Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 645–648

Axially Symmetrical Solutions of the Klein–Gordon–Fock Equation

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It is shown that axially symmetrical solutions of the Klein–Gordon–Fock equation are classified by eigenfunctions of the wave operator that change the sign of eigenfunctions, i.e., the energy spectrum of the wave operator contains a point with a zero energy (frequency). The thermodynamics of the Bose–Einstein condensation is discussed on the basis of the energy spectrum under consideration. There is a phase transition in this mathematically integrable model.

In this note we discuss the thermodynamics of free relativistic spinless particles which are described by the Klein–Fock–Gordon equation (KFG)

$$(\hat{L} + \Delta_\perp - k_0^2)\Psi\left(x, y, z, t\right) = 0, \tag{1}$$

where $\hat{L} = (-1/c^2)\partial^2/\partial t^2 + \partial^2/\partial z^2$ is a wave (hyperbolic) operator, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is an elliptical operator, and $k_0 = (mc/h)$ is the inverse Compton wavelength.

As the eigenfunctions of equation (1) we put those that change the sign of operator $\hat{L}$ [1]

$$\Psi_{(+)} = \phi_{(+)}\left(x, y\right) \exp i\left(-ct\sqrt{k_z^2 + Q^2} + zk_z\right)$$

and

$$\Psi_{(-)} = \phi_{(-)}\left(x, y\right) \exp i\left(-\omega t + z\sqrt{(\omega^2/c^2) + \tilde{Q}^2}\right).$$

It is easy to verify that $\hat{L}\Psi_{(+)} = Q^2\Psi_{(+)}$, $\hat{L}\Psi_{(-)} = -\tilde{Q}^2\Psi_{(-)}$. Substitution of the eigenfunctions $\Psi_{(\pm)}$ in equation (1) gives equations for a definition of functions $\phi_{(\pm)}\left(x, y\right)$ as eigenfunctions of the operator $\Delta_\perp$:

$$\left(\Delta_\perp^2 - k_0^2 + Q^2\right)\phi_{(+)}\left(x, y\right) = 0, \qquad \left(\Delta_\perp^2 - k_0^2 - \tilde{Q}^2\right)\phi_{(-)}\left(x, y\right) = 0,$$

where functions $\phi_{(\pm)}\left(x, y\right)$ have the form

$$\phi_{(+)}\left(x, y\right) = \exp\left(i\left(k_x x + k_y y\right)\right), \qquad k_x^2 + k_y^2 = Q^2 - k_0^2 \geqslant 0,$$
$$\phi_{(-)}\left(x, y\right) = K\left(k_\perp x_\perp\right), \qquad x_\perp = \sqrt{x^2 + y^2}$$

here $k_\perp = \sqrt{k_x^2 + k_y^2}$, $k_\perp^2 = k_0^2 + \tilde{Q}^2 > 0$. The function K_0 is expressed by the modified Bessel function or McDonald function. The wave numbers and angular frequency k_z, $k_\perp$, ω satisfy the dispersion relations for the function $\Psi_{(+)}$:

$$\omega/c = \sqrt{k_z^2 + k_\perp^2 + k_0^2} > 0, \tag{2}$$

for the functions $\Psi_{(-)}$:

$$\omega/c = \sqrt{k_z^2 - k_\perp^2 + k_0^2} \geqslant 0. \tag{3}$$

Dispersion relations of equation (2) and equation (3) are represented by the hyperbolic branches in $(\omega/c, k_z)$ plane at a fixed transversal momentum. It should be noted that the dispersion relation of equation (3) contains the point $\omega = 0$ at $|k_z| = \sqrt{k_\perp^2 - k_0^2} \geqslant 0$, $k_\perp \geqslant k_0$.

The group velocity $\upsilon_z = \partial\omega/\partial k_z = ck_z/\omega$ tends to infinity at $\omega = 0$. The dispersion expression of the type of equation (3) was studied by Migdal [2], where the relativistic limit was considered in the nucleon environment (in the gas approximation). The nucleon environment is a thermostat. In the paper [2] was pointed that there exists such k at which $\omega = 0$ and therefore $\partial\omega/\partial k_z \to \infty$ [2]. Migdal called this phenomenon the effect of a disappearing rest mass and a π-condensation [2].

The aim of this note is to consider the thermodynamics of the model described by dispersion expressions of equation (2) and equation (3) and to study the Bose–Einstein condensation within it.

Let us write the full number of free particles N in the volume V in the Bose–Einstein statistics (BES) [3]

$$N = \frac{Vk_0^3}{2\pi^2}\left(\int g_1(\xi)\, f(\xi, T, \mu)\, d\xi + \frac{1}{2}\int g_2(\xi)\, f(\xi, T, \mu)\, d\xi\right). \tag{4}$$

The following notations have been used: $g_1(\xi) = \xi\sqrt{\xi^2 - 1}$, $\xi = h\omega/mc^2 = \omega/k_0 c$, $g_2(\xi) = \xi\sqrt{\xi^2 + \alpha^2} - \xi^2$, $\alpha = m^*/m$, $m^* = (h/c)\sqrt{Q^2 - k_0^2}$, where m^* is the value depending on the cutting parameter Q. The distribution function f

$$f = (\exp((\xi\theta/T) + (\mu/k_B T)) - 1)^{-1} = \sum_1^\infty \exp(-n((\xi\theta/T) + (\mu/k_B T))),$$

here $\theta = mc^2/k_B$ is the characteristic temperature of the relativistic thermodynamics. For electrons with mass $m \approx 10^{-27}$ g and temperature $\theta \approx 10^{10^0}$ K, k_B is the Boltzmann constant and μ is the chemical potential. An integral of equation (4) with the density $g_1(\xi)$ was calculated by Pauli [4]. The second integral of equation (4) is calculated by the substitution $\xi = \alpha\,\mathrm{sh}(\theta)$ and using the table integral [5]

$$\int_0^\infty d\theta \exp(-q\,\mathrm{sh}\,\theta) = (\pi/2)(\mathbf{H}_0(q) - N_0(q)) = (\pi/2)\, A_0, \tag{5}$$

where $\mathbf{H}_0$ is the Struve function of zero-th order and N_0 is the Bessel function of the second kind.

With the table integral of equation (5) we represent the full number of particles N as the series:

$$N = \frac{Vk_0^3}{2\pi^2}\sum_{n=1}^\infty (K_2(n\theta/T)/(n\theta/T) + (\alpha^3/2)\, B(n\theta^*/T))\exp(-\mu n/k_B T), \tag{6}$$

where $B(q) = (\pi/2)\left((2A_1(q)/q^2 - A_0(q)/q)\right)$, $\theta^* = (m^* c^2/k_B)$. The series of equation (6) can be summarized in the asymptotic at low temperature $T \ll \theta$, therefore we have

$$N \approx \frac{Vk_0^3}{2\pi^2}\left(\sqrt{\frac{\pi}{2}}(T/\theta)^{3/2} F_{3/2}((\theta/T) + \mu/k_B T) + \frac{\alpha}{2}\left(\frac{T}{\theta}\right)^2 F_2\left(\frac{\mu}{k_B T}\right)\right), \tag{7}$$

where $F_p(x) = \sum_1^\infty (\exp(-nx))/n^p$, $F_p(0) = \sum_1^\infty 1/n^p = \zeta(p)$, and $\zeta(p)$ is the zeta Riemannian function. Taking into account that $F_p(x) \to 0$ at $x \gg 1$, we can see that the first term in equation (7) is negligibly small compared with the second one. Therefore in the asymptotic limit at low temperatures $T \ll \theta$ we get an approximate relation

$$N \approx \frac{Vk_0^3}{2\pi^2}\alpha\left(\frac{T}{\theta}\right)^2 F\left(\mu/k_BT\right) = \frac{Vk^*}{4\pi^2}\left(\frac{k_BT}{hc^2}\right)^2 F_2\left(\mu/k_BT\right) \tag{8}$$

with $k^* = (m^*c/h) = \sqrt{Q^2 - k_0^2}$.

The critical temperature of condensation $T = T_c$ is determined from equation (8) on condition that the chemical potential μ is zero and the full number of particles N is a constant

$$N = \frac{Vk^*}{4\pi^2}\left(\frac{k_BT}{hc^2}\right)^2 \zeta\left(2\right) = \text{const}, \qquad T_c \sim \left(\frac{N}{V}\right)^{1/2}, \tag{9}$$

where $\zeta\left(2\right) = \sum\limits_1^\infty 1/n^2 = 1.64$

Define as N_0 a number of particles (condensate) with the zero-th energy and the temperature $T > T_c$: $N = N_0 + N_1$, where

$$N_1 = N\left(T/T_c\right)^2, \qquad T \geqslant T_c,$$
$$N_0 = N\left(1 - T^2/T_c^2\right), \qquad T \leqslant T_c.$$

In the nonrelativistic thermodynamics the condensation depends on the temperature according to the following law: $(N_1/N) = (T/T_c)^{3/2}$.

Let us calculate the internal energy of condensate U, at $T \ll \theta$

$$U = \frac{Vk_0^3}{4\pi^2}mc^2 \int \xi g_2(\xi) f\left(\xi, \theta, \mu = 0\right) d\xi \simeq 1.5Nk_BT\left(T/T_c\right)^2. \tag{10}$$

In equation (10) we have used an approximate equality $2\zeta(3)/\zeta(2) \approx (2 \cdot 1.2)/1.6 = 1.5$, respectively, the capacity C_V will be: $C_V = 4.5Nk_B\left(T/T_c\right)^2$. Expressions of equation (8) and (9) enable us to calculate the chemical potential μ:

$$\left(T/T_c\right)^2 \zeta\left(2\right) = \sum_{n=1}^{\infty} (1/n^2) \exp\left(-\mu n/k_BT\right). \tag{11}$$

In equation (11) we can make an approximate summation up in the vicinity of the critical temperature $\left(T - T_c\right)/T_c \ll 1$ on condition that $(\mu/k_BT) \ll 1$. It gives:

$$\mu \approx 2k_BT\ln(T/T_c), \qquad \text{at} \qquad T - T_c \ll T_c.$$

Now we can calculate the internal energy U and capacity C_V of overcondensated particles in the temperature interval $T - T_c \ll T_c$; $U = 1.5Nk_BT$, $C_v = 1.5Nk_B$. The overcondensate consisting of relativistic bosons with energy $U = 1.5Nk_BT$ has the same behavior as an ideal undegenerated gas.

Using the definition of a Heaviside step we write the internal energy of the condensate of relativistic bosons in the form

$$U = 1.5Nk_BT\left(\left(T/T_c\right)^2 \theta(T_c - T) + \theta(T - T_c)\right). \tag{12}$$

The internal energy U in equation (12) is continuous at temperature $T = T_C$ and has an angular point (break). The capacity C_V has the discontinuity at $T = T_C$ and the value of this discontinuity will be $\Delta C = C_V(T_C + 0) - (T_C - 0) = -3k_bN$. In the nonrelativistic thermodynamics the capacity C_V versus temperature dependence is continuous.

To write an equation of state of a condensate we calculate the thermodynamic potential Ω:

$$\Omega = k_B T \frac{V k_0^3}{4\pi^2} \int g_2(\xi) \ln\left(1 - \exp\left(-\left(\xi\theta/T\right) - \left(\mu/k_B T\right)\right)\right) \approx -U/3. \tag{13}$$

In equation (13) integration by parts was made and an asymptotic of special functions at large values of the argument (i.e., at low temperatures) used. From equation (13) it follows that relations between thermodynamic potential Ω and internal energy U are the same as for massless relativistic particles.

Entropy S is obtained by formula: $S = \int dT C_V(T)/T + \text{const}$, where the constant is determined from the condition of the continuous entropy at $T = T_C$.

$$S = \frac{3}{2} N k_B \left(\frac{3}{2}\left(T/T_C\right)^2 \theta\left(T_C - T\right) + ((3/2) + \ln(T/T_C))\theta\left(T - T_C\right)\right). \tag{14}$$

Equation (14) is true in the interval of temperatures $\left(T - T_C\right)/T_C \ll 1$. Entropy of condensate is smaller than the entropy of an overcondensate, i.e., the condensate is a more ordered state.

Now we have to consider the density of the number of particles N of equation (6) at high temperatures $\theta/T < 1$. Using the asymptotes of special functions at small values of the argument, we have

$$N \approx \frac{V k_0^3}{2\pi^2}\left(\left(T/\theta\right)^3 F_3\left(\mu/k_B T\right) + \frac{\alpha^2}{2}\left(k_B T/\hbar c\right) F_1\left(\mu/k_B T\right)\right). \tag{15}$$

There is no critical temperature in equation (15) because $F_1(0) = \sum\limits_1^\infty \frac{1}{n} \to \infty$.

In conclusion, we have shown that axially symmetrical solution of the Klein–Gordon–Fock equation with equation (3) as the dispersion relation yields a Bose–Einstein condensation at low temperatures.

Acknowledgements

The authors would like to thank Mikhail Belogolovskii for useful discussions and kind cooperation.

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