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On Equivariant Boundary Value Problems

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We describe an investigation method of general equivariant boundary value problem for PDE of general form in a domain and show how it works for the case of simplest group $O(n,\mathbb{R})$.

1 Solvability conditions of equivariant expansion

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\mathcal{L} = \sum\limits_{|\alpha|\le m} a_\alpha(x)D^\alpha$ be some arbitrary differential operation with smooth coefficients $a_\alpha(x)$, $\mathcal{L}^+$ be a formally adjoint differential operation. Let L_0, L_0^+ be minimal operators (i.e., for example, $D(L_0)$ is the clozure of $C_0^\infty(\Omega)$ in the norm of the graph $\|u\|_L^2 = \|u\|^2_{L_2(\Omega)} + \|Lu\|^2_{L_2(\Omega)}$), and L, L^+ be maximal expansions of $\mathcal{L}$, $\mathcal{L}^+$ in the space $L_2(\Omega)$ respectively (i.e. $L = (L_0^+)^*$, $L^+ = (L_0)^*$), $\tilde L = L|_{D(\tilde L)}$ where $D(\tilde L)$ is the clozure of $C^\infty(\bar\Omega)$ in the norm of the graph $\|u\|_L$ and it is analogous for $\tilde L^+$.

M.Yo. Vishik introduced the conditions:

V_1) the operator L_0: $D(L_0) \to L_2(\Omega)$ has a continuous left-inverse,

V_2) the operator L_0^+: $D(L_0^+) \to L_2(\Omega)$ has a continuous left-inverse

and proved that

1) these conditions are necessary and sufficient for the existence of a solvable expansion L_B: $D(L_B) \to L_2(\Omega)$, (that is $D(L_0) \subset D(L_B)$, $\exists\, L_B^{-1}$: $L_2(\Omega) \to D(L_B)$);

2) under conditions V_1), V_2) for any solvable expansion L_B the following decomposition of the domain $D(L)$ is valid: $D(L) = D(L_0) + \ker L + B$, and L: $B \to \ker L^+$ is an isomorphism.

Let G be some Lie group, smoothly acting in the closed domain $\bar\Omega$. It means, that there is a group of diffeomorphisms U_g: $\bar\Omega \ni x \to g\cdot x = U_g(x) \in \bar\Omega$ of domain $\bar\Omega$ onto itself, group, smoothly depending on an element of G, and mapping $g \to U_g$ is a homomorphism of groups. Thus the contraction of diffeomorphisms U_g on boundary $\partial\Omega$ induces a smooth action of group G on boundary $\partial\Omega$.

The action of group G on domain $\bar\Omega$ generates a representation of the group G in function spaces: $(gu)(x) = u(g^{-1}x)$ (homomorphism of group G into group of converted operators). Such representation is induced on spaces $C_0^\infty(\Omega)$, $C^\infty(\Omega)$, $H^m(\Omega)$, $H^{-m}(\Omega)$, $\mathcal{D}'(\Omega)$, $H^{(m)}(\Omega)$, $H^{(-m)}(\Omega)$ and others. Let the differential operation $\mathcal{L}$ be invariant with respect to the action of group G, that is $g(\mathcal{L}u) = \mathcal{L}(gu)$. Then spaces $D(L)$, $D(L_0)$, $C(L)$, $\ker L$ are invariant with respect to the action of the group.

If the action of group preserves the volume of the domain Ω then the scalar product in the space $L_2(\Omega)$ is invariant with respect to the action of group G, and consequently the representation of the group G is unitary in this space. In this case the operation $\mathcal{L}^+$ is also invariant with respect to an action of group G, the spaces $D(L^+)$, $D(L_0^+)$, $C(L^+)$, $\ker L^+$ are invariant.

Boundary value problem

$$Lu = f, \qquad \Gamma u \in B, \tag{1}$$

generated by a subspace $B \subset C(L)$ of the boundary space $C(L) = D(L)/D(L_0)$ we shall name G-**invariant**, if the space B is invariant with respect to the indicated action of group G. A G-invariant boundary value problem we will name **equivariant**, if it is clear what group acts.

If the group G is compact (and is continuous), then, as it is well known, the Hilbert space of representation is decomposed in the direct sum of finite-dimensional invariant subspaces, in which the irreducible representations of group G are induced. And if the group is also commutative, the irreducible representations are one-dimensional.

Let space of a representation of the group G be the boundary space $C(L)$. For the case of compact group we have decompositions

$$C(L) = \sum_{k=0}^{\infty} \oplus \tilde{C}^k, \qquad C(\ker L) = \sum_{k=0}^{\infty} \oplus C^k(\ker L), \qquad B = \sum_{k=0}^{\infty} \oplus B^k.$$

If our G-invariant boundary value problem is well-posed, the decompositions in the direct sum $C(L) = C(\ker L) \oplus B$ imply decompositions in the direct sum $C^k := C^k(\ker L) \oplus B^k = \sum_l \tilde{C}^{k_l}$ with finite-dimensional projectors Π^k: $C^k \to C^k(\ker L)$ along B^k and now check of the well-posedness of a G-invariant boundary value problem can be reduced to check of two properties:

1) $C^k(\ker L) \cap B^k = 0$;

2) $\exists\, \kappa > 0, \quad \forall\, k, \quad \|\Pi^k\|_{C^k} < \kappa$.

Below we shall study a spectrum of an operator of a general well-posed equivariant boundary value problem for the Poisson equation in a disk and in a ball, detecting cases of violation of the well-posedness of the problem, which are expressed in violation of property 1). Thus the fulfilment of property 2) will be assured by the assumed property of the well-posedness of this problem for the Poisson equation.

2 Equivariant boundary value problems for the Helmholtz equation in a disk

Let us conduct evaluations on check of two properties of the well-posedness of a general equivariant boundary value problem in a simplest case. As the group we will choose group of rotations of the plane $SO(2,\mathbb{R})$. It is compact commutative group.

Let us consider the problem (1), where $\mathcal{L} = \Delta$ and L is the maximum operator generated by the Laplace operator Δ, invariant with respect to the action of rotations group, domain $\Omega = K = \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$ is the disk. Let us remark, that here we have $L = \tilde{L}$, i.e. each function from $D(L)$ can be approximated by smooth functions. Let us assume that this boundary value problem is G-invariant and is well-posed.

And we study such boundary value problem for the Helmholtz equation

$$L_\lambda v = \Delta v + \lambda^2 v = g, \qquad \Gamma v \in B,$$

where λ is a complex number. But in the beginning we study the boundary space $C(L_\lambda)$ of the Helmholtz operator and its subspace $C(\ker L_\lambda)$.

Boundary space consists of some pairs of functions $(u|_{\partial\Omega}, u'_\nu|_{\partial\Omega}) \in H^{-1/2}(\partial K) \times H^{-3/2}(\partial K)$, therefore a general boundary condition must have the form

$$Au|_{\partial K} + Bu'_\nu|_{\partial K} = 0$$

with some operators A, B. The G-invariance of this condition means the commutativity of operators A and B with all the representation operators. But, as it is well-known, a rotation invariant linear operator has a form of the convolution with a function. Therefore we will consider boundary value problems of the type:

$$\alpha * u|_{\partial K} - \beta * u'_\nu|_{\partial K} = 0,$$

where $\alpha = \sum \alpha_k e^{ik\tau}$; $\beta = \sum \beta_k e^{ik\tau}$ are functions on the boundary ∂K, $*$ is the convolution on ∂K: $\alpha * \psi = \sum \alpha_k \psi_k e^{ik\tau}$.

The boundary condition for Fourier coefficients of functions $u|_{\partial\Omega}$, $u'_\nu|_{\partial\Omega}$ from the space B can be written in the form

$$\forall\, k \in \mathbb{Z}, \qquad \alpha_k a_k + \beta_k b_k = 0. \tag{2}$$

Let us designate by C^k an image of an enclosure I_k: $\mathbb{C}^2 \to C(L)$ acting by a rule I_k: $(a, b) \to (ae^{ik\tau}, be^{ik\tau})$. The boundary problem sets a subspace B of the space $C(L)$, which, as we see, intersects each space C^k in a straight line. The well-posedness of our problem, i.e. expansion in the direct sum $C(L_\lambda) = B \oplus C(\ker L_\lambda)$, means now that

$$\exists\, A > 0, \quad \forall\, k \in \mathbb{Z}, \quad |\sin(B^k, C^k(\ker L_\lambda))| > A,$$

i.e.

$$\forall\, k, \quad \frac{|\beta_k \lambda J'_k(\lambda) - \alpha_k J_k(\lambda)|}{\sqrt{|\lambda J'_k(\lambda)|^2 + |J_k(\lambda)|^2}} > A > C \quad \text{at} \quad \lambda \neq 0$$

and

$$\frac{|k\beta_k - \alpha_k|}{\sqrt{k^2+1}} > A > 0 \quad \text{at} \quad \lambda = 0.$$

Proposition 1. *The problem* (2)*, which is well-posed for the equation $\Delta u = g$, is well-posed for the equation $\Delta u + \lambda^2 u = g$ at $\lambda \neq 0$ if and only if the following condition holds*

$$\forall\, k, \quad |k\beta_k J_k^1(\lambda) - \alpha_k J_k^2(\lambda)| \neq 0.$$

Proposition 2. *Spectrum of the operator of well-posed boundary value problem* (2) *for the equation $\Delta u = g$ is a set $\bigcup_k \Sigma_k$, where Σ_k is the set of proper values of a form $-\lambda^2$ and λ runs all zeros of the equation*

$$\beta_k \lambda J'_k(\lambda) - \alpha_k J_k(\lambda) = 0 \quad \text{at} \quad \lambda \neq 0. \tag{3}$$

Proposition 3. *Spectrum of the operator of well-posed boundary value problem* (2) *for the equation $\Delta u = g$ is finite-to-one.*

Proposition 4. *Every well-posed G-invariant boundary value problem for the Poisson equation is quite correct, i.e. its solving operator is compact.*

Propositions concerning the same equation on n-dimensional ball have similar formulations but the equation (3) has the following form:

$$\beta_l^1 \lambda J'_{\nu+l}(\lambda) - \beta_l^1 \nu J_{\nu+l}(\lambda) - \alpha_l^1 J_{\nu+l}(\lambda) = 0,$$

where $\nu = \frac{n}{2} - 1$, $l \in \mathbb{N} \cup 0$.

And the corresponding equivariant boundary value problem must have the following form:

$$u|_{\partial\Omega} * \alpha + u'_\nu|_{\partial\Omega} * \beta = 0,$$

where $\alpha = \sum\limits_{l=0}^{\infty} \sum\limits_k \alpha_l^k H_l^k$, $\beta = \sum\limits_{l=0}^{\infty} \sum\limits_k \beta_l^k H_l^k$ are functions on sphere S^{n-1}, which are decomposed in Fourier series, $*$ is convolution on $\partial\Omega$: $\psi * \alpha = \sum\limits_{l=0}^{\infty} \sum\limits_k \psi_l^k \alpha_l^1 H_l^k$, what means, in particular, that we can omit tesseral (not zonal) parts in decomposition α and β.

[1] Burskii V.P., Investigation methods of boundary value problem for general differential equations, Kyiv, Naukova Dumka, 2002 (in Russian).