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Asymptotic Irrelevance of the KdV Hierarchy

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All the equations of the KdV hierarchy share many common features, like single and multiple soliton solutions, singular algebraic potentials, etc.; in particular the phase shifts for soliton-soliton scattering are all the same in the chain KdV_1, KdV_2, etc., depending only on the momenta. We exhibit the reason for this behaviour: as these equations are originated as isopectral deformations of the Schrödinger equation in one dimension, the above features are really properties of the Schrödinger operator, and the different KdV equations just prescribe different "speeds", given by the dispersion law, for reaching the asymptotic regime. In particular, we show how the "interaction" is sum of two-body forces, what is the key for the solubility of these equations in the first place, and then calculate the phase shifts from inverse scattering (as a double-Darboux factorization), obtaining the known result, independent of the hierarchy, without ever using any of the KdV equations. We also obtain the Hirota formula for N solitons as a Wronskian, by creating them one by one from the $u(x) = 0$ potential.

1 Isospectral problem for finite matrices

Consider a Hermitian $n \times n$ matrix M and an automorphism $M \to M^{\#}$; as in any mathematical category, we are interested in the orbit $M/\{\mathrm{Aut}\}$. Now in a complete matrix algebra any automorphism is *inner*, that is

$$M \to M^{\#} = UMU^{-1} \tag{1}$$

with U unitary to preserve Hermiticity. The spectrum of M, $\mathrm{Spec}\, M$, is preserved, and we call it a hard datum. The orbit of a prefixed M is

$$\{M\} = U(n)/(\text{isotropy subgroup for } M) \equiv U(n)/H(M); \tag{2}$$

e.g. if $\mathrm{Spec}\, M$ is nondegenerate, that is for a generic M,

$$H(M) = U(1) \times \cdots \times U(1) \qquad \text{and} \qquad \{M\} = \text{Flag manifold} = U(n)/(U(1)^n). \tag{3}$$

So a generic Hermitian matrix is characterised by n different real numbers, the Spectrum, the hard datum, which is deformation invariant and $n^2 - n$ real numbers, variable under deformation, which specify the Orientation of M, and constitute the soft datum. The isospectral deformation changes the orientation but maintains the spectrum; it translates the matrix along the flag manifold.

At any moment the spectrum is retrieved from the powers of traces of the matrix by

$$I_k = (1/k)\,\mathrm{Tr}\, M^k, \tag{4}$$

i.e. $c_1 := \sum \lambda_i = I_1$, $c_2 := \sum_{i<j} \lambda_i \lambda_j = I_1^2/2 - I_2$, etc., where c_i are the elementary symmetric polynomials on the eigenvalues. Alternatively the spectrum is given by the poles of the resolvent

$$\mathrm{Res}\, M(\lambda) = [M - \lambda]^{-1}. \tag{5}$$

In the following we apply this analysis to the Schrödinger operator as a (infinite dimensional local) Hermitian operator.

2 Hard and soft data for the Schrödinger operator

Consider the Schrödinger equation on the real line $\mathbb{R}$; the operator is

$$H = -D^2 + u(x), \tag{6}$$

where $x \in \mathbb{R}$, $D \equiv d/dx$, and $u(x)$ is a real function decaying "fast enough" at $x \to \pm\infty$. There are *three* differences with the above finite case: infinite dimension, presence of continuous spectrum, and differential operator (the most general operator would be an *integral* operator). The data for the Discrete Spectrum are the bound states

$$H\Psi_j(x) = E_j\Psi_j(x), \qquad E_j < 0, \qquad j : 1, 2, \ldots, N, \qquad N+1 \in \mathbb{N}. \tag{7}$$

For $x \to +\infty$

$$\Psi_j(x) \to c_j \exp(-\kappa_j x), \qquad \|\Psi\| = 1, \qquad \kappa_j^2 = -E_j. \tag{8}$$

The *eigenvalues* E_i are the (discrete) hard data; the norming constants c_j localise the bound states; there are position (soft data), not spectral data. Note we take *normalised* discrete eigenfunctions.

Continuous Spectrum. It is contained in the *S-matrix*, which in our $1D$ case is a 2×2 matrix: we have forward/backward amplitudes for the direct and the *zurdo* scattering (zurdo or $\hat{}$: incoming particle form the right):

$$S(k) = \begin{pmatrix} t(k) & b(k) \\ \hat{b}(k) & \hat{t}(k) \end{pmatrix}, \qquad k \in [0, \infty). \tag{9}$$

Our potential is real so time reversal holds; hence $t(k) = \hat{t}(k)$; also by causality $t(k)$ is related to the *discrete* spectrum [1]; hence the forward amplitude is a hard datum, as part of the spectral density. There are also restrictions coming from the *unitarity* of S:

$$\begin{aligned} &S^\dagger S = 1 = SS^\dagger \Rightarrow |t|^2 + |b|^2 = |\hat{t}|^2 + |\hat{b}|^2 = 1, \\ &\operatorname{Arg} t + \operatorname{Arg} \hat{t} (= 2\operatorname{Arg} t) - \operatorname{Arg} b - \operatorname{Arg} \hat{b} = \pi. \end{aligned} \tag{10}$$

Hence the *argument* of either reflected amplitude (say, in direct scattering) is the soft datum for the continuous spectrum. Hence the complete data for the Schrödinger operator are

$\kappa_j (j : 1, \ldots, N)$ and $|b(k)|$ are hard data, fixed by $\operatorname{Spec} H$,
c_j and $\operatorname{Arg} b(k)$ are soft data, fixed by the Orientation of H.

In particular κ_j and $|b(k)|$ fix the *spectral density*. It is to be remarked that the natural "observables": energies E_j and scattering *coefficients* $\sigma_+ = |t(k) - 1|^2$ and $\sigma_- = |b(k)|^2$ are *hard data*.

3 Inverse problem

We just recall the usual procedure to recover the potential from the scattering data, cf. e.g. [2]. There are three steps:

1) Fourier transform of the total (soft + hard) data:

$$B(x) \equiv \frac{1}{2\pi} \int b(k)\, dk \exp(ikx) + \sum_j c_j^2 \exp(-\kappa_j x). \tag{11}$$

2) Solve for the kernel $K(x,y)$ in the Marchenko integral equation

$$K(x,z) + B(x+z) + \int_x^\infty K(x,y)B(y+z)dy = 0. \tag{12}$$

3) Retrieve the potential by

$$u(x) = -2K_x(x,x), \qquad \text{where} \quad K_x \equiv dK/dx. \tag{13}$$

Notice the solution is complete, i.e. we recover the full potential, and hence both the hard and the soft data are used. For example, for the *solitonic potentials* (defined as reflectionless potentials by $b(k) \equiv 0$), it is

$$u(x) = -2\big(d^2/dx^2\big) \log \det A, \tag{14}$$

where

$$A_{mn} = \delta_{mn} + c_n^2[\exp -(\kappa_m + \kappa_n)x]/(\kappa_m + \kappa_n), \tag{15}$$

where the discrete spectrum data κ_j, c_j are given arbitrarily, subject only to all eigenvalues κ_j different. For example for $N = 1$ $\{\kappa, c\}$ we recover the one soliton potential $u(x) = -2\kappa^2 \operatorname{sech}^2\kappa(x - x_0)$, where $E = -\kappa^2$ and $c = \sqrt{2\kappa}\exp(\kappa x_0)$.

The exponential part in (15) is related to the soliton scattering phase shift for any of the KdV hierarchy equations (see later); we have approached it here from inverse scattering theory, with no reference to the hierarchy at all! (15) was first obtained by Kay and Moses [3] from inverse scattering and later by Hirota [4] from his bilinear method. We shall obtain it below also by the Darboux method.

4 General isospectral problem

We know that the KdV equations are isospectral evolution equations; hence, their solutions $u(x) = u(x, t = 0) \to u(x,t)$ are a particular case of the general isospectral problem. It is this: *given a potential $u(x)$, find the most general potential $v(x)$ with the same hard data.* The formal solution of the inverse problem sketched above allows us to set and solve, in principle, the general isospectral problem: There are again three steps:

1) Solve the direct problem for $u(x)$: find

$$\kappa_j, \quad c_j, \quad |b(k)| \quad \text{and} \quad \operatorname{Arg} b(k). \tag{16}$$

2) Change arbitrarily the soft data c_j and $\operatorname{Arg} b(k)$.

3) Solve the inverse problem from the new c and $b(k)$.

Then the new potential $v(x)$ is the most general potential strictly isospectral to $u(x)$.

Some restrictions apply: for example, the new $b(k)$ has to be "nice", i.e. to admit Fourier transform guaranteeing that the new potential $v(x)$ falls off at infinity fast enough. Also c_j have to be positive (by an initial convention).

From the above it is clear that the KdV equations realize part of this program; before seeing it in detail, let us consider the simplest case: to obtain the most general potential isospectral to the standard one-soliton potential

$$u(x) = -2\operatorname{sech}^2(x); \tag{17}$$

it has a single bound state with $\kappa = 1$, $c = \sqrt{2}$ and it is reflectionless, $b(k) = 0$. To change the soft data means just to change $\sqrt{2}$ to an arbitrary c; then the inverse method produces inmediately

$$v(x) = -2\operatorname{sech}^2(x - x_1), \qquad \text{with} \quad x_1 = \log\big(c/\sqrt{2}\big). \tag{18}$$

The only effect on the potential is just a traslation of the "well", where the soliton is located.

We shall see now that two methods: the Double Darboux method (DD) and the KdV hierarchy are equivalent to finding deformed isospectral potentials in somewhat complementary ways from a given potential. We first dispose of the DD method [5].

In the simplest application of the DD method one kills the ground state and reinserts it in another "position"; therefore, it changes only the first norming constant c_1. The procedure is constructive, i.e. the new ground state wavefunction is calculated completely from the original one (see e.g. [6]):

$$\Psi(x,\lambda) = \Psi(x)/[1+\lambda\int_{-\infty}^{x}\Psi^2(x)\,dx], \tag{19}$$

and therefore the new norming constant is (as $\|\Psi\|=1$)

$$c_1' = c_1/(1+\lambda). \tag{20}$$

The iterated DD method repeats the process changing simultaneously the positions of two or more arbitrary bound states (by killing and reinstating them somewhere else), and leaving the phases of the reflection amplitudes unchanged. The procedure can be always implemented (and indeed, not only for fast-decaying potentials), and the solutions are also algebraic, but the procedure is cumbersome and tedious!

The original method was devised by Darboux (1882) [5] to relate linear differential operators, not necessarily of second order. The general application to solutions of the Schrödinger equation was found by Abraham and Moses (1980) [6], Mielnick (1984) [7] and others.

5 The KdV hierarchy

The Korteweg–de Vries equation ($\mathrm{KdV}=\mathrm{KdV}_1$) was discovered in 1895 as an equation exhibiting soliton behaviour, and the hierarchy KdV_n was first presented by P. Lax (1968) [8]. The time evolution of the solutions of the equations of the KdV hierarchy, $u(x)\to u(x,t)$ represents the isospectral change of the Hamiltonian of the Schrödinger equation by an uniparametric group of automorphims whose generator is polynomial in the derivatives.

In the beautiful exposition of Lax (1970) [8], we have

$$H \to H(t) = U(t)HU^{-1}(t). \tag{21}$$

Define $B(t)$ by $B(t)=(dU(t)/dt)U^{-1}(t)$. Then

$$H_t = [B,H], \qquad u_t = K[u], \qquad \text{and} \qquad \Psi_t = B\Psi \tag{22}$$

with $a_t = \partial a/\partial t$ throughout. Then the first equation defines the problem (i.e., to find B). The second equation is the Non-Linear evolution equation (KdV hierarchy). The third equation is important for the evolution of the scattering data.

The KdV hierarchy obtains solving for B order by order in derivatives:

$B=D$: The equation is $u_t=u_x$; the chiral wave equation.

$B=D^3+\{u,D\}$. The equation is, of course

$$u_t = 6uu_x - u_{xxx}, \qquad (\text{Korteweg–de Vries}, 1895) \tag{23}$$

with a cubic dispersion law. There are at least one equation of each odd order of dispersion; for systematics we just write the (natural) quintic equation [9]

$$u_t = -30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx}. \tag{24}$$

The key point why these hierarchical equations can be solved *inspite of being partial and nonlinear* is that *the scattering data of the potential change with the linear part of the operator B only*, which has the highest derivative: the rest is function of the potential u, and goes to zero faster, as u itself decays fast.

Let us interpret the evolution in the original KdV equation, with cubic dispersion law, as an one-parameter isospectral deformation; at time t (=deformation parameter), the hard data are of course invariant:

$$\kappa_j(t) = \kappa_j(0), \qquad |b(k,t)| = |b(k,0)|, \tag{25}$$

but the soft data evolve in a simple manner:

$$c_j(t) = \exp\left(4\kappa_j^3 t\right) c_j(0), \qquad b(k,t) = \exp\left(8ik^3 t\right) b(k,0), \tag{26}$$

i.e. the "bumps" of the bound state wavefunctions move all harmonically, and only the phase of the reflected amplitudes changes. As it is expected, the other equations in the KdV hierarchy just trade, as far as soft data are concerned, the cubic "3" of KdV_1 for any greater odd number.

It is interesting the comparison of the Double Darboux method and the KdV solutions for new potentials:

– In the DD method, we change one position of the discrete spectrum at a time arbitrarily.

– With the KdV equations, we change all the soft data with a particular dispersion law.

6 Irrelevance of the KdV hierarchy

It is common lore that the "nonlinear" character of the KdV equations is the agent responsible for the phase shift in soliton-soliton scattering; the shift is obtained from the asymptotic behaviour, of course; now the motion of the solitons through space means the centers x_i (or the norming constants c_i) move with a particular dispersion law, but of course they are irrelevant at the asymptotic limit: initial $x_i \to -\infty$, final $x_i \to +\infty$. We conclude, therefore, that the phase shift is a hard datum; hence, we should be able to get it directly from the Schrödinger equation; this we show next, limiting ourselves to the case of two solitons.

To create two arbitrary solitons we use the direct Darboux method twice starting from the $u(x) = 0$ solution, the vacuum [10]. The procedure is well known, so we just quote some formulas and results:

Take the $E_1 = -\kappa_1^2$, unphysical solution $\phi_1(x) = \cosh \kappa_1(x - x_1)$. Factorize $H - E_1 = A^\dagger A$, where $A = D + W' = \phi_1 \cdot D \cdot \phi_1^{-1}$ with $\phi_1(x) = \exp(-W)$. The new, first step potential is

$$v_1(x) = -2\kappa_1^2 \operatorname{sech}^2(x - x_1). \tag{27}$$

To create the second solution, with $E_2 < E_1$, take now the free solution $\phi_2(x) = \sinh \kappa_2(x - x_2)$, which becomes a zeroless unphysical solution of the first step potential by

$$\tilde{\phi}_2(x) = A\phi_2(x) = (D - \tanh \kappa_1(x - x_1))\phi_2(x) = \phi_1 \cdot D \cdot \phi_1^{-1}\phi_2(x). \tag{28}$$

The new potential is now

$$v_2(x) = -2D^2 \log \phi_1(x) - 2D^2 \log \tilde{\phi}_2(x) = -2D^2 \log W(\phi_1(x), \phi_2(x)), \tag{29}$$

where W is the *Wronskian* $W = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x)$.

This is an *exact* solution of the two soliton potential; to compute the phase shift we let now $x_2 \to -\infty$ and compare the new center of the remaining (original) soliton x_1' with the original x_1; the phaseshift is twice this value times κ_1 (as Φ is dimensionless): as the second soliton is created

at $x = -\infty$, the first gets repelled and moves a distance $x_1' - x_1$: when the bigger soliton moves to $+\infty$, the old one recedes to $-x_1' - x_1$.The messy algebra leads to a well known expression (recall $\kappa_2 > \kappa_1$)

$$\Phi = \log(\kappa_2 + \kappa_1)/(\kappa_2 - \kappa_1) \tag{30}$$

checked numerically (it is enough to take $\kappa_1 = 1$, and $x_1 = 0$). The second soliton repels the first by a sort of exclusion principle: the bound eigenvalues of the Schrödinger equations are nondegenerate, hence $\kappa_1 = \kappa_2$ would lead to an infinite phase shift, and "by continuity" we get (30); writing it in the form

$$\Phi = 2\,\mathrm{Arg}\,\tanh(\kappa_1/\kappa_2) \tag{31}$$

looks like a very common formula is scattering theory (with arc tan instead; recall κ is like an imaginary wavenumber; our shift is not really a phase shift, as it is not periodic; it is more a space (vs time) delay). For instance, scattering for a point impurity of strength g produces an even phase shift [11] $\mathrm{tg}\,\delta_+ = g/k$.

Formula (30) is of course obtained from the KdV_n equations: the generalization for n solitons leads, from KdV, to the Hirota formula [4]; in our method we get directly the Wronskian for n independent $E < 0$ free solutions; the identity of the Kay–Moses & Hirota formula with the Wronskian we leave to the reader to check; for this is essential to use the affine invariance of the equation (14), namely $\det A = \det BA$, where B is a column matrix with entries $\exp(a_i x + b_i)$, up to terms which die in calculating the potential.

Notice also the full n-soliton interaction is sum of independent two-body "forces", which accounts for the completely solubility of the system.

7 Singular solutions

We can see also how many special solutions, common to all KdV hierarchy equations, are obtained directly by the Darboux method starting again from the $u(x) = 0$ potential. For example, the first step of the $u(x) = 0$, $E = 0$ solution $\phi(x) = ax + b$ produces the centrifugal potential

$$u(x) = +2/(x - x_1)^2 \tag{32}$$

which is also a rational solution of KdV. Iteration with the $\phi_2(x) = x^2$ solution produces e.g.

$$u(x) = 6x\big(x^3 - 24x_1\big)/\big(x^3 + 12x_1\big)^2 \tag{33}$$

which again is a rational solution of KdV (with $t = x_1$) (cf. [9]).

8 Final remarks

Soliton properties, including scattering, are really properties of the Schrödinger operator. In particular, the one-soliton potential is just the partner potential to the vacuum potential $u(x) = 0$ doubled by the (unphysical, negative energy) solution $\phi(x) = \cosh(x)$. The iteration, always with negative energy, will produce the most general multi-soliton solution at any x, hence should reproduce the phase shifts, as we checked for the two soliton case; solitons "forces" are exclusively the effect of non-degeneracy of the normalizable eigenfunctions.

As the solitons move in one direction only, they behave as chiral fields. In the Lagrangian formulation, these chiral fields are rather peculiar, because the space of potentials $\{u\}$ (or rather

of the integrals $\varphi = \int u$, which is what enters in the local Lagrangian) is already the phase space, not the configuration space. This implies, in particular, that in the quantum version the solitons should behave like fermions (Jackiw proved this directly for the chiral, KdV_0 case [12]). This fermi character is a translation, we believe, of the purely classical effect: namely two solitons of close wavenumber cannot superpose (P. Lax), reflecting the case that for equal wavenumber they should be at infinite distance, because the Schrödinger equation does not allow degeneracy in the bound states in one dimension.

We believe also that upon quantization these solitons should behave like nonrelativistic Majorana–Weyl fermions, reflecting the "chiral" behaviour of the classical counterpart. But we have not looked carefully yet to the quantization problem.

[1] Lamb G.L., Elements of soliton theory, New York, J. Wiley, 1980.

[2] Dodd R.K., Eilbeck J.C., Gibbon J.D. and Morris H.C., Solitons and nonlinear wave equations, Academic Press, 1982.

[3] Kay I. and Moses H.B., Reflectionless transmission through dielectrics and scattering potentials, *J. Appl. Phys.*, 1956, V.27, 1503–1508.

[4] Hirota E., Exact solutions for the KdV with multiple solitons, *Phys. Rev. Lett.*, 1971, V.27, 1992–1994.

[5] Ince E.L., Ordinary differential equations, New York, Dover, 1956.

[6] Abraham P.B. and Moses H.E., Changes in potentials due to changes in the point spectrum, *Phys. Rev. A*, 1980, V.22, 1333–1340.

[7] Mielnick B., Factorization method and new potentials with the oscillator spectrum, *J. Math. Phys.*, 1984, V.25, 3387–3389.

[8] Lax P., Integrals of nonlineaar equations of evolution and solitary waves, *Comm. Pur. Appl. Math.*, 1968, V.21, 467–490; Nonlinear partial differential equations, in Actes Int. Congr. Maths. (1970, Nice), 1970, V.2, 831–849.

[9] Drazin P.G. and Johnson R.S., Solitons: an introduction, Cambridge University Press, 1989.

[10] Blecua P., Boya L.J. and Segui A., New soluble potentials related to $V(x) = 0$, *Il Nuov. Cim. B*, 2003, V.118, 535–546; quant-ph/0311139.

[11] Lapidus I.R., Phase shifts by a one-dimensional delta-function potential, *Am. J. Phys.*, 1969, V.37, 930–931.

[12] Florianini R. and Jackiw R., Selfdual fields as charge-density solitons, *Phys. Rev. Lett.*, 1987, V.59, 1873–1876.