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On the Deformations of Dorfman's and Sokolov's Operators

Jen-Hsu CHANG

Department of General Courses, Chung-Cheng Institute of Technology, National Defense University, Tau-Yuan County, Taiwan
E-mail: *jhchang@ccit.edu.tw*

We deform the Dorfman's and Sokolov's Hamiltonian operators by the quasi-Miura transformation coming from the topological field theory and investigate the deformed operators.

1 Introduction

The Dorfman's and Sokolov's Hamiltonian operators are defined respectively as [2,11] ($D = \partial_x$)

$$J = D\frac{1}{v_x}D\frac{1}{v_x}D, \tag{1}$$

$$S = v_x D^{-1} v_x, \tag{2}$$

which are Hamiltonian operators (or $J^{-1} = D^{-1}v_xD^{-1}v_xD^{-1}$ and $S^{-1} = \frac{1}{v_x}D\frac{1}{v_x}$ are symplectic operators). The Dorfman's operator J (or J^{-1}) and the Sokolov's operator S are related to integrable equations as follows.

- The Riemann hierarchy

$$\begin{aligned} v_{t_n} = v^n v_x = S\delta H_n = \frac{1}{(n+1)(2n+1)}K\delta H_{n+1} = \frac{1}{(n+1)(n+2)}D\delta H_{n+2} \\ = \frac{1}{(n+1)(n+2)(n+3)(n+4)}J\delta H_{n+4}, \end{aligned} \tag{3}$$

where

$$K = Dv + vD, \qquad H_n = \int v^n dx, \qquad n = 1, 2, 3\ldots,$$

and δ is the variational derivative. When $n = 1$, it is called the Riemann equation or dispersionless KdV equation. We notice that it seems that the Riemann hierarchy (3) is a quater-Hamiltonian system. But one can show that S and J is not compatible, i.e., $S + \lambda J$ are not a Hamiltonian operator for any $\lambda \neq 0$ (see below).

- The Schwarzian KdV equation [10,13]

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x} = v_x\{v, x\} = S\delta H_1 = J^{-1}\delta H_2, \tag{4}$$

where $\{v, x\}$ is the Schwartz derivative and

$$H_1 = \frac{1}{2}\int \left(v_x^{-2}v_{xx}^2\right) dx, \qquad H_2 = \frac{1}{2}\int \left(-v_x^{-2}v_{xxx}^2 + \frac{3}{4}v_x^{-4}v_{xx}^4\right) dx.$$

Remark 1. It is not difficult to verify that J^{-1} is also a Hamiltonian operator and, then, J is also a symplectic operator; however, $S^{-1} = \frac{1}{v_x}D\frac{1}{v_x}$ is <u>not</u> a Hamiltonian operator and, then, S is not a symplectic operator.

Next, to deform the operators J and S, we use the free energy in topological field theory of the famous KdV equation

$$u_t = uu_x + \frac{\epsilon^2}{12} u_{xxx} \tag{5}$$

to construct the quasi-Miura transformation as follows. The free energy F of KdV equation (5) in TFT has the form ($F_0 = \frac{1}{6}v^3$)

$$F = \frac{1}{6}v^3 + \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g\left(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)}\right).$$

Let

$$\begin{aligned}\triangle F &= \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g\left(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)}\right)\\ &= F_1(v; v_x) + \epsilon^2 F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx})\\ &\quad + \epsilon^4 F_3\left(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}, \ldots, v^{(7)}\right) + \cdots .\end{aligned}$$

The $\triangle F$ will satisfy the loop equation [4, p. 151]

$$\begin{aligned}&\sum_{r\geq 0} \frac{\partial \triangle F}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \triangle F}{\partial v^{(r)}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}}\\ &\quad = \frac{1}{16\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}\\ &\quad + \frac{\epsilon^2}{2} \sum_{k,l\geq 0} \left[\frac{\partial^2 \triangle F}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \triangle F}{\partial v^{(k)}} \frac{\partial \triangle F}{\partial v^{(l)}}\right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}}\\ &\quad - \frac{\epsilon^2}{16} \sum_{k\geq 0} \frac{\partial \triangle F}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2}.\end{aligned} \tag{6}$$

Then we can determine F_1, F_2, F_3, … recursively by substituting $\triangle F$ into equation (6). For F_1, one obtains

$$\frac{1}{v-\lambda} \frac{\partial F_1}{\partial v} - \frac{3}{2} \frac{v_x}{(v-\lambda)^2} \frac{\partial F_1}{\partial v_x} = \frac{1}{16\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}.$$

From this, we have

$$\kappa_0 = \frac{1}{16}, \qquad F_1 = \frac{1}{24} \log v_x.$$

For the next terms $F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx})$, it can be similarly computed and the result is

$$F_2 = \frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4}.$$

Now, one can define the quasi-Miura transformation as

$$\begin{aligned}u &= v + \epsilon^2 (\triangle F)_{xx} = v + \epsilon^2 (F_1)_{xx} + \epsilon^4 (F_2)_{xx} + \cdots\\ &= v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + \epsilon^4 \left(\frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4}\right)_{xx} + \cdots .\end{aligned} \tag{7}$$

One remarks that Miura-type transformation means the coefficients of ϵ are homogeneous polynomials in the derivatives $v_x, v_{xx}, \dots, v^{(m)}$ [4, p. 37], [5] and "quasi" means the ones of ϵ are quasi-homogeneous rational functions in the derivatives, too [4, p. 109] (see also [12]).

The truncated quasi-Miura transformation

$$u = v + \sum_{n=1}^{g} \epsilon^{2n} \left[F_n \left(v; v_x, v_{xx}, \dots, v^{(3g-2)} \right) \right]_{xx} \tag{8}$$

has the basic property [4, p. 117] that it reduces the Magri–Poisson pencil [6] of KdV equation (5)

$$\{u(x), u(y)\}_\lambda = [u(x) - \lambda] D\delta(x-y) + \frac{1}{2} u_x(x)\delta(x-y) + \frac{\epsilon^2}{8} D^3 \delta(x-y) \tag{9}$$

to the Poisson pencil of the Riemann hierarchy (3):

$$\{v(x), v(y)\}_\lambda = [v(x) - \lambda] D\delta(x-y) + \frac{1}{2} v_x(x)\delta(x-y) + O\left(\epsilon^{2g+2}\right). \tag{10}$$

One can also say that the truncated quasi-Miura transformation (8) deforms the KdV equation (5) to the Riemann equation $v_t = v v_x$ up to $O\big(\epsilon^{2g+2}\big)$.

Remark 2. A simple calculation shows that, under the transformation $u = \frac{\epsilon^2}{4}\{m, x\}$, the KdV equation (5) is transformed into the Schwarzian KdV equation

$$m_t = \frac{\epsilon^2}{12} m_x \{m, x\} = \frac{\epsilon^2}{12} \left(m_{xxx} - \frac{3}{2} \frac{m_{xx}^2}{m_x} \right).$$

Furthermore, after a direct calculation, one can see that the Magri Poisson bracket

$$K(\epsilon) = \{u(x), u(y)\} = u(x) D\delta(x-y) + \frac{1}{2} u_x(x)\delta(x-y) + \frac{\epsilon^2}{8} D^3 \delta(x-y) \tag{11}$$

is transformed into the Dorfman's symplectic operator J^{-1} $(m = v)$

$$\{m(x), m(y)\} = -\frac{\epsilon^2}{8} D^{-1} m_x D^{-1} m_x D^{-1} \delta(x-y).$$

Now, a natural question arises: under the truncated quasi-Miura transformation (8), are the deformed Dorfman's operator $J(\epsilon)$ and Sokolov's operator $S(\epsilon)$ still Hamiltonian operators up to $O\big(\epsilon^{2g+2}\big)$? For simplicity, we consider only the case $g = 1$, i.e.,

$$u = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + O\left(\epsilon^4\right) \tag{12}$$

or

$$v = u - \frac{\epsilon^2}{24} (\log u_x)_{xx} + O\left(\epsilon^4\right). \tag{13}$$

The answer is true for the Dorfman's operator $J(\epsilon)$ but it is false for the Sokolov's operator $S(\epsilon)$. It is the purpose of this article.

2 Deformations under quasi-Miura transformation

In the new "u-coordinate", J and S will be given by the operators

$$J(\epsilon) = M^* D \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} D \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} DM + O\left(\epsilon^4\right), \tag{14}$$

$$S(\epsilon) = M^* \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) M + O\left(\epsilon^4\right), \tag{15}$$

where

$$M = 1 - \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2, \qquad M^* = 1 + \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D, \tag{16}$$

M^* being the adjoint operator of M. Then we have the following

Theorem 1. 1. *$J(\epsilon)$ is a Hamiltonian operator up to $O(\epsilon^4)$.* 2. *$S(\epsilon)$ is <u>not</u> a Hamiltonian operator up to $O(\epsilon^4)$.*

Proof. 1. The fact that $J(\epsilon)$ is a skew-adjoint (or $J^*(\epsilon) = -J(\epsilon)$) differential operator (up to $O(\epsilon^4)$) follows immediately from (14). Rather than prove the Poisson form [7] of the Jacobi identity for $J(\epsilon)$, it is simpler to prove that the symplectic two-form

$$\Omega_J(\epsilon) = \int \{du \wedge J(\epsilon)^{-1} du\} dx + O\left(\epsilon^4\right)$$

is closed [8, 9]: $d\Omega_J(\epsilon) = O\left(\epsilon^4\right)$.

A simple calculation can yield

$$\begin{aligned}
J(\epsilon)^{-1} &= \left(1 + \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2\right) D^{-1} \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \\
&\quad \times \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \left(1 - \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D\right) \\
&= \left(D^{-1} u_x - \frac{\epsilon^2}{24} D^{-1} (\log u_x)_{xxx} + \frac{\epsilon^2}{24} D \frac{1}{u_x} D u_x\right) D^{-1} \\
&\quad \times \left(u_x D^{-1} - \frac{\epsilon^2}{24} (\log u_x)_{xxx} D^{-1} - \frac{\epsilon^2}{24} u_x D \frac{1}{u_x} D\right) + O\left(\epsilon^4\right) \\
&= D^{-1} u_x D^{-1} u_x D^{-1} + \frac{\epsilon^2}{24} \bigg[D \frac{1}{u_x} D u_x D^{-1} u_x D^{-1} - D^{-1} (\log u_x)_{xxx} D^{-1} u_x D^{-1} \\
&\quad - D^{-1} u_x D^{-1} u_x D \frac{1}{u_x} D - D^{-1} u_x D^{-1} (\log u_x)_{xxx} D^{-1} \bigg] + O\left(\epsilon^4\right) \\
&= D^{-1} u_x D^{-1} u_x D^{-1} \\
&\quad + \frac{\epsilon^2}{24} \left[D u_x D^{-1} - D^{-1} u_x D + (\log u_x)_x u_x D^{-1} + D^{-1} (\log u_x)_x u_x \right] + O\left(\epsilon^4\right).
\end{aligned}$$

Let ψ denote the potential function for u, i.e., $u = \psi_x$. Thus, formally,

$$D_x^{-1}(du) = d\psi$$

and hence, after a series of integration by parts, one has

$$\begin{aligned}
\Omega_J(\epsilon) = \int \Bigg\{ &\left[\left(D^{-1} d \left(\frac{\psi_x^2}{2} \right) \right) \wedge d \left(\frac{\psi_x^2}{2} \right) - \psi_x d\psi \wedge d \left(\frac{\psi_x^2}{2} \right) \right] \\
&+ \frac{\epsilon^2}{24} \left[2\psi_{xx} d\psi \wedge d\psi_{xx} + 2\psi_{xxx} d\psi_x \wedge d\psi \right] \Bigg\} dx + O\left(\epsilon^4\right).
\end{aligned}$$

So

$$d\Omega_J(\epsilon) = \int \left\{0 + \frac{\epsilon^2}{12}\left[d\psi_{xxx} \wedge d\psi_x \wedge d\psi\right]\right\} dx + O\left(\epsilon^4\right)$$
$$= \frac{\epsilon^2}{12}\int \{(d\psi_{xx} \wedge d\psi_x \wedge d\psi)_x\}\, dx + O(\epsilon^4) = O\left(\epsilon^4\right).$$

This completes the proof of (1).

2. The skew-adjoint property of the deformed Sokolov's operator $S(\epsilon)$ (15) is obvious. To see whether $S(\epsilon)$ is Hamiltonian operator or not, we must check whether $S(\epsilon)$ satisfies the Jacobi identity up to $O\left(\epsilon^4\right)$. Following [7, 8], we introduce the arbitrary basis of tangent vector Θ, which is then conveniently manipulated according to the rules of exterior calculus. The Jacobi identity is given by the compact expression

$$P(\epsilon) \wedge \delta I = O\left(\epsilon^4\right) \quad \text{(mod. div.)}, \tag{17}$$

where $P(\epsilon) = S(\epsilon)\Theta$, $I = \frac{1}{2}\Theta \wedge P(\epsilon)$ and δ denotes the variational derivative. The vanishing of the tri-vector (17) modulo a divergence is equivalent to the satisfaction of the Jacobi identity.

After a tedious calculation, one can obtain

$$S(\epsilon) = M^*\left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1}\left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) M + O\left(\epsilon^4\right)$$
$$= \left[u_x + \frac{\epsilon^2}{24}\left(D^3 + D^2(\log u_x)_x - (\log u_x)_{xxx}\right)\right] D^{-1}$$
$$\times \left[u_x - \frac{\epsilon^2}{24}\left(D^3 - (\log u_x)_x D^2 + (\log u_x)_{xxx}\right)\right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{24}\left[D^2 u_x + D^2(\log u_x)_x D^{-1} u_x - (\log u_x)_{xxx} D^{-1} u_x - u_x D^2\right.$$
$$\left. + u_x D^{-1}(\log u_x)_x D^2 - u_x D^{-1}(\log u_x)_{xxx}\right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{24}\left[D^2 u_x - u_x D^2 + (\log u_x)_x D u_x + u_x D(\log u_x)_x\right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{12}[D u_{xx} + u_{xx} D] + O\left(\epsilon^4\right).$$

So

$$P(\epsilon) = S(\epsilon)\Theta = u_x D^{-1}(u_x\Theta) + \frac{\epsilon^2}{12}[2u_{xx}\Theta_x + u_{xxx}\Theta] + O\left(\epsilon^4\right).$$

Hence

$$I = \frac{1}{2}\Theta \wedge P(\epsilon) = \frac{1}{2}u_x\Theta \wedge D^{-1}(u_x\Theta) + \frac{\epsilon^2}{12}u_{xx}\Theta \wedge \Theta_x + O\left(\epsilon^4\right)$$

and then

$$\delta I = -\frac{1}{2}[\Theta \wedge D^{-1}(u_x\Theta)]_x - \frac{1}{2}u_x\Theta \wedge D^{-1}(\Theta_x) + \frac{\epsilon^2}{12}[\Theta \wedge \Theta_x]_{xx} + O\left(\epsilon^4\right)$$
$$= -\frac{1}{2}\Theta_x \wedge D^{-1}(u_x\Theta) + \frac{\epsilon^2}{12}[\Theta \wedge \Theta_x]_{xx} + O\left(\epsilon^4\right).$$

Finally,

$$
\begin{aligned}
P(\epsilon)\wedge\delta I &= \left\{u_x D^{-1}(u_x\Theta)+\frac{\epsilon^2}{12}[2u_{xx}\Theta_x+u_{xxx}\Theta]\right\}\\
&\quad\wedge\left\{-\frac{1}{2}\Theta_x\wedge D^{-1}(u_x\Theta)+\frac{\epsilon^2}{12}[\Theta\wedge\Theta_x]_{xx}\right\}+O\left(\epsilon^4\right)\\
&=0+\frac{\epsilon^2}{12}\Big\{-\frac{1}{2}u_{xxx}\Theta\wedge\Theta_x\wedge D^{-1}(u_x\Theta)+u_{xxx}D^{-1}(u_x\Theta)\wedge\Theta\wedge\Theta_x\\
&\quad+3u_{xx}u_x\Theta\wedge\Theta\wedge\Theta_x+u_x^2\Theta_x\wedge\Theta\wedge\Theta_x\Big\}+O\left(\epsilon^4\right)\\
&=0+\frac{\epsilon^2}{24}u_{xxx}\Theta\wedge\Theta_x\wedge D^{-1}(u_x\Theta),
\end{aligned}
$$

which can be easily checked that it cannot be expressed as a total divergence. So $S(\epsilon)$ cannot satisfy the Jacobi identity and therefore $S(\epsilon)$ is not a Hamiltonian operator. This completes the proof of (2). ■

Remark 3. Using the technics of the last proof, one can show that J and S is not compatible. Since J and S are Hamiltonian operators, what we are going to do is show that [7,8]

$$\tilde{Q}(\Theta)\wedge\delta R+Q(\Theta)\wedge\delta\tilde{R}\neq 0\quad(\text{mod.}\quad\text{div.}),$$

where

$$
\begin{aligned}
Q(\Theta)&=v_x D^{-1}(v_x\Theta), &\qquad R&=\frac{1}{2}\Theta\wedge Q(\Theta),\\
\tilde{Q}(\Theta)&=\left(\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\right)_x, &\qquad \tilde{R}&=\frac{1}{2}\Theta\wedge\tilde{Q}(\Theta)=-\frac{1}{2v_x^2}\Theta_x\wedge\Theta_{xx}.
\end{aligned}
$$

Then

$$\delta R=\frac{-1}{2}[\Theta\wedge D^{-1}(v_x\Theta)]_x-\frac{1}{2}v_x\Theta\wedge D^{-1}(\Theta_x)=\frac{-1}{2}\Theta_x\wedge D^{-1}(v_x\Theta)$$

and

$$\delta\tilde{R}=-\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)_x.$$

Hence

$$
\begin{aligned}
&\tilde{Q}(\Theta)\wedge\delta R+Q(\Theta)\wedge\delta\tilde{R}\\
&\quad=\left(\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\right)_x\wedge\left(\frac{-1}{2}\Theta_x\wedge D^{-1}(v_x\Theta)\right)-v_xD^{-1}(v_x\Theta)\wedge\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)_x\\
&\quad=\frac{1}{2}\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\wedge\left[\Theta_{xx}\wedge D^{-1}(v_x\Theta)+v_x\Theta_x\wedge\Theta\right]\\
&\qquad+\left[v_{xx}D^{-1}(v_x\Theta)+v_x^2\Theta\right]\wedge\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)\\
&\quad=\frac{1}{2v_x}\Theta_{xx}\wedge\Theta_x\wedge\Theta-\frac{v_{xx}}{2v_x^3}\Theta_x\wedge\Theta_{xx}\wedge D^{-1}(v_x\Theta)\\
&\qquad+\frac{v_{xx}}{v_x^3}D^{-1}(v_x\Theta)\wedge\Theta_x\wedge\Theta_{xx}+\frac{1}{v_x}\Theta\wedge\Theta_x\wedge\Theta_{xx}\\
&\quad=\frac{1}{2v_x}\Theta\wedge\Theta_x\wedge\Theta_{xx}+\frac{v_{xx}}{2v_x^3}\Theta_x\wedge\Theta_{xx}\wedge D^{-1}(v_x\Theta)\\
&\quad\neq 0\quad(\text{mod. div.}),
\end{aligned}
$$

as required.

3 Concluding remarks

• That $J(\epsilon)$ is a Hamiltonian operator (up to $O(\epsilon^4)$) is proved in [1]. We give another proof here, which remarkably simplifies the proof given in [1].
• We notice that all the deformed operators $J(\epsilon)$ (14), $D(\epsilon)$ $\left(= D + O\left(\epsilon^4\right)\right)$, $K(\epsilon)$ (11) under the quasi-Miura transformation (7) are Hamiltonian operators (up to $O\left(\epsilon^4\right)$). That the deformed Sokolov's operator $S(\epsilon)$ is not Hamiltonian is a little surprising that means that the Poisson bracket of the Hamiltonians $H_m(u;\epsilon)$, $H_n(u;\epsilon)$ for $S(\epsilon)$

$$\{H_m(u;\epsilon), H_n(u;\epsilon)\}_{S(\epsilon)}$$

will not be $O\left(\epsilon^4\right)$ but $O\left(\epsilon^2\right)$, i.e., it cannot be a conserved quantity of the Riemann hierarchy (3).

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[1] Chang J.H., Rational approximate symmetries of KdV equation, *J. Phys. A: Math. Gen.*, 2003, V.36, 8025–8034; nlin.SI/0211032.

[2] Dorfman I., The Krichever–Novikov equation and local symplectic structures, *Dokl. Akad. Nauk SSSR*, 1988, V.302, 792–795.

[3] Dorfman I., Dirac structures and integrability of non-linear evolution equations, England, John Wiley & Sons Ltd, 1993.

[4] Dubrovin B. and Zhang Y.J., Normal forms of hierarchies of integrable PDES, Frobenius manifolds and Gromov–Witten invariants, math.DG/0108160.

[5] Lorenzoni P., Deformations of bi-Hamiltonian structures of hydrodynamic type, *J. Geom. and Phys.*, 2002, V.44, 331–375.

[6] Magri F., A simple construction of integrable systems, *J. Math. Phys.*, 1978, V.19, 1156–1162.

[7] Olver P.J., Applications of Lie groups to differential equations, New York, Springer-Verlag, 1993.

[8] Olver P.J. and Nutku Y., Hamiltonian structures for systems of hyperbolic conservational laws, *J. Math. Phys*, 1988, V.29, 1610–1619.

[9] Olver P.J., On the Hamiltonian structure of evolution equations, *Math. Proc. Camb. Phil. Soc.*, 1980, V.88, 71–88.

[10] Krichever I.M. and Novikov S.P., Holomorphic bundles and nonlinear equations. Finite-gap solutions of rank 2, *Dokl. Akad. Nauk SSSR*, 1979, V.247, 33–37.

[11] Sokolov V.V., Hamiltonian property of the Krichever–Novikov equation, *Dokl. Akad. Nauk SSSR*, 1984, V.277, N 1, 48–50.

[12] Strachan I.A.B., Deformations of the Monge/Riemann hierarchy and approximately integrable systems, nlin.SI/0205051.

[13] Wilson G., On the quasi-Hamiltonian formalism of KdV equation, *Phys. Lett. A*, 1988, V.132, 445–450.