (9)–(10)*, where the density $\rho(\boldsymbol{r})$ and anisotropy factor $\tau(\boldsymbol{r})$ are constant on both magnetic field lines and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\boldsymbol{r})\left(\sum_{k=1}^{3} B_k \frac{1/\mu - \tau}{\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu}(\boldsymbol{V}\cdot\boldsymbol{B})\frac{\partial}{\partial p_\perp}\right), \tag{32}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial p_\perp}, \tag{33}$$

$$X^{(3)} = N(\boldsymbol{r})\left(2\rho\frac{\partial}{\partial\rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k}\right), \tag{34}$$

$$X^{(4)} = L(\boldsymbol{r})\left(2(1/\mu - \tau)\frac{\partial}{\partial\tau} - \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + \frac{B^2}{\mu}\frac{\partial}{\partial p_\perp}\right), \tag{35}$$

$$X^{(5)} = \frac{\partial}{\partial p_\perp}. \tag{36}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$. *Here* $L(\boldsymbol{r})$, $M(\boldsymbol{r})$, $N(\boldsymbol{r})$ *are arbitrary smooth functions constant on both magnetic field lines and streamlines.*

This theorem is proven exactly the same way as Theorem 1, by replacing the arbitrary functions of the transformations (13), (14) with exponents containing a parameter and differentiating by it.

The analog of Theorem 2 is also true for incompressible CGL plasma equilibria:

Theorem 4. *Lie point transformations defined by* (32)–(36) *are equivalent to the group* G *of Bogoyavlenskij transformations* (13), (14).

To prove this theorem, one needs to follow the sequence of steps of proof of Theorem 2, replacing MHD equilibrium infinitesimal operators by operators (32)–(36).

5 Summary

The isotropic and anisotropic plasma equilibrium equations (1), (2), (4), (5) in the incompressible case (3) possess infinite-dimensional families of intrinsic symmetries (11), (13), which are the richest known classes of transformations for these equations. These symmetries form Abelian groups $G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2$ and $G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ with eight and sixteen connected components respectively.

It is important that both of these groups of symmetries are implied by the Lie point transformations of these equations, and are thus obtainable from the standard procedure of Lie group analysis, which is applicable to any system of PDEs with sufficiently smooth coefficients.

The Lie procedure in application to MHD equilibria is described in Section 3. Every solution to a system of PDEs with n variables and m unknown functions represents a manifold Ω^1 in $(m+n+mn)$-dimensional jet space $X \times U \times U_1$ of independent and dependent variables $\boldsymbol{x}$, $\boldsymbol{u}$ and partial derivatives u_i^k (15). The Lie procedure consists in finding vector fields $\boldsymbol{v}$ (17) tangent to Ω^1. These vector fields serve as infinitesimal transformation group generators. Their components ξ^i, η^j (18) are functions of all independent and dependent variables. The equations (22) for determining the tangent vector field components are the conditions of invariance of the solution manifold Ω^1 under the action of $\boldsymbol{v}$.

To find the above Bogoyavlenskij symmetries for MHD and CGL equilibria, one must take into account the fact that in the general case ideal plasma domain is spanned by nested 2-dimensional magnetic surfaces – surfaces tangent to plasma velocity and magnetic field [16]. The arbitrary functions of Bogoyavlenskij symmetries are functions constant on such magnetic surfaces.

The operators (23)–(26) admissible by incompressible MHD equilibria form a basis of the Lie algebra of infinitesimal operators corresponding to the subgroup $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$ of the group (16) of all Lie point transformations.

The same statement is true for the operators (32)–(36) and incompressible MHD equilibrium equations.

Theorems 2 and 4 show that the transformations generated by infinitesimal operators from Theorems 1, 3 are equivalent to the groups G_m, G of Bogoyavlenskij symmetries for MHD and CGL equilibria respectively.

This result illustrates that the general Lie approach of analyzing systems of partial differential equations is capable of revealing highly non-trivial intrinsic transformations that may have great importance in applications, as is the case for Bogoyavlenskij symmetries.

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