Internal Symmetry from Division Algebras in Pure Spinor Geometry

P. BUDINICH

International School for Advanced Studies, Trieste, Italy
E-mail: *fit@ictp.trieste.it*

The É. Cartan's equations defining "simple" spinors (renamed "pure" by C. Chevalley) are interpreted as equations of motions for fermion multiplets in momentum spaces which, in a constructive approach based bilinearly on those spinors, result compact and Lorentzian, naturally ending up with a ten-dimension space. The equations found are most of those traditionally adopted ad hoc by theoretical physics in order to represent the observed phenomenology of elementary particles. In particular, it is shown how the known internal symmetry groups, in particular those of the standard model, might derive from the 3 complex division algebras correlated with the associated Clifford algebras. They also explain the origin of charges, the tendency of fermions to appear in charged-neutral doublets, as well as the origin of families. The adoption of the Cartan's conjecture on the non-elementary nature of Euclidean geometry (bilinearly generated by simple or pure spinors) might throw light on several problematic aspects of particle physics.

1 Foreword

It is well known that fermions and bosons are to be conceived as the quanta of spinor- and of Euclidean tensor-fields, respectively. Now it is generally agreed that bosons may be always bilinearly represented in terms of fermions which then appear as the elementary constituents of matter. The consequent physical assumption is that bosons should be bound states of fermions, having in mind the example of nuclei, say, bound states of nucleons.

However this interpretation appears some times not viable; as in the case of the photon which, notoriously, may not be conceived as a bound state of neutrinos as proposed a long time ago by L. de Broglie. Then one may shift back the attention to the classical fields, before quantization, and try to see if possibly the Euclidean tensor fields might be steadily bilinearly represented in terms of spinor fields, and, consequently, the physical, somehow naive, explanation in terms of bound states might be substituted by the purely geometrical one which, by itself, may offer aspects of great interest and of deep meanings, since it could reveal, among others, a striking parallelism between geometry and physics, as we will see. This way was attempted by several distinguished authors, starting from W. Heisenberg.

In this paper we will try to show that indeed the geometrical way might be the right one, provided the spinors adopted are those which É. Cartan named simple [1], later renamed pure by E. Chevalley [2]. In fact É. Cartan himself advocated the conjecture that Euclidean geometry might have to be conceived as not elementary insofar its elementary constituents might be represented by simple spinor.

In the following we will assume the reader familiar with spinor geometry rich of published literature [1–4] and try to concentrate on the synthetic exposition of some results.

2 Basic definitions and propositions

Given a space $W = \mathbb{C}^{2n}$ with Clifford algebra $\mathbb{C}(2n) = \operatorname{End} S$ with generators γ_a obeying:

$$[\gamma_a, \gamma_b]_+ = 2\delta_{ab}, \qquad a, b, = 1, 2, \dots, 2n,$$

let $\psi_D \in S$ represent a 2^n-dimensional (Dirac) spinor. For $z \in W$, the Cartan's equation:

$$z_a \gamma^a \psi_D = 0, \qquad a = 1, 2, \ldots, 2n \tag{1}$$

defines a totally null plane of dimension d, indicated in the following with $T_d(\psi_D)$.

The volume element γ_{2n+1} defined by: $\gamma_{2n+1} = \gamma_1 \gamma_2 \cdots \gamma_{2n}$ anticommutes with all γ_a and $\gamma_1, \gamma_2, \ldots, \gamma_{2n}, \gamma_{2n+1}$ generate $\mathbb{C}\ell(2n+1)$ whose associated Pauli spinors will be indicated with ψ_P. We will further indicate with φ_W the Weyl spinors defined by:

$$\varphi_W^{\pm} = \frac{1}{2}(1 \pm \gamma_{2n+1})\psi_D; \tag{2}$$

they are 2^{n-1}-dimensional and associated with the even subalgebra $\mathbb{C}\ell_0(2n)$ of $\mathbb{C}\ell(2n)$. The corresponding Cartan–Weyl equations will be:

$$z_a \gamma^a (1 \pm \gamma_{2n+1})\psi_D = 0, \qquad a = 1, 2, \ldots, 2n. \tag{3}$$

Definition 1. A Weyl spinor φ_W associated with $\mathbb{C}\ell_0(2n)$ is simple or pure if the dimension d of the associated totally null plane: $T_d(\varphi_W)$ is maximal; that is equal n.

É. Cartan proved that a simple spinor φ_W is equivalent to T_n (φ_W) (up to a sign) and he stressed the importance of this equivalence; in so far it establishes the fundamental link between spinor-geometry and a specially elegant and simple sector of Euclidean geometry: the projective one; from which presumably derives the qualification "simple" for the corresponding spinors; now however substituted in the literature with the word "pure" later introduced by Chevalley.

Observe that while the dimension of $T_n(\varphi_W)$ increases linearly with n, those of φ_W increase with n as 2^{n-1}; therefore for high n, simple φ_W will have to be subject to constraint relations; and in fact all Weyl spinors are simple for $n = 1, 2, 3$ while for $n = 4, 5, 6, 7$ simple spinors are subject to $1, 10, 66, 364$ constraint relations respectively.

We will represent spinors with one column matrices, and γ_a with square ones, and then let us define the main automorphism B of $\mathbb{C}\ell(2n)$: $B\gamma_a = \gamma_a^t B$; $B\psi = \psi^t B$ where γ_a^t and ψ^t means γ_a and ψ transposed.

For $\psi, \phi \in S$ we have [5]:

$$\psi \otimes B\phi = \sum_{j=0}^{n} T_j = \sum_{j=0}^{n} [\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_j}] T^{a_1 a_2 \cdots a_j}, \tag{4}$$

where the products of γ_a matrices are antisymmetrized and where:

$$T_{a_1 a_2 \cdots a_j} = \frac{1}{2^n} \langle B\phi, [\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_j}] \psi \rangle. \tag{5}$$

Set in (4) $\phi = \psi = \varphi_W$ and we have:

Proposition 1. φ_W is simple iff in equation (4):

$$T_0 = T_1 = \cdots = T_{n-1} = 0, \quad \text{while} \tag{6}$$

$$T_n = \frac{1}{2^n} \langle B\varphi_W, [\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_n}] \varphi_W \rangle \neq 0. \tag{7}$$

Equations (6) represent the constraint relations while T_n represents the maximal t.n. plane bilinear in φ_W and equivalent, up to a sign to φ_W simple.

Let us now multiply equation (4) on the left by γ_a and apply it to $\gamma_a \psi$; if we sum over a and set it to zero we obtain:

$$\gamma_a \psi \otimes B\phi \, \gamma^a \psi = z_a \gamma^a \psi = 0, \qquad a = 1, 2, \ldots, 2n \tag{8}$$

that is equation (1) where:

$$z_a = \frac{1}{2^n}\langle B\phi, \gamma_a\psi\rangle. \tag{9}$$

We have now the fundamental:

Proposition 2. *Let $z \in W = \mathbb{C}^{2n}$ with components $z_a = \langle B\phi, \gamma_a\psi\rangle$; for ϕ arbitrary, $z_a z^a = 0$ iff $\psi := \varphi_W$ is simple or pure.*

The formal proof is in [5]; however there is also a geometrically visible one. In fact it is obvious that z, defined by (9), represents the intersection of the planes $T_d(\phi)$ and $T_{d'}(\varphi_W)$ now for φ_W simple $d' = n$ is maximal and then z must be null, viceversa if this has to be true for every ϕ, it has to be $d' = n$, maximal, and φ_W is simple.

Let us now consider the isomorphism of Clifford algebras:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \tag{10}$$

both central simple, and:

$$\mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \tag{11}$$

both non-simple, from which we have the isomorphisms and subsequent embeddings of Clifford algebras:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \hookrightarrow \mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \hookrightarrow \mathbb{C}\ell(2n+2) \tag{12}$$

and the corresponding ones for the associated spinors:

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W \oplus \psi_W = \Psi_D \simeq \psi_D \oplus \psi_D \tag{13}$$

which implies that a Dirac or Pauli spinor is isomorphic to a doublet of Dirac or Pauli or Weyl spinors. These isomorphisms may be represented explicitly. In fact let γ_a be the generators of $\mathbb{C}\ell(2n)$ and Γ_A with $A = 1, 2, \ldots, 2n+2$, those of $\mathbb{C}\ell(2n+2)$. Then for

$$\Gamma_a^{(m)} = \sigma_m \otimes \gamma_a, \qquad m = 0, 1, 2, 3, \quad a = 1, 2, \ldots, 2n, \tag{14}$$

where $\sigma_0 = 1$ and σ_1, σ_2, σ_3 are Pauli matrices, the corresponding spinor ψ associated with $\mathbb{C}\ell(2n+2)$ is:

$$\Psi^{(m)} = \begin{pmatrix} \psi_1^{(m)} \\ \psi_2^{(m)} \end{pmatrix}, \tag{15}$$

where, for $m = 0$; 1, 2, and 3 $\psi_1^{(m)}$ and $\psi_2^{(m)}$ are Dirac, Weyl and Pauli spinors respectively. Now defining the unitary operators:

$$U_m = 1 \otimes L + \sigma_m \otimes R = U_m^{-1}, \qquad m = 0, 1, 2, 3, \tag{16}$$

where

$$L = \frac{1}{2}(1 + \gamma_{2n+1}), \qquad R = \frac{1}{2}(1 - \gamma_{2n+1}) \tag{17}$$

we have:

$$U_j\Gamma_A^{(0)}U_j^{-1} = \Gamma_A^{(j)}, \qquad U_j\Psi^{(0)} = \Psi^{(j)}, \tag{18}$$

as easily verified [6], from which

Proposition 3. *Dirac and Pauli spinors may be isomorphically represented by Dirac, Pauli or Weyl spinor doublets.*

Propositions 2 and 3 represent the basic geometrical tools for our job. All we have to do is to restrict the above to real spaces, which will result unambiguously Lorentzian, and then to read the Cartan–Weyl equations we obtain in physical terms. We will see that indeed the Euclidean tensor fields will be bilinear in spinor fields as already anticipated by Proposition 2; and then their quanta: the bosons, bilinear in fermions, even in the case they can not be bound states. In fact we will obtain also Maxwell's equations from neutrino Weyl equations. Obviously the geometrical way derives from the basic properties Euclidean spaces as generated by simple spinors in the frame of Cartan's conjecture. Physical fermions may well be represented by non simple spinors, for which the bound state approach will be relevant, as in the case of nuclei.

We will start from the simplest and transparent case of $n = 1$ and then find out the rule for going from n to $n + 1$.

3   The elementary case of $n = 1$. The signature

Start from $\mathbb{C}\ell(2)$ and let $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$ represent two of its associated Dirac spinors, or Pauli spinors of the isomorphic $\mathbb{C}\ell_0(3)$, generated by the Pauli matrices σ_1, σ_2, σ_3. Insert them in equation (4), where now $B = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := \epsilon$, which becomes:

$$\begin{pmatrix} \varphi_0\psi_1 & -\varphi_0\psi_0 \\ \varphi_1\psi_1 & -\varphi_1\psi_0 \end{pmatrix} \equiv \varphi \otimes B\psi = z_0 + z_j\sigma^j \equiv \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix} \tag{19}$$

from which we easily get both the z-vector components bilinear in the spinors ψ and φ : $z_\mu = \frac{1}{2}\psi^t\epsilon\sigma_\mu\varphi$ (compare the matrices) and the nullness of the vector z : $z_\mu z^\mu = z_0^2 - z_1^2 - z_2^2 - z_3^2 \equiv 0$ (compute the determinants of the matrices) in agreement with Proposition 2.

In order to restrict to the real z_0 and z_j, of interest for physics, we need to introduce the conjugation operator C defined by: $C\gamma_a = \bar{\gamma}_a C$, $C\varphi = \bar{\varphi}C$, where $\bar{\gamma}_a$ and $\bar{\varphi}$ mean γ_a and φ complex conjugate. Then equation (19) may be expressed, and uniquely, [6] in the form:

$$\begin{pmatrix} \varphi_0\bar{\varphi}_0 & \varphi_0\bar{\varphi}_1 \\ \varphi_1\bar{\varphi}_0 & \varphi_1\bar{\varphi}_1 \end{pmatrix} = p_0 + p_j\sigma^j = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} \tag{19$'$}$$

and now

$$p_\mu = \varphi^\dagger\sigma_\mu\varphi, \qquad \mu = 0, 1, 2, 3, \tag{20}$$

where $\varphi^\dagger$ means φ Hermitian conjugate. Then we have, again identically:

$$p_\mu p^\mu = p_0^2 - p_1^2 - p_2^2 - p_3^2 \equiv 0 \tag{21}$$

which shows how p_μ are the components of a null or optical vector of a momentum space with Minkowski signature. This is a particular case of application of Proposition 2. In fact imbed $\mathbb{C}\ell_0(3)$ in the non simple $\mathbb{C}\ell(3)$ isomorphic to $\mathbb{C}\ell_0(1,3)$ with generators $\gamma_\mu = \{\sigma_1 \otimes 1, -i\sigma_2 \otimes \sigma_j\}$ and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1$. Then we may identify the above Pauli spinor with one of the two Weyl spinor defined by

$$\varphi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi, \tag{22}$$

where ψ is a Dirac spinor associated with $\mathbb{C}\ell(1,3)$. Then equation (20) identifies with one of the two:

$$p_\mu^{(\pm)} = \tilde{\psi}\gamma_\mu(1 \pm \gamma_5)\psi, \qquad \mu = 0, 1, 2, 3, \tag{23}$$

where $\tilde{\psi} = \psi^\dagger\gamma_0$.

Now the vectors $p^\pm$ are null or optical because of Proposition 2, since the Weyl spinors $\varphi_\pm$ are simple or pure. The corresponding Cartan–Weyl equations will be:

$$p_\mu\gamma^\mu(1 \pm \gamma_5)\psi = 0 \tag{24}$$

which may be expressed in Minkowski space-time if p_μ are interpreted as generators of Poincaré translations: $p_\mu \to i\frac{\partial}{\partial x_\mu}$. They identify, after introduction of the Planck's constant, with the known wave equation of motion for massless neutrinos.

Observe that in this unique derivation, obtained by merely imposing the reality of the p_μ components, Minkowski signature derives from quaternions, as may be seen already from equations (19) and (19′) and from their correlation with Clifford algebras, in fact notoriously $\mathbb{C}\ell(1,3) = H(2)$, where H stands for quaternions. One might then affirm that Minkowski signature is the image in nature of quaternions.

It is interesting to observe that if we define the electromagnetic tensors F with components

$$F_{\mu\nu}^{(\pm)} = \tilde{\psi}[\gamma_\mu, \gamma_\nu](1 \pm \gamma_5)\psi \tag{25}$$

we obtain, from Cartan–Weyl equation (24) the Maxwell's equations in empty space[1] [6]:

$$p_\mu F_+^{\mu\nu} = 0, \qquad \epsilon_{\lambda\rho\mu\nu}p^\rho F_-^{\mu\nu} = 0. \tag{26}$$

It is easy to see that also the electromagnetic potential may be bilinearly expressed in terms of Weyl spinors and then, in the quantised theory, the photon will result bilinear in neutrinos. However, now the bound-state assumption will not be necessary, and furthermore, as well known, it would not work.

In the last part of this section we naturally operated the transition from $n = 1$ to $n = 2$, we have now only to generalize it. To this end it is easy to show that the same results may be obtained starting from the neutral Clifford algebra $\mathbb{C}\ell(1,1)$ which also brings to $\mathbb{C}\ell(1,3)$ as above.

4   The rule: from n to $n + 1$

Let $\psi_D \in S$ for $\mathbb{C}\ell(1, 2n - 1) = \text{End } S$ with generators γ_a. Define the Weyl spinors $\varphi_W^\pm$ as in equation (2) and let them be simple or pure, then because of Proposition 2:

$$p_a^\pm = \tilde{\psi}_D\gamma_a(1 \pm \gamma_{2n+1})\psi_D, \qquad a = 1, 2, \ldots, 2n \tag{27}$$

define null vectors in $\mathbb{R}^{1,2n-1}$.

Now we have $\varphi_W^+ \oplus \varphi_W^- = \psi_D$ and correspondingly: $p_a^+ + p_a^- = p_a = \tilde{\psi}_D\gamma_a\psi_D$ which are the components of a non-null vector, which however is the projection in $\mathbb{R}^{1,2n-1}$ of a null vector of $\mathbb{R}^{1,2n+1}$ with real components:

$$P_A = \tilde{\Psi}\Gamma_A(1 + \Gamma_{2n+3})\Psi, \qquad A = 1, 2, \ldots, 2n + 2 \tag{28}$$

---

[1]Also the inhomogeneous Maxwell's equations in the presence of external electromagnetic sources may be obtained from spinor geometry [7].

with $\Psi \in S$, $\mathbb{C}\ell(1, 2n+1) = \text{End } S$, generated by Γ_A, which defines the Weyl (simple) spinors

$$\psi_W^{\pm} = \frac{1}{2}(1 \pm \Gamma_{2n+1})\Psi_D. \tag{29}$$

It may be shown [6] that the real components P_A may be written in the form: $P_a = p_a = \tilde{\psi}_D \gamma_a \psi_D$; $P_{2n+1} = i\tilde{\psi}_D \gamma_{2n+1} \psi_D$; $P_{2n+2} = \tilde{\psi}_D \psi_D$.

Therefore the rule is:

$$\varphi_W^+ \oplus \varphi_W^- = \psi_D \tag{30}$$

and

$$p_a^+ \oplus p_a^- \hookrightarrow P_A = \{p_a, P_{2n+1}, P_{2n+2}\}. \tag{31}$$

The corresponding Cartan–Weyl equation is:

$$P_A \Gamma^A (1 \pm \Gamma_{2n+3})\Psi_D = 0, \qquad A = 1, 2, \ldots, 2n+2 \tag{32}$$

which, because of Proposition 3, may be set in the form [6]:

$$(P^a \gamma_a^{(m)} + iP_{2n+1}\gamma_{2n+1}^{(m)} \pm P_{2n+2})\psi^{(m)} = 0, \qquad m = 0, 1, 2, 3 \tag{33}$$

and a similar one for the signature $(2n+1, 1)$:

$$(P^a \gamma_a^{(m)} + P_{2n+1}\gamma_{2n+1}^{(m)} \pm iP_{2n+2})\psi^{(m)} = 0, \qquad m = 0, 1, 2, 3. \tag{33'}$$

To interpret physically equations (33) and (33') we have only to interpret the first four P_μ as generators of Poincaré translations: $i\frac{\partial}{\partial x_\mu}$ by which Minkowski space-time is generated as a homogeneous space, and then consider both the spinor $\psi^{(m)}$ and P_j, with $j > 5$ as functions of x_μ; the latter representing external fields (bilinear in spinors).

We see then that the rule which derives from the generalization of the natural first step of Section 3, and coherent with Cartan's conjecture, foresees steadily the appearance of Lorentzian signatures for all values of n.

Remark 1. Observe that in equations (33) or (33') the P_A components, have to define a null vector of $\mathbb{R}^{1,2n+1}$ or $\mathbb{R}^{2n+1,1}$, in order to have for $\psi^{(m)}$ non-null solutions. This condition is geometrically implied by Proposition 2 if we adopt for our momentum space Cartan's conjecture on the fundamental role of simple spinors. The corresponding momentum space will then result compact and equivalent to the Poincaré invariant mass-sphere:

$$\pm P_\mu P^\mu = M_n^2 = P_5^2 + P_6^2 + \cdots + P_{2n+2}^2 \tag{34}$$

whose radius increases with n; that is, with the dimension 2^{n-2} of the fermion multiplet we are dealing with. In such space, in the quantized field theory there will not be ultraviolet divergences.

Observe that in Section 3 we naturally operated the transition from $n = 1$ to $n = 2$ which is a particular case: for $n = 1$, of the general rule defined in this section. In that case in terms of Clifford algebras we went, de facto, from $\mathbb{C}\ell(1, 1)$ to $\mathbb{C}\ell(1, 3)$.

Now let us recall the Bott periodicity theorem on Clifford algebras, stating that: $\mathbb{C}\ell(n + 8, m) = \mathbb{C}\ell(n, m + 8) = \mathbb{C}\ell(n, m) \otimes \mathbb{R}(16)$.

If we apply it to our case we should increase n, step by step from $n = 1$ up to $n = 5$ arriving at

$$\mathbb{C}\ell(1, 1) \otimes \mathbb{R}(16) = \mathbb{C}\ell(1, 9) = \mathbb{C}\ell(9, 1) = \mathbb{R}(32) \tag{35}$$

since $\mathbb{C}\ell(1,1) = \mathbb{R}(2)$, after which the cycle, because of the periodicity theorem, will be repeated. Now it happens that it is precisely $\mathbb{R}^{1,9}$ the higher dimensional space which is generally adopted to explain the main features of elementary particle physics. This coincidence might not be accidental.

We will now concisely list the features which are naturally and uniquely emerging at the various steps of our construction and, somehow surprisingly, we will see that they reproduce most of the known elementary particle properties. Several of these appear as due to the known correlations of Clifford algebras with division algebras.

5   The steps from $n = 2$ to $n = 5$

We will now study equations (33) or (33′) for increasing n: from 2 to 5, and try to interpret them as equations of motions (for $p_\mu \to i\frac{\partial}{\partial x_\mu}$) for fermions or fermions multiplets. In this journey we will naturally find the equations, traditionally postulated ad hoc, in order to represent elementary phenomena of fermions. We will find them more or less in the same order as they were historically postulated: from the isospin symmetry for nuclear forces up to the $SU(3)$ ones for quarks; however now they derive from pure spinor geometry and, in particular, internal symmetries appear to originate from the division algebras correlated with the corresponding Clifford algebras, but we will also obtain indications on the possible geometrical origin of charges, families and some other features.

$n = 2$. Equation (33′) for $\mathbb{C}\ell(3,1) = R(4)$ may represent Majorana spinors [6].

$n = 3$. Equation (33′) for $m = 0$ is: $(p_a\Gamma^a + p_7\Gamma_7 + ip_8)\,N = 0$, where Γ_a $(a = 1, 2, \ldots, 6)$ are the generators of $\mathbb{C}\ell(5,1)$, $N = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, since $m = 0$, is a doublet of $\mathbb{C}\ell(3,1)$ Dirac spinors. It may be written in the form:

$$(p_\mu 1 \otimes \gamma^\mu + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M)N = 0, \tag{36}$$

where the pion field $\vec{\pi} = \frac{1}{8}\tilde{N}\vec{\sigma} \otimes \gamma_5 N$ is bilinear in N. Equation (36) well represents (for $p_\mu \to i\frac{\partial}{\partial x_\mu}$) the pion nucleon equation of motion with isospin symmetry $SU(2)$ clearly of quaternionic origin. However now $SU(2)$ is not the covering of $SU(3)$; it is generated by the reflection operators Γ_5, Γ_6, Γ_7, which are the same as those of the conformal group [6]. Obviously one might also obtain from equation (36) the equation of motion for the pion field in a similar way as Maxwell's equations were derived from massless neutrino equation (24) in Section 3.

From equation (36) one may also derive the role of complex numbers at the origin of the electric charge. In fact, let us write (36) explicitly for ψ_1 and ψ_2:

$$\begin{aligned} (p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,\psi_1 + \gamma_5\,(p_5 - ip_6)\,\psi_2 = 0, \\ (p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\,(p_5 + ip_6)\,\psi_1 = 0\,. \end{aligned} \tag{37}$$

after defining: $p_5 \pm ip_6 = \rho e^{\pm\frac{i\omega}{2}}$, where $\rho^2 = p_5^2 + p_6^2$, we obtain:

$$\begin{aligned} (p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,e^{i\frac{\omega}{2}}\psi_1 + \gamma_5\rho\psi_2 = 0, \\ (p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\rho e^{i\frac{\omega}{2}}\psi_1 = 0 \end{aligned} \tag{37′}$$

which presents a $U(1)$ phase invariance of ψ_1 generated by $J_{56} = \frac{1}{2}[\Gamma_5, \Gamma_6]$ which being local induces a covariant derivative and (36) may be written in the form:

$$\left\{\gamma_\mu\left[i\frac{\partial}{\partial x_\mu} + \frac{e}{2}\,(1 - i\Gamma_5\Gamma_6)\,A_\mu\right] + \vec{\pi}\cdot\vec{\sigma}\otimes\gamma_5 + M\right\}\begin{pmatrix} p \\ n \end{pmatrix} = 0, \tag{38}$$

where we set $\psi_1 = p$, $\psi_2 = n$, well representing the proton-neutron doublet interacting with the pion and with the electromagnetic potential A_μ.

$n = 4$. Equation (33′) for $m = 0$ is:

$$(p_A G^A + p_9 G_9 + ip_{10})\Theta = 0, \qquad A = 1, 2, \ldots, 8, \tag{39}$$

where G_A are generators of $\mathbb{C}\ell(7,1)$ and $\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is a doublet of $\mathbb{C}\ell(5,1)$ Dirac spinors.

It is easy to see [6] that N_1 (or N_2) presents a $U(1)$ covariance generated by $J_{78} = \frac{1}{2}[G_7, G_8]$ at the origin of a charge for N_1 from which N_2 (or N_1) is free. If interpreted as the strong charge, the N_1 may represent a baryon doublet and N_2 a lepton one (from which the similarity of lepton-baryon families, discussed below).

But we may also obtain a non-Abelian gauge field. We will illustrate it in the case of the possible geometrical origin of the electroweak model. In fact supposing $\Theta = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ to represent leptons (see next step: $n = 5$) and, starting from equation (33), assume the generators G_A of the form (we selected the signature $(1, 7)$):

$$\begin{aligned} G_\mu &= 1_2 \otimes \sigma_3 \otimes \gamma_\mu, & G_{5,6} &= i\sigma_1 \otimes \sigma_{1,2} \otimes 1_4, \\ G_7 &= i\sigma_1 \otimes \sigma_3 \otimes \gamma_5, & G_{8,9} &= i\sigma_{2,3} \otimes 1_2 \otimes \gamma_5. \end{aligned} \tag{40}$$

Then from equation (39) we may easily derive:

$$p_\mu \sigma_3 \otimes \gamma^\mu L_{1R} + (p_9 - p_{10})L_{1L} + \left(p_8 + \frac{i}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma}\right) L_{2L} = 0,$$

$$p_\mu \sigma_3 \otimes \gamma^\mu L_{2R} + (p_9 + p_{10})L_{2L} + \left(p_8 - \frac{i}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma}\right) L_{1L} = 0, \tag{41}$$

where $\frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = p_5\sigma_1 + p_6\sigma_2 + p_7\sigma_3$ and L_{jL}, L_{jR}; represent, for $j = 1, 2$, left-handed and right-handed projections of L_1 and L_2 respectively. Now defining

$$p_8 \pm \frac{i}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma} = \rho e^{\pm \frac{i}{2}\boldsymbol{\omega} \cdot \boldsymbol{\sigma}} = \rho e^{\pm \frac{q}{2}}, \tag{42}$$

where $q = i\boldsymbol{\omega} \cdot \boldsymbol{\sigma}$ represents an imaginary quaternion, it is easy to see that L_{1L} presents a phase invariance $e^{i\frac{q}{2}} \to e^{i\frac{q'}{2}}$ from which L_{2L} is free. Since q is local this gives rise to a non-Abelian covariant derivative

$$D_\mu = \frac{\partial}{\partial x_\mu} + i\boldsymbol{\sigma} \cdot \boldsymbol{w}_\mu \tag{43}$$

which if, applied to the electron e and left-handed neutrino ν_L doublet: $L_1 = \begin{pmatrix} e \\ \nu_L \end{pmatrix}$ gives origin to the equation:

$$\left(i\frac{\partial}{\partial x_\mu} + m\right) \begin{pmatrix} e \\ \nu_L \end{pmatrix} + \boldsymbol{\sigma} \cdot \boldsymbol{w}_\mu \gamma^\mu \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} + B_\mu \gamma^\mu e_R = 0, \tag{44}$$

which is the strating equation for the electroweak model (equation (44) may also be obtain directly from Proposition 3 [6]).

$n = 5$. Equation (33′) for $m = 0$ is:

$$(p_\alpha \mathcal{G}^\alpha + P_{11}\mathcal{G}_{11} + iP_{12})\Phi = 0, \qquad \alpha = 1, 2, \ldots, 10. \tag{45}$$

With $\mathcal{G}_\alpha$ generators of $\mathbb{C}\ell(9,1)$ and $\Phi = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$ a doublet of $\mathbb{C}\ell(7,1)$ Dirac spinors. Again, Θ_1 has a $U(1)$ phase invariance generated by $J_{9,10} = \frac{1}{2}[\mathcal{G}_9, \mathcal{G}_{10}]$ not presented by Θ_2; we will then assume $\Theta_1 := \Theta_B$ to represent a quadruplet of baryons and $\Theta_2 := \Theta_\mathcal{L}$ a quadruplet of leptons.

6   The baryon quadruplet Θ_B

Θ_B obey equation (39), therefore it defines the invariant mass of equation (34):

$$-p_\mu p^\mu = M_4^2 = p_5^2 + p_6^2 + p_7^2 + p_8^2 + p_9^2 + p_{10}^2 \tag{46}$$

which defines a sphere S_5 presenting a symmetry $SO(6)$ orthogonal to the Poincaré group. Therefore the maximal internal symmetry for the baryon quadruplet Θ_B could be $SU(4)$. However before this there is an internal symmetry originating from the maximal complex division algebra: that of octonions. In fact it was shown [8] that $\mathrm{Spin}\,(1,9) = \mathrm{Spin}\,(9,1) \cong SL(2,\mathbf{o})$ where $\mathbf{o}$ stands for octonion (even if the isomorphism is restricted to the infinitesimal groups).

In order to set it in evidence let us first remind that equation (39) for Θ_B derives from the Cartan–Weyl equation (32) which in our case is:

$$p_\alpha \mathcal{G}^\alpha (1 + \mathcal{G}_{11})\Phi = 0, \qquad \alpha = 1, 2, \ldots, 10 \tag{47}$$

and, since Θ_B presents an $U(1)$ local covariance it will determine a covariant derivative in space-times such that equation (47) may be written in the form:

$$\left[i\left(\frac{\partial}{\partial x_\mu} - ig A_{(m)}^\mu \right) \mathcal{G}_\mu^{(m)} + \sum_{j=5}^{9} p_j \mathcal{G}_j^{(m)} + ip_{10} \right] \Phi^{(m)} = 0, \tag{48}$$

where the indices m derive from Proposition 3 as in equation (33').

It may be shown [6] that the isomorphism with the octonion algebra may be represented through the $\mathcal{G}_\alpha$ matrices. Precisely $\mathcal{G}_\mu^{(n)}$ and $\mathcal{G}_{6+n}$ for $n = 1, 2, 3$ may represent the first 3 octonion imaginary unit e_1, e_2, e_3 respectively while $i\mathcal{G}_{11}$ represent the seventh one e_7, such that the projector $\frac{1}{2}(1 + \mathcal{G}_{11})$ selects a particular direction in octonion space. This then reduces the automorphism group G_2 of octonions to $SU(3)$. In this way the so-called complex octonions may be defined:

$$U_\pm = (1 \pm \mathcal{G}_{11}), \qquad V_{\mu\pm}^{(n)} = \mathcal{G}_\mu^{(n)} U_\pm, \qquad V_\pm^{(n)} = \mathcal{G}_{6+n} U_\pm. \tag{49}$$

They represent an $SU(3)$ invariant algebra [9]. Precisely $U_\pm$ transform as singlets and $V_{\mu+}^{(n)}$ and $V_+^{(n)}$ transform as the (3) representation of $SU(3)$ while $V_{\mu-}^{(n)}$ and $V_-^{(n)}$ as the ($\bar{3}$) one. In this way equation (48) may be set in the form:

$$\left[i\left(\frac{\partial}{\partial x_\mu} - ig A_{(n)}^\mu \right) V_{\mu+}^{(n)} + p_5 \mathcal{G}_5 + p_6 \mathcal{G}_6 + \sum_{n=}^{3} p_{6+n} V_+^{(n)} ip_{10} \right] U_+ \Phi = 0, \tag{50}$$

where the term with $V_{\mu+}^{(n)}$ may be interpreted to represent $SU(3)$ color and the one with $V_+^{(n)}$ represents $SU(3)$ flavors. Equation (50) has no direct physical interpretation since the vector component like p_{6+n} are bilinear in spinors. Formally we could obtain $SU(3)$ flavor covariance if we would define $p_{6+n} = \Phi^\dagger \mathcal{G}_0 V_-^{(n)} \Phi$.

But it should also be possible to obtain the known Gell–Mann 3×3 representation of the pseudo-octonion algebra [10] by acting with the 3 operators corresponding to $V_+^{(n)}$ on Cartan-standard spinors, or equivalently, on the vacuum of a Fock representation of spinor space as in Ref. [5] to obtain, as minimal left ideals, 3 spinors representing quarks, as will be discussed elsewhere. In this way the $SU(3)$ symmetry both of flavor and color might be obtained in the framework of the algebraic theory of spinors, as already obtained by other authors [11, 12].

According to the present geometrical scheme a fourth quark should exist presenting with the other 3 on $SU(4)$ symmetry. It could be discovered at higher energies.

7 The lepton quadruplet $\Theta_{\mathcal{L}}$

We have seen that $n \to n + 1$ means doubling the dimension of spinor space and adding two more terms to the equations of motion. This means that in our geometrical scheme dimensional reduction means $n \to n - 1$ and is equivalent to reducing to one half the dimension of spinor space and decoupling of the equations of motions. Therefore the quadruplet $\Theta_{\mathcal{L}}$, missing strong charge, will have to be reduced to a doublet; as in fact it appears in nature where lepton always appear as charged-neutral doublets. Now it may be easily seen that this dimensional reduction, because of Proposition 3, may be operated in 3 non-equivalent ways giving origin to 3 lepton-neutrino families differing in the values of the invariant masses [6]. And in nature electron, muon and τ lepton seem to differ mainly in masses. It may be shown that the 3 families may be correlated with the 3 imaginary units of quaternions [13].

Observe that, if $\Theta_{\mathcal{L}} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ where L_1 and L_2 are doublets of leptons, as we have seen in Section 4, for $n = 4$, it is foreseen that if L_1 presents electroweak interactions, L_2 do not. Therefore L_2 needs a further dimensional reduction and decoupling of the equations of motion; that is a reduction from $n = 3$ to $n = 2$ and $n = 1$ obtaining the equations for Majorana fermions and neutrinos. They could be the candidates for the explanation of the origin of black matter [6].

8 Concluding remarks and outlook

We have seen how Cartan's simple spinors are appropriate to represent most of the observed properties of fermions. They plainly explain internal symmetries as due to their correlation with Clifford algebras; they explain the origin of charges and the tendency of fermions to appear in charged-neutral pairs or pairs of multiplets, as the baryon-leptons ones, with similar properties; in particular the origin of families.

In this framework the main role of simplicity is to render compact (due to Proposition 2) the momentum spaces where the equations of motion are naturally formulated. This implies that the corresponding quantized theory (in second quantization) will be free from one of the most severe difficulties of quantum field theories; that of ultraviolet divergences.

In this preliminary approach there appear several aspects which deserve further study. One is the physical meaning of equation (34) defining invariant masses (which implicitly contains also the charges) and the possibility it may offer to compute their values. Another is the constraint relations which simple or pure spinors must obey. In our case they are in number 66 for the spinor appearing for $n = 5$ in Section 5 (and in our interpretation representing 4 baryons and 4 leptons), and in number 10 for the $\mathbb{C}\ell(1,9)$-pure spinors considered in Section 6 (in our interpretation representing 4 baryons). It is interesting to observe that this case has already been studied with unexpected positive results in superstring theory [14].

However, this approach might present further interesting aspects. In fact let us assume as a postulate what it indicates: that for the appropriate description of the wave mechanics of fermions we need the geometry of Cartan's simple spinors. Now this mechanics is generally conceived as the constituent of classical mechanics of macroscopic bodies of which fermions are in turn the elementary constituents. Let us now remind that, classical mechanics is well representable with Euclidean geometry (or with its Riemanian generalizations) in space-time and, in this framework, the physical interpretation of quantum mechanics presents known and widely discussed difficulties. But now there could be another possibility, once again obtained by shifting the attention from physics to geometry. In fact if the geometry to be adopted for dealing with fermions is that of Cartan's simple spinors, then, following Cartan, those spinors might represent the elementary constituents of Euclidean geometry and the problem may be

shifted on how, from simple spinors, one may construct the elements of Euclidean geometry. All this may be synthetically represented in the following table, where with Perceptible World we mean that which can be directly perceived with out senses and is then accessible to our ordinary intuition. The arrows indicate embeddings.

	PHYSICAL OBJECTS	MECHANICS	GEOMETRICAL INSTRUMENTS
PERCEPTIBLE WORLD	Macroscopic bodies	Classical mechanics	Euclidean geometry
	↑ 1	↑ 2	↑ 3
ELEMENTARY CONSTITUENTS	Fermions	Quantum mechanics	Cartan's simple spinors

For the embedding 1 there is no problem, (bosons are bilinears of fermions). The embedding 2 is problematic; it could be substituted with the embedding 3 following Cartan's conjecture[2].

In other words the problem of how quantum mechanics may be embedded in classical mechanics, and then understood in its frame, would be shifted from physics to geometry.

Now notoriously, this problem when dealt with in the frame of the embedding 2, that is in the frame of physics, gives rise to known paradoxes; that is to propositions contradicting our ordinary intuition. If we shift the problem to the embedding 3 that is to how one may construct Euclidean geometry from that of Cartan's simple spinors we will have to deal only with abstract mathematical and geometrical objects for which we do not need the steady control of our ordinary intuition[3].

Furthermore, when the progress of theoretical physics is guided by geometry or mathematics it may well happen that apparent paradoxes may arise, as it happened in relativity with the apparent paradoxes of dilatation of proper time[4], and then they are accepted as a appropriate corrections to the errors and limitations of our ordinary intuition.

Some tentative consequence of such a geometrization of the problem may be set in evidence already at this preliminary stage. In fact we have seen that with simple or pure spinors we may only obtain, bilinearly (according to Proposition 2) Euclidean null vectors; the ordinary vectors may then be obtained only as sums or integrals of them; and the latter may represent strings [15]. This might explain the motivation of the necessity of strings, in quantum mechanics. In fact coherently with this approach we cannot introduce in our quantum mechanics the concept of point-event which is a concept of Euclidean geometry in space-time valid only for classical mechanics. Neither we can construct, via Fourier transforms, this concept, since for that we would need an infinite momentum space of which we do not dispose, as seen above. Therefore the resulting quantum mechanics in space-time will have to be fundamentally non local. The rigorous way to represent this non locality should be studied in dealing with the geometrical problem of embedding Cartan's simple spinors in Euclidean geometry and will be dealt with elsewhere.

---

[2]This, among others, would set in evidence a striking parallelism between physics and geometry.

[3]As an example, mentioned above, a simple spinor associated with a complex Euclidean space of dimension $2n$ is equivalent (up to a sign) to a totally null plane of dimension n whose vectors are all null and mutually orthogonal, which is mathematically correct but no easily accessible to our ordinary intuition.

[4]Like the one: it takes about 3 years for light to arrive here from α centauri. I could arrive there tomorrow if I could dispose of a vehicle travelling with a velocity slightly <u>smaller</u> than that of light.

[1] Cartan É., Lecons sur la Theorie des Spineurs, Paris, Hermann, 1937.

[2] Chevalley C., The algebraic theory of spinors, New York, Columbia University Press, 1954.

[3] Budinich P. and Trautman A., The spinorial chessboard, New York, Springer, 1989.

[4] Penrose R. and Rindler W., Spinors and space-time, Cambridge, Cambridge University Press, 1984.

[5] Budinich P. and Trautman A., Fock space description of simple spinors, *J. Math. Phys.*, 1989, V.30, 2125–2131.

[6] Budinich P., From the geometry of pure spinors with their division algebras to fermion physics, *Found. Phys.*, 2002, V.32, 1347–1398.

[7] Hestenes D., Space-time algebra, New York, Gordon Breach, 1987.

[8] Sudbery A., Division algebras, (pseudo)orthogonal groups and spinors, *J. Phys. A: Math. Gen.*, 1984, V.17, 939–955.

[9] Gürsey F. and Tze C.H., On the role of division, Jordan, and related algebras in particle physics, Singapore, World Scientific, 1996.

[10] Okubo S., Introduction to octonion and other non-associative algebras in physics, Cambridge, Cambridge University Press, 1995.

[11] Dixon G.M., Division algebras: octonions quaternions and complex numbers and the algebraic design of physics, Kluwer, 1994.

[12] Trayling G. and Baylis W.E., A geometric basis for the standard-model gauge group, *J. Phys. A: Math. Gen.*, 2001, V.34, 3309–3324.

[13] Manogue C.A. and Dray T., Dimensional reduction, *Mod. Phys. Lett. A*, 1999, V.14, 99–103.

[14] Matone M., Mazzucato L., Oda I., Sorokin D. and Tonin M., The superembedding origin of the Berkovits pure spinor covariant quantization of superstrings, *Nucl. Phys. B*, 2002, V.639, 182–202.

[15] Budinich P. and Rigoli M., Cartan spinors, minimal surfaces and strings, *Nuovo Cimento B*, 1988, V.102, 609–647.