

On the Use of the Lie–Bäcklund Groups in the Context of Asymptotic Integrability

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A new approach to the problem of asymptotic integrability of physical systems is developed and applied to the KdV equation with higher-order corrections. A central object of the approach is an integrable reference equation, which is constructed by defining a proper Lie–Bäcklund group of transformations and applying it to the leading order equation. It is shown that the solitary wave solutions of the asymptotically integrable equations, derived using asymptotic transformations, fail to approximate solutions of those equations in some (rather wide) range of the soliton and system parameters.

1 Introduction

For many important physical systems, the leading order term in an asymptotic perturbation expansion is given by an integrable nonlinear equation. This implies that those physical systems are integrable at the nontrivial leading orders in an asymptotic sense. The meaning of the term “asymptotic integrability” is that the equation with higher-order corrections is integrable up to a certain order in the asymptotic sense. Asymptotic integrability at certain order is commonly ascertained by developing an asymptotic transformation relating this higher-order equation to an integrable equation. If such a transformation is impossible, the nonintegrable effects (the “obstacles” to integrability [1]) are defined as additional terms appearing in the target equation.

The most extensively studied example is that of weakly nonlinear long waves, when the leading order equation is the Korteweg–de Vries (KdV) equation. The KdV equation first arose as an approximate equation governing the unidirectional small amplitude long waves in inviscid incompressible fluid [2] but later it was introduced in many different physical contexts. If effects of higher-order are of interest, then extension of terms up to the next orders in the (small) wave amplitude, with the weakly nonlinear and weakly dispersive effects being in balance, leads to the extended KdV equation including higher-order corrections. The KdV equation with the first-order corrections is integrable in the asymptotic sense as it has been shown by Kodama [3] (see also [4]) who has found a transformation which maps the perturbed KdV equation to the integrable equation (normal form) obtained by combining the KdV equation with its first commuting flow. In [4], the normal form was introduced up to the second-order corrections. It was found that, in general, the asymptotic integrability cannot be extended to the second-order, and the asymptotic transformation from the second order KdV equation to the equation representing a normal form plus one obstacle was defined.

It is commonly accepted that the asymptotic transformations from the high-order KdV equations to the corresponding integrable equation (if there are no obstacles), or to the form given by the symmetries plus obstacles, can be used for investigating the properties of the high-order KdV solitons (see, e.g. [5, 6]).

In the present paper, we develop an approach allowing us to study the approximation properties of the asymptotic solutions of the higher-order KdV equations that are obtained by Kodama’s asymptotic transformation from the normal form solitary waves. Our approach, as

applied in the context of asymptotic integrability, is aimed at constructing some asymptotically integrable higher-order KdV equations, for which the asymptotic solutions, alternative to those derived via Kodama's transformation, may be defined. The central object of the approach is some reference integrable equation, which depends on a small parameter of the physical system in such a way that its asymptotic perturbation expansion, to a certain order, has a form of the corresponding order KdV equation. The reference integrable equation is constructed by applying the properly defined Lie–Bäcklund group of transformations to the leading order equation.

2 The reference equation

We will consider a physical system that can be described by an asymptotic perturbation expansion with the leading order term given by an integrable nonlinear equation. We will assume the following expansion form with a small parameter ϵ and a leading order term representing a scalar differential equation:

$$F^{(0)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) + \epsilon F^{(1)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k+r)}) \\ + \epsilon^2 F^{(2)}(x, u, u_{(1)}, u_{(2)}, \dots) + \dots + \epsilon^N F^{(N)}(x, u, u_{(1)}, u_{(2)}, \dots) = O(\epsilon^{N+1}), \quad (1)$$

where $x = (x^1, x^2, \dots, x^n)$ are n independent variables, u is the dependent variable, and $u_{(j)}$ denotes the set of all j th-order partial derivatives of u with respect to x .

Following the ideas of the approach, developed in [7] for the point transformations, we will consider the one-parameter (a) group of the Lie–Bäcklund transformations

$$\bar{u} = \phi(x, u, u_{(1)}, u_{(2)}, \dots; a) \quad (2)$$

with (canonical) Lie–Bäcklund infinitesimal generator

$$U = \eta(x, u, u_{(1)}, u_{(2)}, \dots) \frac{\partial}{\partial u}, \quad (3)$$

where the order of derivatives appearing in the generator is, in general, not restricted.

The group (2) is defined from the requirement that the leading order equation transform, under an infinitesimal action of the group (for small values of the group parameter $a = \epsilon$), into the first-order perturbed equation obtained by retaining only the leading order and first-order terms in (1), as follows

$$F^{(0)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) + \epsilon F^{(1)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k+r)}) = 0. \quad (4)$$

To define the group the following steps are made:

(i) The group (2) is applied to the *unperturbed equation* $F^{(0)} = 0$ written in the variables \bar{u} , $\bar{u}_{(1)}$ and so on, as

$$F^{(0)}(x, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \dots, \bar{u}_{(k)}) = 0. \quad (5)$$

As the result, equation (5) is transformed to

$$H(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}; a) = 0 \quad (6)$$

or infinitesimally

$$F^{(0)}(x, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \dots, \bar{u}_{(k)}) = F^{(0)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) \\ + a U^{(k)} F^{(0)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) \Big|_{[F^{(0)}]} + O(a^2) \quad (a \ll 1), \quad (7)$$

where $U^{(k)}$ is the k th extended infinitesimal generator of (3) defined by

$$U^{(k)} = \eta \frac{\partial}{\partial u} + D_j(\eta) \frac{\partial}{\partial u_j} + D_i D_j(\eta) \frac{\partial}{\partial u_{ij}} + \dots \quad (8)$$

and $[F^{(0)}]$ denotes the set of equations including the equation $F^{(0)} = 0$ and its differential consequences:

$$D_i F^{(0)} = 0, \quad D_i D_j F^{(0)} = 0, \quad \dots \quad (9)$$

(ii) It is required that

$$U^{(k)} F^{(0)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) \Big|_{[F^{(0)}]} = F^{(1)}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k+r)}). \quad (10)$$

It yields determining equations for the Lie–Bäcklund group generator $\eta(x, u, u_{(1)}, u_{(2)}, \dots)$.

(iii) With the group generator defined, the finite transformations (2) can be determined as a solution of the Lie–Bäcklund equation

$$\frac{d\bar{u}}{da} = \eta(x, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \dots), \quad \bar{u}|_{a=0} = u(x). \quad (11)$$

It is evident from (i)–(iii) that the group (2) defined in such a way is neither a symmetry group of equation (1) nor a symmetry group of the unperturbed equation.

In some cases, for specific relations among the coefficients of differential monomials in the first-order term of (1), the Lie–Bäcklund equation (11) can be solved in a closed form so that the finite transformation (2) is represented by an analytical function. Using this closed form transformation one can construct the equation (6) in an explicit form (not as a series). This new equation is dependent on a parameter a (for the sake of brevity we will name it the “ a -equation”) and possesses the following properties:

a) The a -equation is integrable, since there exists an exact transformation (inverse to (2)) that converts equation (6) into the leading order integrable equation (5). Therefore any exact solution of equation (5) yields the exact solution of the a -equation.

b) When $a \ll 1$, the a -equation coincides with the initial perturbed equation (1) up to first-order in $a = \epsilon$.

c) In the case when (1) is an evolution equation and the x -derivative terms at each order represent differential polynomials of specific weights, the corresponding order terms in an expansion of the a -equation in series with respect to a have the same differential structure as those in the original equation but with the monomial coefficients specified in a certain way.

In view of the property (c), the a -equation can be considered as a model for the original physical system in a sense, since it contains impacts of all orders of the asymptotic perturbation expansion on the solution. This model may correspond to a real physical system if there are enough freedoms to satisfy the required conditions on the coefficients.

From another point of view, an expansion of the a -equation up to a certain order produces equations that can be treated as asymptotically integrable at this order since the equation expanded is exactly integrable and the terms dropped may be considered as having an asymptotically negligible influence upon the solution.

3 Applications to the higher-order KdV equations

It can be shown with the use of an appropriate perturbation method (e.g., see [2]) that the higher-order correction to the KdV equation has the following expansion form with a small parameter ϵ :

$$u_t + K^{(0)}[u] + \epsilon K^{(1)}[u] + \epsilon^2 K^{(2)}[u] + \dots + \epsilon^N K^{(N)}[u] = O(\epsilon^{N+1}), \quad (12)$$

where $K^{(0)}[u]$ gives the KdV flow in its standard form

$$K^{(0)}[u] = u_3 + 6uu_1 \quad (13)$$

and the next two terms in (12) are given by

$$K^{(1)}[u] = a_1^{(1)}u_5 + a_2^{(1)}u_3u + a_3^{(1)}u_2u_1 + a_4^{(1)}u_1u^2, \quad (14)$$

$$K^{(2)}[u] = a_1^{(2)}u_7 + a_2^{(2)}u_5u + a_3^{(2)}u_4u_1 + a_4^{(2)}u_3u^2 + a_5^{(2)}u_3u_2 + a_6^{(2)}u_2u_1u + a_7^{(2)}u_1u^3 + a_8^{(2)}u_1^3. \quad (15)$$

The subscript t denotes derivatives with respect to time variable and other subscripts denote derivatives of the corresponding order with respect to the space variable x . Each polynomial $K^{(n)}[u]$ has the scaling property with the homogeneous weight $2n + 5$ if one assigns the weight 2 to u and 1 to $\partial/\partial x$.

Following the approach described in the previous section, we consider the one-parameter (a) Lie-Bäcklund group of transformations

$$\bar{u} = \phi(x, t, u, p, u_1, u_2, \dots; a), \quad U = \eta(x, t, u, p, u_1, u_2, \dots) \frac{\partial}{\partial u}, \quad (16)$$

where the nonlocal terms containing

$$p = D^{-1}(u), \quad D^{-1}(\cdot) = \int_{-\infty}^x (\cdot) dx' \quad (17)$$

may appear. The transformation (16) is defined by the requirement that it converts the leading order equation $u_t + K^{(0)}[u] = 0$ written in the variables \bar{u} , \bar{u}_1 and so on, as

$$F^{(0)}(\bar{u}) = \bar{u}_t + \bar{u}_3 + 6\bar{u}\bar{u}_1 = 0 \quad (18)$$

into a new equation $H(u; a) = 0$ which possesses the property that, for small values of the group parameter a , it coincides with the perturbed equation (12) up to first-order in $a = \epsilon$:

$$H(u; a) = u_t + K^{(0)}[u] + aK^{(1)}[u] + O(a^2) \quad (a \ll 1). \quad (19)$$

Applying this requirement (expressed in terms of the group generator by (10)) yields determining equations for the Lie-Bäcklund group generator $\eta(x, t, u, p, u_1, u_2, \dots)$.

It is evident that the generator $\eta(x, t, u, p, u_1, u_2, \dots)$ defined in such a way is not unique since, in addition to the terms depending on the coefficients $a_1^{(1)}, \dots, a_4^{(1)}$ of the differential polynomial $K^{(1)}[u]$, it may include the terms representing symmetries of the leading order equation $F^{(0)}(u) = 0$ with arbitrary coefficients. In what follows, we will deal with a minimal form of the group generator excluding symmetries, since only it yields the transformation generating the a -equation possessing the property (c): the terms of its expansion with respect to a , at each order, represent differential polynomials of the homogeneous weight like (14), (15) and so on. This minimal form is uniquely defined from the determining equations yielded by (10) as

$$\eta = \lambda_1 u^2 + \lambda_2 u_2 + \lambda_3 u_1 p + \lambda_4 x(u_3 + 6uu_1) \quad (20)$$

or equivalently as

$$\eta = \lambda_1 u^2 + \lambda_2 u_2 + \lambda_3 p u_1 - \lambda_4 x u_t, \quad (21)$$

where the parameters $\lambda_1, \dots, \lambda_4$ are expressed through the coefficients $a_1^{(1)}, \dots, a_4^{(1)}$ of $K^{(1)}[u]$ by the relations

$$\begin{aligned}\lambda_1 &= \frac{1}{6} \left(a_4^{(1)} - a_2^{(1)} - 4a_1^{(1)} \right), & \lambda_2 &= \frac{1}{12} \left(a_4^{(1)} - a_3^{(1)} + 6a_1^{(1)} \right), \\ \lambda_3 &= \frac{1}{3} \left(a_2^{(1)} - 8a_1^{(1)} \right), & \lambda_4 &= \frac{1}{3} a_1^{(1)}.\end{aligned}\tag{22}$$

It is seen that the group generator depends explicitly on x but it does not contain t .

With the group generator defined, the finite transformations (16) are determined as a solution of the Lie–Bäcklund equation

$$\frac{d\bar{u}}{da} = \lambda_1 \bar{u}^2 + \lambda_2 \bar{u}_2 + \lambda_3 \bar{u}_1 \bar{p} + \lambda_4 x (\bar{u}_3 + 6\bar{u}\bar{u}_1), \quad \bar{u}|_{a=0} = u(x, t),\tag{23}$$

where, alternatively, the expression for η on the right-hand side, may be defined as in (21). We will concentrate on the cases where the Lie–Bäcklund equation (23) can be solved in a closed form, which correspond to the specific relations between the coefficients $a_1^{(1)}, \dots, a_4^{(1)}$ of the differential polynomial $K^{(1)}[u]$.

In the present paper, we will consider only the following case

$$a_2^{(1)} = 8a_1^{(1)}, \quad a_3^{(1)} = a_4^{(1)} + 6a_1^{(1)}\tag{24}$$

which, according to (22), corresponds to the generator (21) with the coefficients λ_2 and λ_3 vanishing. Then the Lie–Bäcklund equations (23) can be solved to give

$$\bar{u}(x, t; a) = \frac{u(\xi, \tau)}{1 - a\lambda_1 u(x, \tau)}, \quad \xi = x, \quad \tau = t - a\lambda_4 x,\tag{25}$$

where

$$\lambda_1 = \frac{a_4^{(1)}}{6} - 2a_1^{(1)}, \quad \lambda_4 = \frac{a_1^{(1)}}{3}.\tag{26}$$

In (25), \bar{u} is a solution of the KdV equation (18) and u is a solution of the reference equation $H(u; a) = 0$ which possesses the property (19). The equation for $u(\xi, \tau)$ obtained by substituting (25) into (18) is

$$\begin{aligned}H(u; a) &= \frac{1}{a^4 \lambda_1^4 v^4} \left\{ 6a^2 \lambda_1^2 [L(v)]^3 - 6vL(v) [1 + a^2 \lambda_1^2 L^2(v)] \right. \\ &\quad \left. + a\lambda_1 v^2 [a\lambda_1 v_\tau - 6L(v) + a\lambda_1 L^3(v)] \right\} = 0,\end{aligned}\tag{27}$$

where

$$v = u - \frac{1}{a\lambda_1}, \quad L = \frac{\partial}{\partial \xi} - a\lambda_4 \frac{\partial}{\partial \tau}.\tag{28}$$

The a -equation (27) treated as an equation for $u(x, t)$ is dependent on a – not only explicitly but also through the independent variable τ as is defined in (25). It possesses the properties (a)–(c) formulated in the previous section. Solutions of the a -equation (27) can be obtained from solution of the integrable KdV equation (18) by the transformation (inverse to (25)):

$$u(x, t; a) = \frac{\bar{u}(x, z)}{1 + a\lambda_1 \bar{u}(x, z)}, \quad z = t + a\lambda_4 x,\tag{29}$$

where $\bar{u}(x, z)$ satisfies $\bar{u}_z + \bar{u}_3 + 6\bar{u}\bar{u}_1 = 0$.

Despite the fact that the maximal order of derivatives in equation (27) is three – the same as in the leading order equation, its expansion in to series with respect to a includes the terms with higher-order derivatives (which appear due to the a -dependence of the independent variable τ), and those high-order terms have the same differential structure as the corresponding order terms in the original equation (12). However, the monomial coefficients in the higher-order terms are now specified in a certain way.

In the expansion of equation (27), the coefficients in the first-order term are specified according to (24). We will also write out the relations defining the monomial coefficients in the second-order term

$$\begin{aligned} a_1^{(2)} &= \frac{4}{3}a_1^{(1)2}, & a_2^{(2)} &= 14a_1^{(1)2}, & a_3^{(2)} &= \frac{1}{3}a_1^{(1)} \left(84a_1^{(1)} + 5a_4^{(1)} \right), \\ a_4^{(2)} &= \frac{4}{3}a_1^{(1)} \left(18a_1^{(1)} + a_4^{(1)} \right), & a_5^{(2)} &= \frac{2}{3}a_1^{(1)} \left(66a_1^{(1)} + 5a_4^{(1)} \right), \\ a_6^{(2)} &= 36a_1^{(1)2} + 10a_1^{(1)}a_4^{(1)} + \frac{1}{6}a_4^{(1)2}, & a_7^{(2)} &= \frac{1}{6}a_4^{(1)2}, & a_8^{(2)} &= \frac{1}{6}a_4^{(1)} \left(12a_1^{(1)} + a_4^{(1)} \right). \end{aligned} \quad (30)$$

4 Asymptotic integrability

The analysis made in [4] is based on a near identity transformation $T_\epsilon : v \rightarrow u$

$$u = T_\epsilon(v) = v + \epsilon\Phi^{(1)}(v) + \dots \quad (31)$$

such that the perturbed KdV equation (12) is transformed to

$$v_t + K^{(0)}[v] + \epsilon G^{(1)}[v] + \epsilon^2 G^{(2)}[v] + \dots + \epsilon^N G^{(N)}[v] = O(\epsilon^{N+1}), \quad (32)$$

where $G^{(n)}(v)$ are differential polynomials of the same structure as the corresponding polynomials in (12). In particular, if the calculations are aimed at defining obstacles to the asymptotic integrability, then $G^{(n)}(v)$ are represented as

$$G^{(n)}(v) = a_1^{(n)} K_0^{(n)}(v) + R^{(n)}(v), \quad R^{(n)}(v) = \sum_{i=1}^{\Delta(n)} \mu_i^{(n)} R_i^{(n)}(v). \quad (33)$$

Here $K_0^{(n)}(v)$ are symmetries of the leading order KdV equation. The number $\Delta(n)$ in (33) is the total number of the conditions $\mu_i^{(n)} = 0$ (number of obstacles) needed to be satisfied for existence of a transformation to the normal form containing only symmetries.

At order ϵ , the transform T_ϵ is given by the first-order term $\Phi^{(1)}$ of the generating function. There is no obstacles to asymptotic integrability at the first-order. The transform T_ϵ at the next orders is defined via an expansion of the generating function Φ in the power series of ϵ [4]. At order ϵ^2 , there is one obstacle to asymptotic integrability, which means that one condition on the coefficients $a_1^{(1)}, \dots, a_4^{(1)}, a_1^{(2)}, \dots, a_8^{(2)}$ needs to be satisfied for existence of a transformation to the normal form.

We will study the approximate solitary wave solutions for the KdV equation (12) of a specific form, namely, one with the first- and second-order corrections given by the terms of an expansion of the a -equation (27) in series with respect to a . There is no obstacle to asymptotic integrability of this equation via Kodama's transformation to the normal form – the relations (24) and (30), defining the expansion of the a -equation (27) satisfy the condition for vanishing the obstacle.

The one-soliton solution of the leading order KdV equation and the corresponding solution of the second-order normal form are

$$u = 2k^2 \operatorname{sech}^2 [k(x - 4k^2 t)], \quad (34)$$

$$v = 2k^2 \operatorname{sech}^2 \left\{ k \left[x - \left(4k^2 + 16\epsilon a_1^{(1)} k^4 + 64\epsilon^2 a_1^{(2)} k^6 \right) t \right] \right\}, \quad (35)$$

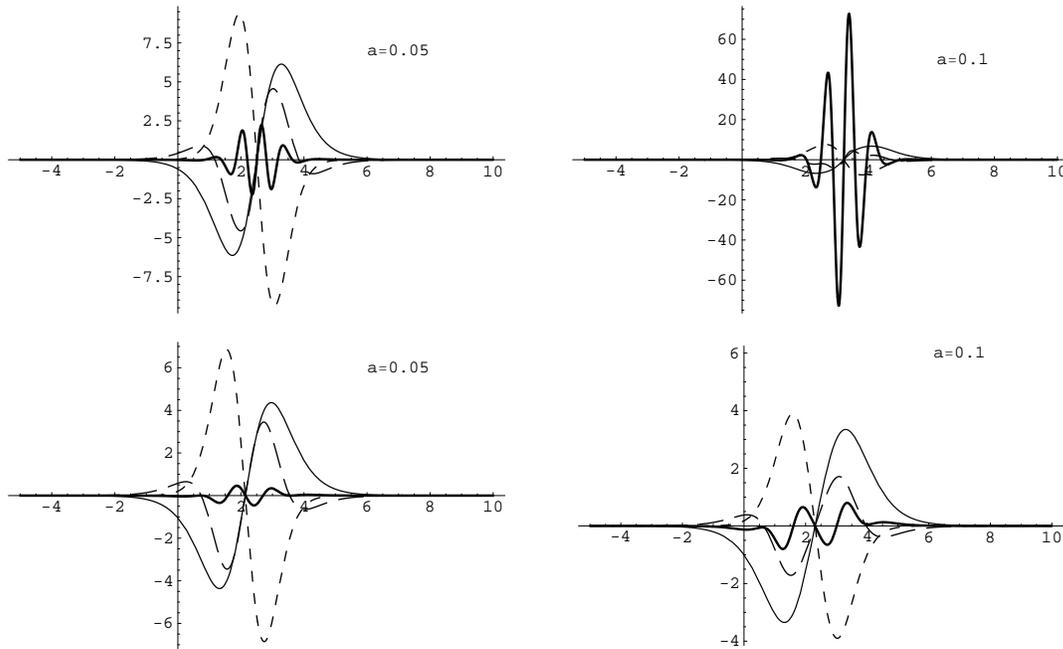


Figure 1. A residual error for the solutions of the second-order KdV equation defined by (24) and (30) in comparison with the terms of the leading order KdV equation (thick solid for the error, thin solid for u_t , short-dashed for $6uu_x$ and long-dashed for u_{xxx}): for the solution (36) obtained by Kodama’s transformation ($\epsilon = a$) from the normal form solitary wave (top) and for the solution (37) of the a -equation (bottom). For both solutions $a_1^{(1)} = 1, a_4^{(1)} = 30, k = 1$.

where a phase constant has been omitted for the sake of simplicity. Applying Kodama’s transformation to the solution (35) yields

$$\begin{aligned}
 u(x, t) = & 2k^2 \operatorname{sech}^2 Z + \frac{2}{3} \epsilon k^4 \left(16a_1^{(1)} - a_4^{(1)} + 4a_1^{(1)} \cosh 2Z \right) \operatorname{sech}^4 Z \\
 & + \frac{2}{9} \epsilon^2 k^6 \left(282a_1^{(1)2} - 32a_1^{(1)} a_4^{(1)} + a_4^{(1)2} + 8a_1^{(1)} \left(19a_1^{(1)} - a_4^{(1)} \right) \cosh 2Z \right. \\
 & \left. + 14a_1^{(1)2} \cosh 4Z \right) \operatorname{sech}^6 Z, \quad Z = k \left[x - \left(4k^2 + 16\epsilon a_1^{(1)} k^4 + \frac{256}{3} \epsilon^2 a_1^{(1)2} k^6 \right) t \right].
 \end{aligned} \tag{36}$$

It is readily checked that this solution satisfies the second-order KdV equation with the coefficients defined by (24) and (30) if the terms of the order ϵ^3 are dropped.

Applying the transformation (29) to the solution (34) yields the corresponding one-soliton solution of the a -equation (27), as follows:

$$u = \frac{12k^2}{3 + 2a \left(a_4^{(1)} - 12a_1^{(1)} \right) k^2 + 3 \cosh \left\{ 2k \left[x - 4k^2 \left(t + \frac{aa_1^{(1)}x}{3} \right) \right] \right\}}. \tag{37}$$

The results presented in Fig. 1 and Fig. 2 demonstrate that the solitary wave solutions, constructed from the normal form solitons using a near identity transformation, although being formally asymptotic solutions of the higher-order KdV equations, have very bad approximation properties for the values of the soliton wave-number k larger than one. Even for $k = 1$ an acceptable approximation is achieved only for values of the expansion parameter $\epsilon \sim 10^{-2}$. For $k = 2$ (the second soliton in Fig. 2) the solution is invalid already for $\epsilon < 10^{-2}$.

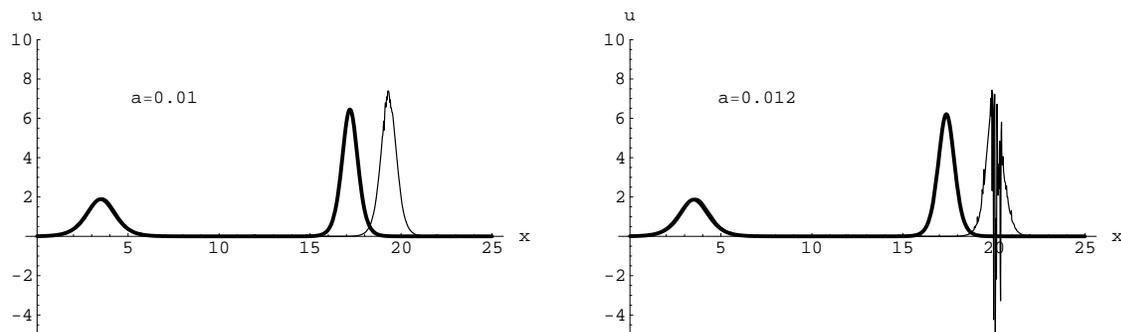


Figure 2. Comparison of different two-soliton ($k_1 = 1$; $k_2 = 2$) solutions of the KdV equation defined by (24) and (30) at $t = 1$: thick solid – the solution of the a -equation; thin solid – the solution obtained by Kodama's transformation ($\epsilon = a$) from the second-order normal form two-soliton solution. For both solutions $a_1^{(1)} = 1$, $a_4^{(1)} = 30$.

It should be noted in this connection that the solutions constructed via Kodama's transformation are ones of an excessive accuracy. For example, in the one-soliton solution (36), one should have taken $Z = k \left[x - \left(4k^2 + 16\epsilon a_1^{(1)} k^4 \right) t \right]$ in the terms multiplied by ϵ , and $Z = k (x - 4k^2 t)$ in the terms multiplied by ϵ^2 . However, the approximation properties for such a solution were found to be even worse than those for (36).

We also conclude, based on some features of the solitary wave solutions of the a -equation (not discussed here), that the solutions constructed via Kodama's transformation cannot (because of the k -dependence of their approximation properties) reflect some solution features intrinsic for the original physical system.

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