

# Non-Lie Reductions of Nonlinear Reaction-Diffusion Systems with Variable Diffusivities

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Non-Lie reductions to systems of first-order ordinary differential equations are performed for a class of systems of two quasilinear reaction-diffusion equations having variable diffusivities. Moreover, families of exact solutions of a diffusive Lotka–Volterra type system are constructed.

## 1 Introduction

In this communication, semilinear systems of two reaction-diffusion (RD) equations of the form

$$\begin{aligned} U_t &= d_1(U^{\alpha_1}U_x)_x + U(a_1 + b_1U^{\alpha_1}) + U^{1-\alpha_1}(h_1 + c_1V^{\alpha_2}), \\ V_t &= d_2(V^{\alpha_2}V_x)_x + V(a_2 + b_2V^{\alpha_2}) + V^{1-\alpha_2}(h_2 + c_2U^{\alpha_1}) \end{aligned} \quad (1)$$

are considered. Here the subscripts  $t$  and  $x$  to the functions  $U$  and  $V$  denote differentiation with respect to these variables and all the coefficients are constants. The communication is organized as follows. In Section 2, a set of new non-Lie ansätze are constructed which reduce RD system (1) to systems of four or more ordinary differential equations. It is shown that these ansätze cannot be obtained using Lie symmetries of this system. In Section 3, families of exact solutions of the RD system

$$\begin{aligned} U_t &= (UU_x)_x + U(a_1 + b_1U) + h_1 + c_1V, \\ V_t &= (VV_x)_x + V(a_2 + b_2V) + h_2 + c_2U. \end{aligned} \quad (2)$$

One can note that system (2) is a system of Lotka–Volterra type, with variable (degenerate) diffusivities (see, e.g. [1]), in which the standard terms  $c_1UV$  and  $c_2UV$  are replaced by the terms  $h_1 + c_1V$  and  $h_2 + c_2U$ , respectively. Hereafter this RD system will be referred as the degenerate diffusive Lotka–Volterra system (DDLV system).

## 2 New non-Lie ansätze for the RD system (1)

This section is devoted to non-Lie reductions of RD systems with power-law nonlinearities of the form (1). Hereafter I assume that *both* equations contain diffusion coefficients with variable diffusivities and that the system is coupled, i.e.,  $d_1d_2\alpha_1\alpha_2 \neq 0$ , and  $c_1 \neq 0$  or  $c_2 \neq 0$ .

It is easily checked that the linear substitution

$$U = d_1^{-1/\alpha_1}U^*, \quad V = d_2^{-1/\alpha_2}V^* \quad (3)$$

reduces (1) to the same form with  $d_1 = d_2 = 1$ ; therefore, without losing generality, I can consider DDLV systems of the form

$$\begin{aligned} U_t &= (U^{\alpha_1}U_x)_x + U(a_1 + b_1U^{\alpha_1}) + U^{1-\alpha_1}(h_1 + c_1V^{\alpha_2}), \\ V_t &= (V^{\alpha_2}V_x)_x + V(a_2 + b_2V^{\alpha_2}) + V^{1-\alpha_2}(h_2 + c_2U^{\alpha_1}). \end{aligned} \quad (4)$$

According to the results of group classification for systems of two reaction-diffusion equations with variable diffusivities [2,3], system (4) is invariant only with respect to the trivial Lie algebra with the basic operators

$$P_t = \partial_t, \quad P_x = \partial_x \tag{5}$$

if the coefficients are arbitrary constants. However, there are the following special cases leading to non-trivial algebras of invariance:  $b_1 = b_2 = h_1 = h_2 = 0$  (see case 6 of Table 1 in [3]);  $b_1 = b_2 = h_1 = h_2 = 0$  and  $\alpha_1 = \alpha_2 = -4/3$  (see case 9 of Table 1 in [3]);  $b_1 = b_2 \neq 0$ ,  $h_1 = h_2 = 0$  and  $\alpha_1 = \alpha_2 = -4/3$  (see cases 10 and 12 of Table 1 in [3]). In particular, the DDLV system (4) admits only the trivial Lie algebra (5) if  $h_1 \neq 0$  or  $h_2 \neq 0$ .

With  $U = u^{1/\alpha_1}$ ,  $V = v^{1/\alpha_2}$  this system is reduced to the form

$$\begin{aligned} u_t &= uu_{xx} + \frac{1}{\alpha_1}u_x^2 + \alpha_1 [h_1 + a_1u + b_1u^2 + c_1v], \\ v_t &= vv_{xx} + \frac{1}{\alpha_2}v_x^2 + \alpha_2 [h_2 + a_2v + b_2v^2 + c_2u], \end{aligned} \tag{6}$$

where  $u(t, x)$  and  $v(t, x)$  are new unknown functions. One observes that system (6) contains only quadratic nonlinearities. Several new approaches were recently suggested to find exact solutions of single evolution equation with quadratic nonlinearities (see [6–10] and references cited therein). Those approaches lead to so called non-Lie ansätze which cannot be found using the classical Lie method and the non-classical Bluman–Cole method [11] (it was, however, shown in [12] that some solutions constructed by those ansätze can be found by a modification of the Bluman–Cole method).

However, it is not easy to apply those approaches for finding exact solutions of systems of equations. To my knowledge there are only a few papers devoted to the case of evolution systems [13, 10, 14].

It turns out that it is possible to construct non-Lie ansätze and to interpret the relevant exact solutions of the system (6) using the notion of additional generating conditions [10, 15].

Consider an additional generating condition of the following non-coupled system of third order ordinary differential equations (ODEs)

$$\beta_1(t)\frac{du}{dx} + \beta_2(t)\frac{d^2u}{dx^2} + \frac{d^3u}{dx^3} = 0, \quad \beta_1(t)\frac{dv}{dx} + \beta_2(t)\frac{d^2v}{dx^2} + \frac{d^3v}{dx^3} = 0, \tag{7}$$

where  $\beta_1(t)$  and  $\beta_2(t)$  are arbitrary smooth functions and the variable  $t$  is considered as a parameter. Depending on the coefficients, the solution to the linear ODE system (7) can take the following forms:

$$u = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2, \quad v = \psi_0(t) + \psi_1(t)x + \psi_2(t)x^2 \tag{8}$$

if  $\beta_1 = \beta_2 = 0$ ;

$$u = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x), \quad v = \psi_0(t) + \psi_1(t)x + \psi_2(t)\exp(\gamma(t)x) \tag{9}$$

if  $\beta_1 = 0$ ;

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x), \\ v &= \psi_0(t) + \psi_1(t)\exp(\gamma_1(t)x) + \psi_2(t)\exp(\gamma_2(t)x) \end{aligned} \tag{10}$$

if  $\gamma_{1,2}(t) = \frac{1}{2}(\pm\sqrt{D} - \beta_2)$ ,  $D = \beta_2^2 - 4\beta_1 > 0$  and  $\gamma_1 \neq \gamma_2$ ;

$$u = \varphi_0(t) + \exp\left(-\frac{\beta_2x}{2}\right) \left[ \varphi_1(t)\cos\left(\frac{\sqrt{-D}}{2}x\right) + \varphi_2(t)\sin\left(\frac{\sqrt{-D}}{2}x\right) \right],$$

$$v = \psi_0(t) + \exp\left(-\frac{\beta_2 x}{2}\right) \left[ \psi_1(t) \cos\left(\frac{\sqrt{-D}}{2}x\right) + \psi_2(t) \sin\left(\frac{\sqrt{-D}}{2}x\right) \right] \quad (11)$$

if  $D < 0$ ; and, finally,

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t) \exp(\gamma(t)x) + x\varphi_2(t) \exp(\gamma(t)x), \\ v &= \psi_0(t) + \psi_1(t) \exp(\gamma(t)x) + x\psi_2(t) \exp(\gamma(t)x) \end{aligned} \quad (12)$$

if  $D = 0$ , i.e.  $\gamma_1 = \gamma_2 = \gamma \neq 0$ .

Let me consider relations (8)–(12) as a chain of ansätze for the system (6). It is important to note that each ansatz contains 6 yet-to-be determined functions  $\varphi_i$ ,  $\psi_i$ ,  $i = 0, 1, 2$ . This enables us to reduce the given system of PDEs to a nonlinear system of first-order ODEs for the unknown functions  $\varphi_i$  and  $\psi_i$  ( $\gamma$ ,  $\beta_1$  and  $\beta_2$  can provide additional unknowns, depending on the ansatz in question). Indeed, calculating the derivatives  $U_t$ ,  $U_x$ ,  $U_{xx}$  with the help of ansatz (8) and substituting them into system (6), one obtains an expression, which can be splitted into separate parts for the powers of  $x$ . Simultaneously the restriction on coefficients

$$b_1 = b_2 = 0 \quad (13)$$

is required (otherwise  $\varphi_1 = \psi_1 = \varphi_2 = \psi_2 = 0$ ) and then a correctly-specified ODE system for the functions  $\varphi_i$  and  $\psi_i$  has the form

$$\begin{aligned} \dot{\varphi}_0 &= 2\varphi_0\varphi_2 + \frac{1}{\alpha_1}\varphi_1^2 + \alpha_1(h_1 + a_1\varphi_0 + c_1\psi_0), \\ \dot{\psi}_0 &= 2\psi_0\psi_2 + \frac{1}{\alpha_2}\psi_1^2 + \alpha_2(h_2 + a_2\psi_0 + c_2\varphi_0), \\ \dot{\varphi}_1 &= \frac{2\alpha_1 + 4}{\alpha_1}\varphi_1\varphi_2 + \alpha_1(a_1\varphi_1 + c_1\psi_1), \\ \dot{\psi}_1 &= \frac{2\alpha_2 + 4}{\alpha_2}\psi_1\psi_2 + \alpha_2(a_2\psi_1 + c_2\varphi_1), \\ \dot{\varphi}_2 &= \frac{2\alpha_1 + 4}{\alpha_1}\varphi_2^2 + \alpha_1(a_1\varphi_2 + c_1\psi_2), \\ \dot{\psi}_2 &= \frac{2\alpha_2 + 4}{\alpha_2}\psi_2^2 + \alpha_2(a_2\psi_2 + c_2\varphi_2). \end{aligned} \quad (14)$$

Finally, given any solution of the ODE system (14), one constructs an exact solution of DDLV system (4), (13) of the form

$$U = (\varphi_0 + \varphi_1 x + \varphi_2 x^2)^{1/\alpha_1}, \quad V = (\psi_0 + \psi_1 x + \psi_2 x^2)^{1/\alpha_2}. \quad (15)$$

In the case of ansatz (9), two different subcases are obtained, namely: (i)  $\alpha_1 \neq -1$  and  $\alpha_2 \neq -1$ ; (ii)  $\alpha_1 = \alpha_2 = -1$ .

The first subcase leads to the requirement  $\varphi_1 = \psi_1 = 0$  and then the ansatz reduces to a particular case of the ansatz (10) that will be considered below. Consider the second subcase. In a similar way to the above calculation for the ansatz (8), I arrive at the coefficient restrictions (13) and also  $a_1 + c_1 = a_2 + c_2$ , together with the following correctly-specified ODE system for the functions  $\gamma$ ,  $\varphi_i$  and  $\psi_i$

$$\begin{aligned} \dot{\gamma} &= \varphi_1 \gamma^2, & \psi_1 &= \varphi_1, & \dot{\varphi}_1 &= -(a_1 + c_1)\varphi_1, \\ \dot{\varphi}_0 &= -(h_1 + a_1\varphi_0 + c_1\psi_0 + \varphi_1^2), \\ \dot{\psi}_0 &= -(h_2 + a_2\psi_0 + c_2\varphi_0 + \psi_1^2), \\ \dot{\varphi}_2 &= \gamma^2\varphi_0\varphi_2 - 2\gamma\varphi_1\varphi_2 - (a_1\varphi_2 + c_1\psi_2), \end{aligned}$$

$$\dot{\psi}_2 = \gamma^2 \psi_0 \psi_2 - 2\gamma \psi_1 \psi_2 - (a_2 \psi_2 + c_2 \varphi_2). \quad (16)$$

Thus, any solution of (16) generates an exact solution of the form

$$U = (\varphi_0 + \varphi_1 x + \varphi_2 \exp \gamma x)^{-1}, \quad V = (\psi_0 + \psi_1 x + \psi_2 \exp \gamma x)^{-1} \quad (17)$$

of the DDLV system

$$\begin{aligned} U_t &= (U^{-1} U_x)_x + U(a_1 + h_1 U + c_1 U/V), \\ V_t &= (V^{-1} V_x)_x + V(a_2 + h_2 V + c_2 V/U), \end{aligned} \quad (18)$$

where  $a_1 + c_1 = a_2 + c_2$ .

Analogously, the ansätze (10) and (11) have been analysed and the two different subcases listed above were found too. Considering the subcase (i), I have obtained the restriction on coefficients

$$\gamma_1 = -\gamma_2 = \sqrt{-\frac{b_1 \alpha_1^2}{\alpha_1 + 1}} = \sqrt{-\frac{b_2 \alpha_2^2}{\alpha_2 + 1}} \equiv \gamma \quad (19)$$

for both ansätze. To find real solutions ansatz (10) can be applied only for  $\frac{b_k}{\alpha_k+1} < 0$ ,  $k = 1, 2$  while ansatz (11) works only for  $\frac{b_k}{\alpha_k+1} > 0$ ,  $k = 1, 2$ . The relevant ODE systems for finding the functions  $\varphi_i$  and  $\psi_i$  have the form

$$\begin{aligned} \dot{\varphi}_0 &= \alpha_1 b_1 \varphi_0^2 + \alpha_1 (h_1 + a_1 \varphi_0 + c_1 \psi_0) - \frac{4}{\alpha_1} \gamma^2 \varphi_1 \varphi_2, \\ \dot{\psi}_0 &= \alpha_2 b_2 \psi_0^2 + \alpha_2 (h_2 + a_2 \psi_0 + c_2 \varphi_0) - \frac{4}{\alpha_2} \gamma^2 \psi_1 \psi_2, \\ \dot{\varphi}_1 &= (2\alpha_1 b_1 + \gamma^2) \varphi_0 \varphi_1 + \alpha_1 (a_1 \varphi_1 + c_1 \psi_1), \\ \dot{\psi}_1 &= (2\alpha_2 b_2 + \gamma^2) \psi_0 \psi_1 + \alpha_2 (a_2 \psi_1 + c_2 \varphi_1), \\ \dot{\varphi}_2 &= (2\alpha_1 b_1 + \gamma^2) \varphi_0 \varphi_2 + \alpha_1 (a_1 \varphi_2 + c_1 \psi_2), \\ \dot{\psi}_2 &= (2\alpha_2 b_2 + \gamma^2) \psi_0 \psi_2 + \alpha_2 (a_2 \psi_2 + c_2 \varphi_2) \end{aligned} \quad (20)$$

and

$$\begin{aligned} \dot{\varphi}_0 &= \alpha_1 b_1 \varphi_0^2 + \alpha_1 (h_1 + a_1 \varphi_0 + c_1 \psi_0) - \frac{\gamma^2}{\alpha_1} (\varphi_1^2 + \varphi_2^2), \\ \dot{\psi}_0 &= \alpha_2 b_2 \psi_0^2 + \alpha_2 (h_2 + a_2 \psi_0 + c_2 \varphi_0) - \frac{\gamma^2}{\alpha_2} (\psi_1^2 + \psi_2^2), \\ \dot{\varphi}_1 &= (2b_1 \alpha_1 + \gamma^2) \varphi_0 \varphi_1 + \alpha_1 (a_1 \varphi_1 + c_1 \psi_1), \\ \dot{\psi}_1 &= (2b_2 \alpha_2 + \gamma^2) \psi_0 \psi_1 + \alpha_2 (a_2 \psi_1 + c_2 \varphi_1), \\ \dot{\varphi}_2 &= (2b_1 \alpha_1 + \gamma^2) \varphi_0 \varphi_2 + \alpha_1 (a_1 \varphi_2 + c_1 \psi_2), \\ \dot{\psi}_2 &= (2b_2 \alpha_2 + \gamma^2) \psi_0 \psi_2 + \alpha_2 (a_2 \psi_2 + c_2 \varphi_2) \end{aligned} \quad (21)$$

respectively. The resulting exact solutions of the DDLV system (4) take the form

$$\begin{aligned} U &= [\varphi_0(t) + \varphi_1(t) \exp(-\gamma x) + \varphi_2(t) \exp(\gamma x)]^{1/\alpha_1}, \\ V &= [\psi_0(t) + \psi_1(t) \exp(-\gamma x) + \psi_2(t) \exp(\gamma x)]^{1/\alpha_2} \end{aligned} \quad (22)$$

and

$$U = [\varphi_0(t) + \varphi_1(t) \cos(|\gamma|x) + \varphi_2(t) \sin(|\gamma|x)]^{1/\alpha_1},$$

$$V = [\psi_0(t) + \psi_1(t) \cos(|\gamma|x) + \psi_2(t) \sin(|\gamma|x)]^{1/\alpha_2}. \quad (23)$$

In the subcase (ii), the restrictions

$$\gamma_1 = -\gamma_2 = \gamma \in \mathbb{R}, \quad b_1 = b_2 = 0 \quad (24)$$

arise instead of (19). The corresponding systems of ODEs coincide with (20) and (21), in which one needs to set  $b_k = 0$ ,  $\alpha_k = -1$ ,  $k = 1, 2$ .

Finally, I have established that ansatz (12) can be applied only in the case  $\varphi_2 = \psi_2 = 0$ , i.e. if one is reduced to a particular case of (10).

I have also investigated the direct generalization of (8) of the form

$$u = \varphi_0 + \varphi_1 x + \cdots + \varphi_{m_1} x^{m_1}, \quad v = \psi_0 + \psi_1 x + \cdots + \psi_{m_2} x^{m_2}. \quad (25)$$

It can be shown that ansatz (25) reduces system (6) to a correctly-specified ODE system only in the cases  $m_1 = m_2 = 3$  and  $m_1 = m_2 = 4$  (otherwise  $\varphi_i = \psi_i = 0$ ,  $i > 2$ ). The corresponding restrictions on the coefficients are as follows:  $\alpha_1 = \alpha_2 = -\frac{3}{2}$ ,  $b_1 = b_2 = 0$  for  $m_1 = m_2 = 3$  and  $\alpha_1 = \alpha_2 = -\frac{4}{3}$ ,  $b_1 = b_2 = 0$  for  $m_1 = m_2 = 4$ . The relevant ODE systems are omitted here but one can find them in [4].

It should be noted that both exponents of power nonlinearities  $\alpha = -\frac{3}{2}$  and  $\alpha = -\frac{4}{3}$  were earlier found (see [16, 17] and [9]) to provide reductions of the single RD equation

$$U_t = (U^\alpha U_x)_x + a_1 U + c_1 U^{1-\alpha}. \quad (26)$$

### 3 Exact solutions of the DDLV system (2)

Consider system (2) and assume  $b_1 b_2 \neq 0$  and  $c_1 c_2 \neq 0$ , i.e. the system contains quadratic nonlinearities in the reaction terms and the equations are coupled. Taking into account [2, 3], one easily observes that under these typical restrictions on coefficients, the DDLV system (2) is invariant only with respect to the trivial algebra (5). So, one can find in that manner only plane wave solutions of the form

$$U = \varphi(\omega), \quad V = \psi(\omega), \quad \omega = k_1 x - k_0 t, \quad (27)$$

where  $k_0, k_1 \in \mathbb{R}$  and the functions  $\varphi$  and  $\psi$  are solutions of the reduced ODE system

$$\begin{aligned} k_1^2 (\varphi \varphi_\omega)_\omega + k_0 \varphi_\omega + \varphi (a_1 + b_1 \varphi) + h_1 + c_1 \psi &= 0, \\ k_1^2 (\psi \psi_\omega)_\omega + k_0 \psi_\omega + \psi (a_2 + b_2 \psi) + h_2 + c_2 \varphi &= 0. \end{aligned} \quad (28)$$

The ODE system (28) is not integrable for  $k_1 \neq 0$ , and only particular solutions can be found (some of them are presented below). The case  $k_1 = 0$  of course leads to solutions which do not depend on the space variable.

It turns out that a much wider class of exact solutions of the DDLV system (2) can be constructed using the non-Lie reductions presented in Section 3. Consider the ODE system (20) with  $\alpha_1 = \alpha_2 = 1$ . As it was shown above, this system is obtained from system (2) using the ansatz

$$\begin{aligned} U &= \varphi_0(t) + \varphi_1(t) \exp(-\gamma x) + \varphi_2(t) \exp(\gamma x), \\ V &= \psi_0(t) + \psi_1(t) \exp(-\gamma x) + \psi_2(t) \exp(\gamma x), \end{aligned} \quad (29)$$

where  $\gamma = \sqrt{\frac{b}{2}}$ ,  $b = -b_1 = -b_2 > 0$ .

It can be noted that the ODE system (20) admits essential simplification for  $\varphi_2 = \psi_2 = 0$ , reducing to

$$\begin{aligned} \dot{\varphi}_0 &= -b\varphi_0^2 + h_1 + a_1\varphi_0 + c_1\psi_0, & \dot{\psi}_0 &= -b\psi_0^2 + h_2 + a_2\psi_0 + c_2\varphi_0, \\ \dot{\varphi}_1 &= -\frac{3}{2}b\varphi_0\varphi_1 + a_1\varphi_1 + c_1\psi_1, & \dot{\psi}_1 &= -\frac{3}{2}b\psi_0\psi_1 + a_2\psi_1 + c_2\varphi_1. \end{aligned} \quad (30)$$

Now one observes that the first two equations of (30) decouple. Formally speaking, this subsystem has four steady-state solutions. I assume that there is at least one real solution among them, say  $(U_0, V_0)$ . In other words,  $(U_0, V_0)$  is a real solution of the system of algebraic equations

$$bU_0^2 = h_1 + a_1U_0 + c_1V_0, \quad bV_0^2 = h_2 + a_2V_0 + c_2U_0. \quad (31)$$

Substituting  $\varphi_0 = U_0, \psi_0 = V_0$  into the last two equation of (30), one arrives at a linear ODE system

$$\dot{\varphi}_1 = \left(a_1 - \frac{3}{2}bU_0\right)\varphi_1 + c_1\psi_1, \quad \dot{\psi}_1 = c_2\varphi_1 + \left(a_2 - \frac{3}{2}bV_0\right)\psi_1. \quad (32)$$

According to the classical theory of linear ODE systems, the form of the general solution of (32) depends on  $\Delta_1 = [(a_1 - a_2) + \frac{3}{2}b(V_0 - U_0)]^2 + 4c_1c_2$ . So I obtain the following general solutions of (32):

if  $\Delta_1 > 0$  then

$$\begin{aligned} \varphi_1 &= e_1c_1 \exp(s_1t) + e_2 \left(s_2 - a_2 + \frac{3}{2}bV_0\right) \exp(s_2t), \\ \psi_1 &= e_1 \left(s_1 - a_1 + \frac{3}{2}bU_0\right) \exp(s_1t) + e_2c_2 \exp(s_2t), \end{aligned} \quad (33)$$

where  $s_{1,2} = \frac{1}{2} [a_1 + a_2 - \frac{3}{2}b(U_0 + V_0) \pm \sqrt{\Delta_1}]$ ;

if  $\Delta_1 < 0$  then

$$\begin{aligned} \varphi_1 &= c_1 \exp(pt)(e_1 \sin(qt) + e_2 \cos(qt)), \\ \psi_1 &= \exp(pt) \left[ \left( e_1 \left( p - a_1 + \frac{3}{2}bU_0 \right) - e_2q \right) \sin(qt) \right. \\ &\quad \left. + \left( e_1q + e_2 \left( p - a_1 + \frac{3}{2}bU_0 \right) \right) \cos(qt) \right], \end{aligned} \quad (34)$$

where  $p = \frac{1}{2} (a_1 + a_2 - \frac{3}{2}b(U_0 + V_0))$ ,  $q = \frac{1}{2}\sqrt{-\Delta_1}$ ;

if  $\Delta_1 = 0$  then

$$\begin{aligned} \varphi_1 &= (e_1c_1t + c_2) \exp(st), \\ \psi_1 &= \left[ e_1(1 \pm \sqrt{-c_1c_2t}) \pm e_2\sqrt{\frac{-c_2}{c_1}} \right] \exp(st)m, \end{aligned} \quad (35)$$

where  $s = \frac{1}{2} [a_1 + a_2 - \frac{3}{2}b(U_0 + V_0)]$ . Here  $e_1$  and  $e_2$  are arbitrary constants.

Thus, three two-parameter families of exact solutions of the DDLV system

$$\begin{aligned} U_t &= (UU_x)_x + U(a_1 - bU) + h_1 + c_1V, \\ V_t &= (VV_x)_x + V(a_2 - bV) + h_2 + c_2U \end{aligned} \quad (36)$$

of the form

$$U = U_0 + \varphi_1(t) \exp\left(-\sqrt{\frac{b}{2}}x\right), \quad V = V_0 + \psi_1(t) \exp\left(-\sqrt{\frac{b}{2}}x\right) \quad (37)$$

can be constructed. Of course, each family arises only if the coefficients of the DDLV system (36) satisfy the relevant conditions on  $\Delta_1$ .

It should be stressed that any exact solution (37) with  $\varphi_1(t)$  and  $\psi_1(t)$  given by (33)–(35) is a non-Lie solution of the DDLV system (36) if  $e_1 e_2 \neq 0$ . In fact, it is easily seen that this solution cannot be reduced to the form (27), i.e. it cannot be obtained using the Lie symmetries of the DDLV system (36). However, one observes that all solutions obtained, except the solution generated by formulas (37) and (34), are reduced to the form (27) if  $e_1 = 0$ . In other words, these two-parameter families of solutions contain as subclasses one-parameter families of Lie solutions.

Some other families of exact solutions of DDLV (2) are constructed in [4].

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- [1] Murray J.D., *Mathematical biology*, Berlin, Springer, 1989.
- [2] Knyazeva I.V. and Popov M.D., A system of two diffusion equations, in *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 1, Editor N.H. Ibragimov, Boca Raton, CRC Press, 1994, 171–176.
- [3] Cherniha R.M. and King J.R., Lie symmetries of nonlinear reaction-diffusion systems with variable diffusivities, in *Group 24 (15–20 July, 2002, Paris)*, *Institute of Physics Conference Series*, Vol. 173, 2003, 717–720.
- [4] Cherniha R.M. and King J.R., Nonlinear reaction-diffusion systems with variable diffusivities: Lie symmetries, ansätze and exact solutions, submitted.
- [5] Dorodnitsyn V.A., On invariant solutions of non-linear heat conduction with a source, *USSR Comput. Math. and Math. Phys.*, 1982, V.22, 115–122.
- [6] Olver P., Direct reduction and differential constraints, *Proc. R. Soc. A*, 1994, V.46, 509–523.
- [7] Fokas A.S. and Liu Q.M., Nonlinear interaction of travelling waves of nonintegrable equations, *Phys. Rev. Lett.*, 1994, V.72, 3293–3296.
- [8] Zhdanov R., Conditional Lie–Bäcklund symmetry and reduction of evolution equations, *J. Phys. A: Math. Gen.*, 1995, V.28, 3841–3850.
- [9] Galaktionov V.A., Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities, *Proc. Royal Society of Edinburgh A*, 1995, V.125, 225–246.
- [10] Cherniha R., A constructive method for construction of new exact solutions of nonlinear evolution equations, *Rep. Math. Phys.*, 1996, V.38, 301–312.
- [11] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042.
- [12] Nucci M.C., Iterations of the non-classical symmetries method and conditional Lie–Bäcklund symmetries, *J. Phys. A: Math. Gen.*, 1996, V.29, 8117–8122.
- [13] King J.R., Mathematical analysis of a model for substitutional diffusion, *Proc. R. Soc. Lond. A*, 1990, V.430, 377–404.
- [14] Cherniha R. and Fehribach J., New exact solutions for a free boundary system, *J. Phys. A: Math. Gen.*, 1998, V.31, 3815–3829.
- [15] Cherniha R., New non-Lie ansätze and exact solutions of nonlinear reaction-diffusion-convection equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 8179–8198.
- [16] King J.R., Exact multidimensional solutions to some nonlinear diffusion equations, *Q. J. Mech. Appl. Math.*, 1993, V.46, 419–436.
- [17] Fushchych W. and Zhdanov R., Antireduction and exact solutions of nonlinear heat equations, *J. Nonlinear Math. Phys.*, 1994, V.1, 60–64.