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Separation of Variables in the Null Hamilton–Jacobi Equation

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This is a summary of the main results contained in a forthcoming joint paper with S. Benenti and G. Rastelli, where we revisit the theory of the separation of variables in the Hamilton–Jacobi equation with a fixed value of the energy. We extend the classical Levi-Civita conditions by using Lagrangian multipliers. By applying the general results to the case of natural Hamiltonians we find conditions generalizing those of Stäckel.

1 Introduction

The study of the Hamilton–Jacobi equation with a fixed value of the energy is a classical topic which is treated in many textbooks. Interesting and deeply examined examples of this kind of equations are for instance

– the Kepler motions with a fixed value of the total energy,

$$\frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 - \frac{1}{r} = E;$$

– the so called homogeneous formalism used to deal with time dependent Hamiltonians,

$$\frac{\partial W}{\partial t} + H\left(t, q^i, \frac{\partial W}{\partial q^i}\right) = 0;$$

– the null geodesics in pseudo-Riemannian manifolds,

$$g^{ii}p_i^2 = 0;$$

– the dynamical systems which are Hamiltonian only on a hypersurface. As a simple example, we consider the dynamical system on T^*E_2

$$\dot{x}_i = y_i + f_i(x_i, y_i)\big(y_1^2 + y_2^2 + x_1^2 + x_2^2 - c\big),$$
$$\dot{y}_i = -x_i + g_i(x_i, y_i)\big(y_1^2 + y_2^2 + x_1^2 + x_2^2 - c\big),$$

where $c \in \mathbb{R}$, (x_1, x_2) are Cartesian coordinates in the Euclidean plane E_2, (y_1, y_2) are the associated momenta, f_i and g_i arbitrary functions on T^*E_2. It is a Hamiltonian system only on the hypersurface $\big(y_1^2 + y_2^2 + x_1^2 + x_2^2\big) = 2c$.

All these examples can be considered as special cases of the *null Hamilton–Jacobi equation*

$$\mathcal{H}\left(q^i, \frac{\partial W}{\partial q^i}\right) = 0, \tag{1}$$

where $\mathcal{H}$ is a smooth function on T^*Q. Although separation of variables for such a kind of equations has been studied since the end of 19th century, in our opinion, we still need a better understanding of the theoretical and geometrical framework of this kind of separation of variables.

Our approach that leads to consideration of a new family of equations characterizing the separation of variables of a null Hamiltonian (1), is based upon three elements:

i) a suitable definition of complete solution for a null Hamilton–Jacobi equation;

ii) a geometrical object: the separable connections (as developed by Benenti), which are related to the complete integrability of a special kind of first-order differential systems;

iii) a technical instrument: the Hadamard Lemma.

2 Complete solutions of the null Hamilton–Jacobi equation

First of all, we revisit the concept of complete solution of a null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ on the cotangent bundle T^*Q of a real n-dimensional manifold Q. We give two definitions, in a sense equivalent, but both essential for the following discussion. The first (and commonly adopted) definition is the "internal complete solution" $W^{(i)}$ which depends on $n-1$ parameters:

Definition 1. An *internal complete solution* of the null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ is a solution $W^{(i)}(\underline{q}, c_\alpha)$ depending on $n-1$ parameters (c_α) satisfying the *completeness condition*:

$$\operatorname{rank}\left[\frac{\partial^2 W^{(i)}}{\partial q^i \partial c_\alpha}\right] = n - 1.$$

The second definition is the "extended complete solution" $W^{(e)}$ depending on n parameters:

Definition 2. An *extended complete solution* of the null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ is a function $W^{(e)}(\underline{q}, \underline{c})$ depending on n real parameters $\underline{c} = (c_i)$ satisfying the *completeness condition*

$$\det\left[\frac{\partial^2 W^{(e)}}{\partial q^i \partial c_j}\right] \neq 0 \tag{2}$$

and which is a solution of $\mathcal{H} = 0$ for $c_n = 0$.

Remark 1. We recall that in the usual definition a complete solution $W^{(o)}$ of the Hamilton–Jacobi equation $\mathcal{H} = \text{const}$ depends on n parameters and satisfies the completeness condition (2), but is a solution for all values of the parameters.

Remark 2. From a geometrical point of view, $W^{(i)}$ is a Lagrangian foliation of the submanifold $\mathcal{H} = 0$, while $W^{(e)}$ is a Lagrangian foliation of an open neighborhood of $\mathcal{H} = 0$ which is reducible to a foliation of $\mathcal{H} = 0$. An ordinary complete solution $W^{(o)}$ is a Lagrangian foliation compatible with all the submanifolds $\mathcal{H} = h$ $(h \in \mathbb{R})$.

These objects are in a sense equivalent. Indeed, it can be shown that starting from an internal complete solution it is possible to construct an extended one, and vice versa.

Proposition 1. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *admits an extended complete solution* $W^{(e)}$ *if and only if it admits an internal complete solution* $W^{(i)}$.

It is a remarkable fact that the transition from internal to extended complete solutions is compatible with the separation of variables. Indeed, it can be proved that the transition between $W^{(i)}$ and $W^{(e)}$ preserves the separated form of the complete solutions $W^{(i)} = \sum_{j=1}^{n} W_j(q^j, c_\alpha)$, $W^{(e)} = \sum_{j=1}^{n} W_j(q^j, \underline{c})$. Thus,

Proposition 2. *The separation of variables for the null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is characterized by the existence of one of the two kind of separable complete solutions, internal or extended.*

3 Separable connections and Hadamard Lemma

The second geometrical tool for our analysis are the separable connections, which arise from a general theory, developed by Benenti in [3], related to the connections on cotangent bundles and their links with the theory of separation of variables. It can be proved that the separation of variables in a given coordinate system (q^i) on Q is related to the complete integrability of a special kind of first-order differential systems. The geometrical interpretation of these systems is a special kind of distribution transversal to the fibers Δ which is called *separable connection.* With a given coordinate system (q^i) on Q and with n functions R_i on T^*Q we associate

- the 1-st order differential system
$$\partial_j p_i = \delta_{ij} R_i(\underline{p}, \underline{q}), \tag{3}$$
- n independent vector fields
$$D_i = \partial_i + R_i(\underline{q}, \underline{p})\partial^i,$$
generating a distribution Δ transversal to the fibers and called *separable connection.*

There is a link between the separation for Hamilton–Jacobi equation $\mathcal{H} = 0$ and the complete integrability of (3), which is equivalent to the complete integrability of the Δ. Indeed, the following properties hold:

i) the integral manifolds of Δ are described by $p_i = f_i(q^h)$ where f_i are solutions of the differential system,

ii) the functions f_i are of the form $f_i = \partial_i W(q^h)$,

iii) the function W is additively separable i.e., $W = \sum_i W_i(q^i)$.

By the Frobenius theorem, we find that Δ is completely integrable on an open subset of T^*Q if and only if $[D_i, D_j] = 0$. The commutation of the generators D_i is equivalent to

$$D_i R_j = 0 \qquad (i \neq j).$$

So far no link with the Hamilton–Jacobi equation (1). We add now the requirement that Δ is compatible with the submanifold $\mathcal{H} = 0$ i.e., that the generators D_i are tangent to $\mathcal{H} = 0$,

$$D_i H|_{\mathcal{H}=0} = 0.$$

Then, the conditions for the existence of a separated extended solution $W^{(e)}$ are

$$\begin{aligned} &D_i R_j = 0 \qquad (i \neq j), \\ &D_i \mathcal{H}|_{\mathcal{H}=0} = 0. \end{aligned} \tag{4}$$

The conditions for the existence of a separated internal solution $W^{(i)}$ are

$$\begin{aligned} &D_i \mathcal{H}|_{\mathcal{H}=0} = 0, \\ &D_i R_j|_{\mathcal{H}=0} = 0 \qquad (i \neq j) \end{aligned} \tag{5}$$

since in this case we require that the reduced distribution $\Delta|_{\mathcal{H}=0}$ is completely integrable.

The last tool we use is the following form of the Hadamard Lemma that allows us to deal with differential conditions restricted to a submanifold such as (4) and (5). A proof of the lemma for the one-dimensional case can be found in [1].

Lemma 1. *A smooth function F on T^*Q which vanishes on the submanifold defined by equation $\mathcal{H} = 0$ is of the form*

$$F = \mathcal{H}\lambda,$$

where λ is a suitable smooth function on T^*Q.

4 The general case

By applying the Hadamard Lemma to conditions (4), equivalent to the existence of an extended solution $W^{(e)}$, we get equations

$$L_{ij}(\mathcal{H}) + \mathcal{H}\Big[\lambda_i(\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial_j\mathcal{H}\partial^i\partial^j\mathcal{H}) + \lambda_j(\partial^i\mathcal{H}\partial_i\partial^j\mathcal{H} - \partial^i\partial^j\mathcal{H}\partial_i\mathcal{H}) + \lambda_i\lambda_j(\mathcal{H}\partial^i\partial^j\mathcal{H} - \partial^i\mathcal{H}\partial^j\mathcal{H}) - \partial_j\lambda_i\partial^i\mathcal{H}\partial^j\mathcal{H} + \partial^j\lambda_i(\partial_j\mathcal{H}\partial^i\mathcal{H} - \lambda_j\mathcal{H}\partial^i\mathcal{H})\Big] = 0, \quad (6)$$

where

$$L_{ij}(\mathcal{H}) = \partial_i\mathcal{H}\partial_j\mathcal{H}\partial^i\partial^j\mathcal{H} + \partial^i\mathcal{H}\partial^j\mathcal{H}\partial_i\partial_j\mathcal{H} - \partial_i\mathcal{H}\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial^i\mathcal{H}\partial_j\mathcal{H}\partial_i\partial^j\mathcal{H},$$

are the classical Levi-Civita operators i.e., the left-hand sides of the Levi-Civita separability condition characterizing the ordinary separation of variables

$$L_{ij}(\mathcal{H}) = 0, \qquad (i \neq j).$$

In (6) the additional unknown functions λ_i play the role of Lagrangian multipliers. We call equations (6) *Levi-Civita conditions with Lagrangian multipliers* (LCL). Thus, due to Proposition 2, we have proved

Theorem 1. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable if and only if there exist* n *functions* $\lambda_i(\underline{q}, \underline{p})$ *on* T^*Q *such that Levi-Civita conditions with Lagrangian multipliers* (6) *are satisfied.*

By requiring the existence of an internal separated solution $W^{(i)}$, we get the following criterion equivalent to Theorem 1:

Theorem 2. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable if and only if the Levi-Civita conditions* $L_{ij}(\mathcal{H}) = 0$*, for all* $i \neq j$*, are satisfied on the submanifold* $\mathcal{H} = 0$*.*

It is remarkable that, from Theorem 1, we can derive another characterization of the separation of variables, where the additional functions λ_i are replaced by a single function Λ on T^*Q and the LCL conditions reduce to the usual Levi-Civita separability conditions on the new Hamiltonian $\mathcal{H}/\Lambda$.

Theorem 3. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable in the coordinates* $\underline{q} = (q^i)$ *if and only if there exists a nowhere vanishing function* $\Lambda = \Lambda(\underline{q}, \underline{p})$ *on* T^*Q *such that the conformal Hamiltonian* $\mathcal{H}/\Lambda$ *is separable i.e., for any* $i \neq j$*,*

$$L_{ij}\left(\frac{\mathcal{H}}{\Lambda}\right) = 0.$$

Definition 3. We call the function $\mathcal{J} = \mathcal{H}/\Lambda$ the *conformal Hamiltonian* associated with $\mathcal{H}$ and the function Λ the *conformal factor*.

Remark 3. The transition from a Hamiltonian $\mathcal{H}$ to a conformal Hamiltonian $\mathcal{J} = \mathcal{H}/\Lambda$ is an extension of the so-called *Jacobi transformation* or *Maupertuis transformation* for natural Hamiltonians (see [9] and references cited therein).

The link between the two Hamiltonian systems associated with $\mathcal{H}$ and $\mathcal{J}$ is given by the following properties:

Proposition 3. *On the submanifold $\mathcal{H} = 0$ the Hamiltonian vector fields $X_\mathcal{H}$ and $X_\mathcal{J}$ are parallel and differ by the factor Λ,*

$$(\Lambda X_\mathcal{J})\big|_{\mathcal{H}=0} = X_\mathcal{H}\big|_{\mathcal{H}=0},$$

so that the corresponding affine parameters t and $\bar{t}$ are related by equation

$$d\bar{t} = \Lambda dt.$$

Proposition 4. *If we know an ordinary complete solution of the Hamilton–Jacobi equation $\mathcal{J} = h$ for the conformal Hamiltonian $\mathcal{J}$, then, for $h = 0$, we get the orbits on $\mathcal{H} = 0$ of the Hamiltonian vector field $X_\mathcal{H}$.*

5 Orthogonal separation for natural Hamiltonians

We apply these general results to the analysis of an important particular case: the natural Hamiltonian in orthogonal coordinates $\frac{1}{2}\, g^{ii} p_i^2 + V(\underline{q})$ on a Riemannian (or pseudo-Riemannian) manifold. The corresponding Hamilton–Jacobi equation with a fixed value E of the energy is associated to the null Hamilton–Jacobi equation

$$\mathcal{H}(\underline{q}, \underline{p}) = \tfrac{1}{2}\, g^{ii} p_i^2 + V(\underline{q}) - E = 0.$$

We examine the following three cases

$$\begin{array}{lll} V = 0, \quad E \neq 0, & \text{non-null geodesics}, & \\ V = 0, \quad E = 0, & \text{null-geodesics}, & \\ V - E \neq 0, & \text{dynamical trajectories with total energy } E. & \end{array} \tag{7}$$

Theorem 4. *A common necessary condition for the orthogonal separability of a natural Hamiltonian in all cases* (7) *is that equations*

$$\frac{1}{g^{hh}} S_{ij}(g^{hh}) = \frac{1}{g^{kk}}\, S_{ij}(g^{kk}), \tag{8}$$

be satisfied for all indices h, k and $i \neq j$, where

$$S_{ij} \colon f \to \partial_i \partial_j f - \partial_j \log g^{ii} \partial_i f - \partial_i \log g^{jj} \partial_j f.$$

Remark 4. The $S_{ij}(\cdot)$ are the second-order differential operators on functions $f(\underline{q})$ on T^*Q that we called *Stäckel operators*. They can be used to characterize the ordinary geodesic orthogonal separation. Indeed, since for the geodesic orthogonal Hamiltonian $H = \frac{1}{2}\, g^{ii} p_i^2$ the Levi-Civita separability conditions are third-degree polynomials in the momenta whose coefficients are (up to a factor) $S_{ij}(g^{kk})$, the Stäckel metrics are characterized by equations

$$S_{ij}(g^{kk}) = 0. \tag{9}$$

Conditions (8) are a non-trivial generalization of the classical Stäckel conditions (9).

Proposition 5. *The necessary conditions* (8) *are equivalent to the existence of a nowhere vanishing function σ on Q such that the conformal metric $\bar{g}^{ii} = g^{ii}/\sigma$ is a Stäckel metric.*

Definition 4. We call *conformal separable coordinates* the orthogonal coordinates satisfying conditions (8).

Necessary and sufficient conditions for the separation of a natural Hamiltonian in the cases (7) are given in the following three theorems:

Theorem 5. *The separation of variables of the Hamilton–Jacobi equation for the non-null geodesic* $g^{ii}p_i^2 = 2E$ *occurs in given orthogonal coordinates* (q^i) *if and only if the metric is a Stäckel metric i.e., if and only if* (9) *hold.*

Theorem 6. *The separation of variables of the Hamilton–Jacobi equation for the null geodesic* $g^{ii}p_i^2 = 0$ *occurs in given orthogonal coordinates* (q^i) *if and only if* (8) *hold.*

Theorem 7. *The separation of variables of the Hamilton–Jacobi equation with a fixed value* E *of the energy and for* $E - V \neq 0$ *occurs in given orthogonal coordinates* (q^i) *if and only if*

$$\forall\, h,\ \forall\, i \neq j \quad \frac{1}{g^{hh}} S_{ij}(g^{hh}) = \frac{1}{V - E} S_{ij}(V).$$

Remark 5. For a non-separable natural Hamiltonian H, there exists at most one value for the energy E such that $H = E$ is separable.

Remark 6. It can be proved [4] that, in a given conformal separable coordinate system (q^i), the most general form for the conformal factor $E - V$ is

$$E - V = \sum_{i=1}^{n} \psi_i(q^i) g^{ii}. \tag{10}$$

Remark 7. By using the conformal Killing two-tensors, we can derive intrinsic versions of the above theorems (see [6, 4]).

6 An illustrative example

In this section we summarize what happens in the two-dimensional case and then we examine a three-dimensional example of conformal separable coordinates in the Euclidean three-space. We recall that (see e.g. [8, 4])

Proposition 6. *A two-dimensional orthogonal coordinate system is conformal separable if and only if, up to a rescaling,*

$$g^{11} = \pm g^{22}.$$

Proposition 7. *Up to a rescaling, every conformal separable system* (q^1, q^2) *of the Euclidean plane is generated by a non-constant analytic function*

$$F(z) = F(x + iy) = q^1(x, y) + iq^2(x, y).$$

Moreover, (q^1, q^2) *are conformal separable coordinates compatible with a natural Hamiltonian* $H = G + V$ *for a fixed value of the energy* E *if and only if* $E - V$ *is of the form*

$$E - V = \left(\psi_1(q^1) + \psi_2(q^2)\right) g^{11}.$$

Many examples of conformal separable coordinates associated with transformations of the complex plane can be found in [7]. Moreover, in [8] the authors examine different cases of potentials $V \neq 0$ compatible with the coordinates generated by a conformal transformation on $\mathbb{C}$. An explicit solution of a natural Hamilton–Jacobi equation on E_2 that is separable only for a single value of the energy is presented in [4].

Example 1. Let us consider the following natural Hamiltonian on the cotangent bundle of the Euclidean three-space

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) + \frac{1}{r^2}f_1\left(\frac{\cos\theta}{r}\right) + \frac{1}{r^2}f_2\left(\frac{\sin\theta}{r}\right) + \frac{r^2}{\sin^2\theta}f_3(\phi),$$

where (r, θ, ϕ) are standard spherical coordinates and f_i arbitrary functions. The corresponding Hamilton–Jacobi equation $H = E$ is not solvable by separation of variables (we can check that condition $d(\mathbf{K}dV) = 0$ is never satisfied by a characteristic Killing tensor). However, for $E = 0$ the corresponding null Hamilton–Jacobi equation is separable in the coordinates (μ, ν, ϕ) defined by

$$x = \frac{\mu}{\mu^2+\nu^2}\cos\phi, \qquad y = \frac{\mu}{\mu^2+\nu^2}\sin\phi, \qquad z = \frac{\nu}{\mu^2+\nu^2}, \tag{11}$$

where (x, y, z) are Cartesian coordinates. With respect to these new coordinates we get

$$g^{11} = g^{22} = (\mu^2+\nu^2)^2, \qquad g^{33} = \frac{(\mu^2+\nu^2)^2}{\mu^2}.$$

Since ϕ is ignorable, it is easy to check that the new coordinates fulfill conditions (8). Moreover, according to (10), we have that a natural Hamiltonian is separable in coordinates (μ, ν, ϕ) for the fixed value E of the energy if and only if

$$E - V = (\mu^2+\nu^2)^2\left(\psi_1(\mu) + \psi_2(\nu) + \frac{\psi_3(\phi)}{\mu^2}\right)$$

that for $E = 0$ and $\psi_i = -f_i$ is satisfied by the potential of the considered Hamiltonian. The coordinates (μ, ν, ϕ) defined by (11) are called tangent-sphere coordinates (see [7]). They are constructed by a rotation around a symmetry axis of a planar conformal separable system associated to the transformation of the complex plane $1/\bar{z}$ $(z = x + iy)$.

Remark 8. As seen in the previous example, it can be shown that the orthogonal rotational coordinate systems given in [7] (apart from the so-called Hyperbolic coordinates) are conformal separable and the general form of compatible potentials is immediately deducible. All these coordinates systems have $g^{11} = g^{22}$ and $q^3 = \phi$ ignorable. Moreover, all of them allow the R-separation of the Laplace equation. More general conformal separable coordinates of E_3 are the so called confocal cyclides (see the classical book [5]). The corresponding coordinate hypersurfaces are fourth-order surfaces such that the inversion with respect to a sphere of a cyclide is also a cyclide.

7 Final remarks and open problems

For the proofs of theorems and propositions and for more references the reader is referred to [4]. The application of the general theory to other well-known cases leads to many open problems. Work about the case of general non-orthogonal separation and of natural Hamiltonians with vector potential is in progress. However, both problems are very complicated to deal with, since we have to repeat and adapt the study done by Benenti [2] about the reduction of separable coordinates to normal forms.

Another interesting topic currently under investigation is the link between separation for a fixed value of the energy, the multiplicative separation and the R-separation for 2-nd order differential equations (Laplace, Schrödinger, Helmholtz).

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