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On 2 + 2 Locally Compact Quantum Groups

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We construct new examples of cocycle bicrossed products of Lie groups by using a relation between 3-cocycles on a Lie algebra and the cocycle-matched pair of the Lie algebras.

1 Introduction

A method for constructing examples of non-commutative and non-cocommutative examples of finite quantum groups, that is Kac algebras, was proposed by G.I. Kac in [1]. This method consists in extending a finite group G_1 with a dual to a finite group G_2. Each such an extension can be given by a group G that contains G_1 and G_2 as subgroups subject to the conditions $G_1 \cap G_2 = \{e\}$ and $G_1 G_2 = G$, called a matched pair of groups, and a pair of 2-cocycles. This method, now known as a cocycle bicrossed product construction, was generalized to bialgebras in [2], on the one hand, and to locally compact quantum groups [3] in [4], on the other hand. The cohomology theory for cocycle bicrossed products was developed in [5].

In the case where G_1 and G_2 are Lie groups, the corresponding notion is that of a cocycle matched pair of Lie algebras [2]. A cohomology theory for cocycle matched pairs of Lie algebras was developed in [6].

In terms of cocycle matched pairs of Lie algebras, an extensive study of low-dimensional locally compact quantum groups has been carried out in [7] for the cocycle bicrossed products. In particular, a complete classification of all locally compact quantum groups that arise from the bicrossed product construction starting with 2- and 1-dimensional Lie algebras is given there. It turned out that one of the cocycles in the constructed cocycle matched pair of groups was equivalent to a trivial one.

The purpose of this article is to construct nontrivial examples of cocycle bicrossed products, in particular, the ones where both cocycles are not equivalent to the trivial ones. We do it by considering two 2-dimensional Lie groups, find the cocycles on the level of the Lie algebras, and then lift them to the matched pairs of the Lie groups. In Section 2 we recall necessary definitions and known facts. We also give results needed for easy calculation of cocycles for matched pairs of Lie algebras. In Section 3 we construct three examples of cocycle bicrossed products. In the first example, we consider $G = \mathbb{R}^2$ and $H = \mathbb{R}^2$ with trivial actions, the second example deals with $G = \mathbb{R} \times \mathbb{R}_+$ and $H = \mathbb{R}_+ \times \mathbb{R}$ with a nontrivial action. In the third example, we take G to be the "$ax + b$" group and $H = \mathbb{R}^2$ with a nontrivial action of G on H.

2 General definitions and properties

Definition 1 ([8]). Let K be a locally compact group, G, H subgroups of K satisfying the conditions

$$G \cdot H = K, \qquad \text{and} \qquad G \cap H = \{e\}. \tag{1}$$

Then (G, H) is called a matched pair of locally compact groups.

Remark 1. For a more general definition of a matched pair of locally compact groups, see [4].

Everywhere in the sequel, elements of G and H are denoted by g, g_1, etc., h, h_1, etc., correspondingly.

Let (G, H) be a matched pair of locally compact groups. It is known [4] that there are right and left actions, $\triangleleft : H \times G \to H$ and $\triangleright : H \times G \to G$, given by

$$h \cdot g = (h \triangleright g) \cdot (h \triangleleft g) \tag{2}$$

and satisfying

$$\begin{aligned} &(h_1 h_2) \triangleleft g = (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g), \\ &h \triangleright (g_1 g_2) = (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2). \end{aligned} \tag{3}$$

Definition 2 ([4]). Let (G, H) be a matched pair of locally compact groups, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A pair of measurable maps (u, v), $u : H \times G \times G \to \mathbb{T}$, $v : H \times H \times G \to \mathbb{T}$ is called a pair of cocycles for the pair (G, H) if

$$u(h \triangleleft g_1; g_2, g_3) u(h; g_1, g_2 g_3) = u(h; g_1, g_2) u(h; g_1 g_2, g_3), \tag{4}$$

$$v(h_1, h_2; h_3 \triangleright g) v(h_1 h_2, h_3; g) = v(h_1, h_2 h_3; g) v(h_2, h_3; g), \tag{5}$$

$$\begin{aligned} v(h_1, h_2; g_1 g_2) u(h_1 h_2; g_1, g_2) &= v(h_1, h_2; g_1) u(h_2; g_1, g_2) \cdot v(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1; g_2) \\ &\quad \cdot u(h_1; h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2). \end{aligned} \tag{6}$$

Definition 3 ([4]). Two pairs of cocycles (u_1, v_1) and (u_2, v_2) for (G, H) are called equivalent if there exists a measurable function $r : H \times G \to \mathbb{T}$ such that

$$\begin{aligned} &u_1(h; g_1, g_2) u_2(h; g_1, g_2)^{-1} = r(h; g_1) r(h \triangleleft g_1; g_2) r(h; g_1 g_2)^{-1}, \\ &v_1(h_1, h_2; g) v_2(h_1, h_2; g)^{-1} = r(h_1 h_2; g) r(h_1; h_2 \triangleright g)^{-1} r(h_2; g)^{-1}. \end{aligned} \tag{7}$$

Let now G and H be Lie groups and $\mathfrak{g}$, $\mathfrak{h}$ be the corresponding Lie algebras. For Lie algebras there is a corresponding notion of a matched pair.

Definition 4 ([2]). A pair $(\mathfrak{g}, \mathfrak{h})$ is called a matched pair of Lie algebras if there are left and right actions $\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\triangleleft : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}$ satisfying

$$\begin{aligned} &X \triangleright [A_1, A_2] = [X \triangleright A_1, A_2] + [A_1, X \triangleright A_2] + (X \triangleleft A_1) \triangleright A_2 - (X \triangleleft A_2) \triangleright A_1, \\ &[X_1, X_2] \triangleleft A = [X_1 \triangleleft A, X_2] + [X_1, X_2 \triangleleft A] + X_1 \triangleleft (X_2 \triangleright A) - X_2 \triangleleft (X_1 \triangleright A), \end{aligned} \tag{8}$$

where $A, A_1, A_2 \in \mathfrak{g}$ and $X, X_1, X_2 \in \mathfrak{h}$.

Definition 5 ([2]). For a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, a pair of linear mappings (U, V), $U : \mathfrak{h} \otimes (\mathfrak{g} \wedge \mathfrak{g}) \to \mathbb{R}$ and $V : (\mathfrak{h} \wedge \mathfrak{h}) \otimes \mathfrak{g} \to \mathbb{R}$ is called a pair of cocycles for $(\mathfrak{g}, \mathfrak{h})$ if

$$\begin{aligned} &U(X \triangleleft A_1; A_2, A_3) + U(X \triangleleft A_2; A_3, A_1) + U(X \triangleleft A_3; A_1, A_2) \\ &\qquad = -U(X; A_1, [A_2, A_3]) - U(X; A_2, [A_3, A_1]) - U(X; A_3, [A_1, A_2]), \end{aligned} \tag{9}$$

$$\begin{aligned} &V(X_1, X_2; X_3 \triangleright A) + V(X_2, X_3; X_1 \triangleright A) + V(X_3, X_1; X_2 \triangleright A) \\ &\qquad = V(X_1, [X_2, X_3]; A) + V(X_2, [X_3, X_1]; A) + V(X_3, [X_1, X_2]; A), \end{aligned} \tag{10}$$

$$\begin{aligned} &U([X_1, X_2]; A_1, A_2) + V(X_1, X_2; [A_1, A_2]) \\ &\qquad = U(X_1; X_2 \triangleright A_1, A_2) + U(X_1; A_1, X_2 \triangleright A_2) - X_1 \leftrightarrow X_2 \\ &\qquad\quad + V(X_1, X_2 \triangleleft A_1; A_2) + V(X_1 \triangleleft A_1, X_2; A_2) - A_1 \leftrightarrow A_2, \end{aligned} \tag{11}$$

where $X_1 \leftrightarrow X_2$ and $A_1 \leftrightarrow A_2$ mean that the corresponding variables are interchanged in the preceding expression.

Definition 6 ([2]). Two pairs of cocycles (U_1, V_1) and (U_2, V_2) for a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ are called equivalent if there is a linear map $R : \mathfrak{h} \otimes \mathfrak{g} \to \mathbb{R}$ such that, for all $A, A_1, A_2 \in \mathfrak{g}$ and $X, X_1, X_2 \in \mathfrak{h}$,

$$\begin{aligned} &U_1(X; A_1, A_2) - U_2(X; A_1, A_2) = R(X \triangleleft A_1; A_2) - R(X \triangleleft A_2; A_1) - R(X; [A_1, A_2]), \\ &V_2(X_1, X_2; A) - V_1(X_1, X_2; A) = R(X_1; X_2 \triangleright A) - R(X_2; X_1 \triangleright A) - R([X_1, X_2]; A). \end{aligned} \tag{12}$$

Let $(\mathfrak{g}, \mathfrak{h})$ be a matched pair of Lie algebras. Then it is known [2] that the vector space $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$ can be endowed with a Lie algebra structure defined by

$$[X, A] = X \triangleright A + X \triangleleft A \tag{13}$$

for $A \in \mathfrak{g}$, $X \in \mathfrak{h}$, and extending it to $\mathfrak{k}$ by linearity.

Let now $(\mathfrak{g}, \mathfrak{h})$ be a matched pair of Lie algebras and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$ be the Lie algebra with the bracket defined by (13). We consider $\mathbb{R}$ as a trivial $\mathfrak{k}$-module.

Proofs of the following two propositions are straightforward, so we omit them.

Proposition 1. *Let $F : \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ be a 3-cocycle on $\mathfrak{k}$ with coefficients in $\mathbb{R}$ such that*

$$F(X_1, X_2, X_3) = 0 \qquad \text{and} \qquad F(A_1, A_2, A_3) = 0 \tag{14}$$

for all $X_1, X_2, X_3 \in \mathfrak{h}$ and $A_1, A_2, A_3 \in \mathfrak{g}$. Then $U : \mathfrak{h} \otimes (\mathfrak{g} \wedge \mathfrak{g}) \to \mathbb{R}$ and $V : (\mathfrak{h} \wedge \mathfrak{h}) \otimes \mathfrak{g} \to \mathbb{R}$ defined by

$$U(X; A_1, A_2) = F(X, A_1, A_2), \qquad V(X_1, X_2; A) = F(X_1, X_2, A) \tag{15}$$

make a pair of cocycles for the matched pair $(\mathfrak{g}, \mathfrak{h})$.

Conversely, let (U, V) be a pair of cocycles for the matched pair $(\mathfrak{g}, \mathfrak{h})$. Define $F : \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ by

$$\begin{aligned} F(A_1 + X_1, A_2 + X_2, A_3 + X_3) &= U(X_3; A_1, A_2) - U(X_2; A_1, A_3) + U(X_1; A_2, A_3) \\ &\quad + V(X_2, X_3; A_1) - V(X_1, X_3; A_2) + V(X_1, X_2; A_3), \end{aligned} \tag{16}$$

where $A_1, A_2, A_3 \in \mathfrak{g}$ and $X_1, X_2, X_3 \in \mathfrak{h}$. Then F is a 3-cocycle on $\mathfrak{k}$ satisfying (14).

Proposition 2. *Let F_1 and F_2 be two 3-cocycles on $\mathfrak{k}$ satisfying (14). Assume that F_1 and F_2 are cohomologous such that $F_1 - F_2 = dS$, where S is a 2-cochain on $\mathfrak{k}$ satisfying*

$$S(A_1, A_2) = 0 \qquad \text{and} \qquad S(X_1, X_2) = 0 \tag{17}$$

for all $A_1, A_2 \in \mathfrak{g}$ and $X_1, X_2 \in \mathfrak{h}$. Then the corresponding pairs of cocycles given by (15) for the matched pair $(\mathfrak{g}, \mathfrak{h})$ are equivalent with

$$R(X; A) = S(X, A). \tag{18}$$

Conversely, let pairs (U_1, V_1) and (U_2, V_2) be equivalent. Then the 3-cocycles F_1 and F_2 on $\mathfrak{k}$ defined by (16) are cohomologous, $F_1 - F_2 = dS$, where S satisfies (17) and is given by

$$S(A_1 + X_1, A_2 + X_2) = R(X_1; A_2) - R(X_2; A_1). \tag{19}$$

3 Examples of cocycle bicrossed products of Lie groups

As was mentioned in Introduction, the bicrossed product construction [4] can be carried out by starting with a matched pair of Lie groups (G, H) and then finding a pair of cocycles satisfying equations of (4), (5), and (6). These cocycles will be sought for in the form

$$u(h; g_1, g_2) = e^{i\tilde{u}(h;g_1,g_2)}, \qquad v(h_1, h_2; g) = e^{i\tilde{v}(h_1,h_2;g)}, \tag{20}$$

where $\tilde{u} : H \times G \times G \to \mathbb{R}$, $\tilde{v} : H \times H \times G \to \mathbb{R}$, so that (4), (5), and (6) become

$$\tilde{u}(h \triangleleft g_1; g_2, g_3) + \tilde{u}(h; g_1, g_2g_3) = \tilde{u}(h; g_1, g_2) + \tilde{u}(h; g_1g_2, g_3) \quad \mod 2\pi \tag{21}$$

$$\tilde{v}(h_1, h_2; h_3 \triangleright g) + \tilde{v}(h_1h_2, h_3; g) = \tilde{v}(h_1, h_2h_3; g) + \tilde{v}(h_2, h_3; g) \quad \mod 2\pi, \tag{22}$$

$$\begin{aligned} &\tilde{v}(h_1, h_2; g_1g_2) + \tilde{u}(h_1h_2; g_1, g_2) = \tilde{v}(h_1, h_2; g_1) + \tilde{u}(h_2; g_1, g_2) \\ &\quad + \tilde{v}(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1; g_2) + \tilde{u}(h_1; h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2) \quad \mod 2\pi. \end{aligned} \tag{23}$$

Then we construct a pair of cocycles (U, V) for the corresponding matched pair of the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ by using Proposition 1, find nonequivalent cocycles in terms of Proposition 2, and consider the corresponding left-invariant forms ω_u and ω_v on $K = GH$. Following the procedure of finding a 3-cocycle on a Lie group from a 3-cocycle on the Lie algebra [9], we consider the 3-simplices σ_u and σ_v in K given by

$$\sigma_v(h_1, h_2, g) = (h_1h_2^{s_2})^{s_1}g^{s_1s_2s_3}, \qquad \sigma_u(h, g_1, g_2) = h^{s_1}(g_1g_2^{s_3})^{s_1s_2}, \tag{24}$$

where $s_i : \Delta_3 \to \mathbb{R}$ are some differentiable functions on the standard 3-simplex $\Delta_3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0,\ t_1 + t_2 + t_3 \leq 1\}$. Then the functions $\tilde{u}$, $\tilde{v}$ are found in the form

$$\tilde{u}(h; g_1, g_2) = \int_{\sigma_u(h,g_1,g_2)} \omega_u, \qquad \tilde{v}(h_1, h_2; g) = \int_{\sigma_v(h_1,h_2,g)} \omega_v. \tag{25}$$

3.1 $G = \mathbb{R}^2$, $H = \mathbb{R}^2$ with trivial actions

Let $G = \mathbb{R}^2$, $H = \mathbb{R}^2$ be a matched pair of Lie groups with addition as the group operation and trivial mutual actions. Then $K = GH = \mathbb{R}^4$ is an Abelian group. We will denote by $g(a, b)$ and $h(x, y)$, $a, b, x, y \in \mathbb{R}$, elements of the groups G and H, respectively.

Let $(\mathfrak{g}, \mathfrak{h})$ be the corresponding matched pair of Abelian two-dimensional Lie algebras with trivial mutual actions. Then $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ is an Abelian Lie algebra of dimension 4. Let A_1, A_2 form a basis in $\mathfrak{g}$ and X_1, X_2 a basis in $\mathfrak{h}$. Propositions 1 and 2 give that non-equivalent pairs of cocycles are of the following form:

$$U(X_i; A_1, A_2) = \lambda_i, \qquad V(X_1, X_2; A_j) = \mu_j, \tag{26}$$

where $\lambda_i, \mu_j \in \mathbb{R}$, $i, j = 1, 2$. The corresponding left-invariant forms on K are given by

$$\omega_u = \lambda_1 da \wedge db \wedge dx + \lambda_2 da \wedge db \wedge dy, \qquad \omega_v = \mu_1 da \wedge dx \wedge dy + \mu_2 db \wedge dx \wedge dy.$$

The following proposition describes cocycles for the matched pair (G, H).

Proposition 3. *The functions*

$$\tilde{u}(h; g_1, g_2) = (\lambda_1 x + \lambda_2 y) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad \tilde{v}(h_1, h_2; g) = (\mu_1 a + \mu_2 b) \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \tag{27}$$

where $\lambda_i, \mu_i \in \mathbb{R}$, $i = 1, 2$, *and* $h = h(x, y)$, $h_i = h(x_i, y_i)$, $g = g(a, b)$, $g_i = g(a_i, b_i)$, $i = 1, 2$, *are solutions of* (21), (22), *and* (23) *thus giving pairs of cocycles for the matched pair of the Lie groups* (G, H).

Proof. One can easily check that if

$$s_1 = 1 - t_1, \qquad s_2 = \frac{1 - t_1 - t_2}{1 - t_1}, \qquad s_3 = \frac{1 - t_1 - t_2 - t_3}{1 - t_1 - t_2},$$

then the functions $\tilde{u}$ and $\tilde{v}$ defined by the formulas (25) and (24) are found to be as in (27) and satisfy the equalities (21), (22), and (23). ■

3.2 $G = \mathbb{R} \times \mathbb{R}_+$, $H = \mathbb{R}_+ \times \mathbb{R}$ with nontrivial actions

Denote

$$g(a, b) = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad h(x, y) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The groups $G = \{g(a, b) : a \in \mathbb{R}, b > 0\}$ and $H = \{h(x, y) : x > 0, y \in \mathbb{R}\}$ form a matched pair of Abelian Lie groups with the mutual actions

$$h(x, y) \triangleright g(a, b) = g(ax, b), \qquad h(x, y) \triangleleft g(a, b) = h\left(x, \frac{y}{b}\right).$$

Thus, the group $K = GH$ is a direct product of two copies of the "$ax + b$" groups, that is, K is the group of matrices of the form

$$\begin{pmatrix} x & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the corresponding matched pair of the Abelian two-dimensional Lie algebras $(\mathfrak{g}, \mathfrak{h})$. Let A_1, A_2 form a basis in $\mathfrak{g}$ and X_1, X_2 a basis in $\mathfrak{h}$ such that the non-trivial actions are

$$X_1 \triangleright A_1 = A_1, \qquad X_2 \triangleleft A_2 = X_2.$$

Thus, the Lie algebra $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ has the following non-zero commutation relations:

$$[X_1, A_1] = A_1, \qquad [X_2, A_2] = -X_2.$$

Using Propositions 1 and 2 we find that the corresponding pair of cocycles modulo equivalent ones has the following form:

$$U(X_i; A_1, A_2) = \delta_{1,i}\lambda, \qquad V(X_1, X_2; A_j) = \delta_{2,j}\mu, \tag{28}$$

where $i, j = 1, 2$, $\lambda, \mu \in \mathbb{R}$, and δ_{ij} denotes Kronecker's symbol. The corresponding left-invariant forms on K are

$$\omega_u = da \wedge db \wedge dx, \qquad \omega_v = db \wedge dx \wedge dy.$$

Proposition 4. *The functions*

$$\tilde{u}(h; g_1, g_2) = \lambda(x - 1) \begin{vmatrix} a_1 & a_2 \\ \ln b_1 & \ln b_2 \end{vmatrix}, \qquad \tilde{v}(h_1, h_2; g) = \mu\left(1 - \frac{1}{b}\right) \begin{vmatrix} \ln x_1 & \ln x_2 \\ y_1 & y_2 \end{vmatrix}, \tag{29}$$

where $\lambda, \mu \in \mathbb{R}$, $h = h(x, y)$, $h_i = h(x_i, y_i)$, $g = g(a, b)$, $g_i = g(a_i, b_i)$, $i = 1, 2$, are solutions of (21), (22), and (23), and define cocycles for the matched pair of the Lie groups (G, H) via (20).

Proof. Using (25) and (24) we obtain that

$$\tilde{u}(h; g_1, g_2) = \begin{vmatrix} a_1 & a_2 \\ \ln b_1 & \ln b_2 \end{vmatrix} \phi(x), \qquad \tilde{v}(h_1, h_2; g) = \begin{vmatrix} \ln x_1 & \ln x_2 \\ y_1 & y_2 \end{vmatrix} \psi(b),$$

where

$$\phi(x) = \int_{\Delta_3} x^{-s_1+1} \ln x \; ds_1 \wedge ds_2 \wedge ds_3, \qquad \psi(b) = \int_{\Delta_3} \frac{\ln b}{b} b^{-s_1 s_2 s_3 - 1} ds_1 \wedge ds_2 \wedge ds_3.$$

Instead of finding the functions s_1, s_2, and s_3 we will find the functions $\phi(x)$, $\psi(b)$ such that the corresponding functions $\tilde{u}$ and $\tilde{v}$ satisfy the relations (21), (22), and (23). equations (21) and (22) are satisfied by all ϕ and ψ. Equation (23) will be satisfied if

$$\phi(x_2) + x_2\phi(x_1) = \phi(x_1 x_2), \qquad \psi(b_1) + \frac{\psi(b_2)}{b_1} = \psi(b_1 b_2).$$

Solving these equations we find that $\phi(x) = (x - 1)$, $\psi(b) = \left(1 - \frac{1}{b}\right)$ up to a constant. ■

3.3 $G =$ "$ax + b$", $H = \mathbb{R}^2$ with nontrivial actions

Denote

$$g(a, b) = \begin{pmatrix} 1 & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h(x, y) = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \qquad k(a, b, x, y) = \begin{pmatrix} 1 & b & y \\ 0 & a & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Let

$$G = \{g(a, b) : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\}, \qquad H = \{h(x, y) : x, y \in \mathbb{R}\},$$
$$K = \{k(a, b, x, y) : a \in \mathbb{R} \setminus \{0\}, b, x, y \in \mathbb{R}\}.$$

Then it follows from (2) that

$$h(x, y) \triangleright g(a, b) = g(a, b), \qquad h(x, y) \triangleleft g(a, b) = h\left(\frac{x}{a}, -\frac{x}{a} b + y\right). \tag{30}$$

The Lie algebra $\mathfrak{k}$ is then

$$\mathfrak{k} = \mathbb{R}\langle \partial_a, \partial_b, \partial_x, \partial_y \rangle$$

with the following commutation relations:

$$[\partial_a, \partial_b] = -\partial_b, \qquad [\partial_a, \partial_x] = \partial_x, \qquad [\partial_a, \partial_y] = 0,$$
$$[\partial_b, \partial_x] = \partial_y, \qquad [\partial_b, \partial_y] = 0, \qquad [\partial_x, \partial_y] = 0.$$

Using Proposition 1 we find that, up to equivalence as in Proposition 2, the linear space of nonequivalent cocycles for the matched pair $(\mathfrak{g}, \mathfrak{h})$ is spanned by V^1 and V^2 that are given by

$$V^1(\partial_a, \partial_1, \partial_2) = 0, \qquad V^1(\partial_b, \partial_x, \partial_y) = 1,$$
$$V^2(\partial_a, \partial_x, \partial_y) = 1, \qquad V^2(\partial_b, \partial_1, \partial_2) = 0,$$

where $\partial_1, \partial_2 \in \{\partial_x, \partial_y\}$. The corresponding left-invariant forms on K are

$$\omega_v^1 = \frac{1}{a}\left(-\frac{b}{a} da + db\right) \wedge dx \wedge dy, \qquad \omega_v^2 = \frac{1}{a^2} da \wedge dx \wedge dy.$$

Proposition 5. *Let* $\tilde{u}^i = 0$, $i = 1, 2$, *and* $\tilde{v}^1$, $\tilde{v}^2$ *be given by*

$$\tilde{v}^1(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{b}{a}, \qquad \tilde{v}^2(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{a-1}{a},$$

where $h_i = h(x_i, y_i)$, $i = 1, 2$, *and* $g = g(a, b)$. *Then* (20) *defines two pairs of cocycles for the matched pair of the groups* (G, H).

Proof. We first consider the connected component of the identity in the group K, $K_0 = \{k(a, b, c, d) : a > 0\}$. Then we find from (24) that

$$\sigma_v = k\left(a^{s_1 s_2 s_3}, \frac{a^{s_1 s_2 s_3 - 1}}{a - 1} b, \, s_1 x_1 + s_1 s_2 x_2, s_1 y_1 + s_1 s_2 y_2\right).$$

It follows from (25) that

$$\tilde{v}^1(h_1, h_2, g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} b \frac{\ln a}{a - 1} \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}} \, ds_1 \wedge ds_2 \wedge ds_3 \tag{31}$$

and

$$\tilde{v}^2(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \ln a \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}} \, ds_1 \wedge ds_2 \wedge ds_3. \tag{32}$$

Let ϕ denote the function

$$\phi(a) = a \ln a \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}} \, ds_1 \wedge ds_2 \wedge ds_3.$$

The equation (22) is satisfied for any ϕ, whereas (23) implies that

$$a_2 \phi(a_1) - \phi(a_1 a_2) + \phi(a_2) = 0.$$

Its unique, up to a constant, differentiable solution satisfying the condition $\phi(1) = 0$ is

$$\phi(a) = a - 1.$$

This, together with (31) and (32), proves the claim for $a > 0$. Extend now $\tilde{v}^1$ and $\tilde{v}^2$ with these formulas over the whole group K. ■

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