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Description of Anti-Fock Representation Set of $*$-Algebras Generated by the Relation $XX^* = f(X^*X)$

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In present paper the method of description of anti-Fock representations of the relation $XX^* = f(X^*X)$ for unimodal mapping f was devised. Examples of description of anti-Fock representation sets for piecewise linear-fractional mappings under condition of existence of attracting cycle were given.

1 Introduction

There were many investigations on the representations of $*$-algebras

$$\mathcal{A}_f = \mathbb{C}\langle X, X^* \,|\, XX^* = f(X^*X)\rangle \tag{1}$$

and their enveloping C^*-algebras. Some of them are purely mathematical, some are motivated by physics [4,5]. For example, a general framework was constructed in the paper [6]. Recently there have appeared papers where attention was focused on $*$-algebras connected with a non-bijective simple dynamical system (f, I). In particular, quantum inharmonic oscillator algebra [7] and "unimodal deformation" of a quantum disk algebra [2] were considered. In the present paper we eliminate the condition of simplicity of a dynamical system, and try to classify the anti-Fock representations of $\mathcal{A}_f$.

Description of representations of $\mathcal{A}_f$ is closely connected with description of positive orbits of dynamical system $(f, \mathbb{R})$ [6]. By positive orbits of dynamical system $(f, \mathbb{R})$ we imply sequences

1) $\omega = (x_k)_{k\in\mathbb{Z}}:\ f(x_k) = x_{k+1},\ x_k > 0$ for all k;
2) $\omega = (x_k)_{k\in\mathbb{N}}:\ x_1 = 0,\ f(x_k) = x_{k+1},\ x_k \geq 0$ for $k > 1$ (Fock orbit);
3) $\omega = (x_{-k})_{k\in\mathbb{N}}:\ x_{-1} = 0,\ f(x_{-k}) = x_{-k+1},\ x_{-k} \geq 0$ for $k > 1$ (anti-Fock orbit).

For every positive orbit one may put in correspondence an irreducible representation of $\mathcal{A}_f$ [6], in particular each anti-Fock representation corresponds to each anti-Fock orbit.

The idea of anti-Fock representation description is to construct one-to-one correspondence among anti-Fock orbits and the paths on a certain graph. To realize this idea we introduce in the Section 2 the definitions of P-partition of the dynamical system (X, σ) and corresponding transition graph Γ_P. After that we prove Theorem 1 that establishes one-to-one correspondence between the set of orbits passing through point $x \in X$ and a set of certain paths on a graph Γ_P.

In the Section 3 of this paper we introduce the definition of negative elements of P-partition and prove Theorem 3. It establishes one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to point 0 and do not pass through this vertex and vertices of the set N.

In the Section 4 we give description of anti-Fock representation set of $\mathcal{A}_f$, where $f = f_{l,s,t}$ is piecewise linear-fractional mapping depending on parameters.

2 Partitions

Definition 1. Let (X, f) be a dynamical system. We say that a partition $X = \bigcup\limits_{j \in T} S_j$, $S_i \cap S_j = \varnothing$, $i \neq j$, where index set T generally is not countable, is a P-partition, if the following conditions are held:

1) $\forall\ i \in T$ either $f^{-1}(S_i) = \varnothing$ or there exists a partition $f^{-1}(S_i) = \bigcup\limits_{j \in Q_i} U_j$, $U_i \cap U_j = \varnothing$, $i \neq j$ such that $\forall\ j \in Q_i$ $f(U_j) = S_i$ and f is one to one on U_j;

2) for any two sets U_{j_1}, U_{j_2}, $j_1, j_2 \in Q_i$ there are sets S_{i_1} and S_{i_2}, such that $U_{j_1} \subseteq S_{i_1}$, $U_{j_2} \subseteq S_{i_2}$, $i_1, i_2 \in T$.

We will also assume in this paper that

2′) if $j_1 \neq j_2$ then $i_1 \neq i_2$.

The following graph can be associated with P-partition: $\Gamma_P = \{\Gamma_{P0}, \Gamma_{P1}\}$, where $\Gamma_{P0} = \{v_j : j \in T\}$ is the set of vertices, and $\Gamma_{P1} = \{s_{ij} : i, j \in T,\ S_i \subseteq \sigma(S_j)\}$ is the set of ages.

Theorem 1. *Let (X, σ) be a dynamical system that possesses P-partition. And let Γ_P be a corresponding graph. Then there exists one-to-one correspondence between the set of orbits passing through the point $x \in X$ and the set of infinite paths on the graph Γ_P starting from the vertex that corresponds to the element of the partition containing point x.*

Proof. Let $\Delta_x = \{\{x_k\}_{k\in\mathbb{Z}} : x_{-1} = x\}$ be a set of orbits passing through the point x. Denote $\{v_{j_i}\}_{i\in\mathbb{N}}$, $v_{j_i} \in \Gamma_{P0}$ is a path that starts at vertex v_{j_1}, K_v is the element of P-partition that corresponds to vertex v and $W_x = \{\{v_{j_i}\}_{i\in\mathbb{N}}, x \in K_{v_{j_1}}\}$, is a set of paths that start at vertex v_{j_1} such that corresponding element of P-partition contains point x. Define mapping $\phi : \Delta_x \to W_x$ in the following way: $\phi(\{x_k\}_{k\in\mathbb{Z}}) = \{v_j\}_{j\in\mathbb{N}}$, $x_{-j} \in K_{v_j}$. From conditions 1, 2 in the definition of P-partition it follows that mapping ϕ is well-defined. Let now $\delta_1, \delta_2 \in \Delta_x$, $\delta_1 \neq \delta_2$ $\delta_1 = \{x_k\}_{k\in\mathbb{Z}}$, $\delta_2 = \{y_k\}_{k\in\mathbb{Z}}$. Let k_0 be the least integer such that $x_k = y_k$ for all $k \geqslant k_0$. Then from conditions 2, 2′ in the definition of P-partition it follows that x_{k_0-1}, y_{k_0-1} belong to different elements of P-partition, therefore $\phi(\delta_1) \neq \phi(\delta_2)$. So ϕ is injection.

Let now $p = \{v_{j_i}\}_{i\in\mathbb{N}} \in W_x$ be a path. Then from construction of the graph and from condition 2 in definition of P-partition it follows that there exists a sequence $\{x_{-k}\}_{k\in\mathbb{N}}$, $\sigma(x_{-k-1}) = x_{-k}$, $x_{-1} = x$ such that $x_{-k} \in v_{j_k}$. Therefore $\phi(\ldots, x_{-3}, x_{-2}, x_{-1}, \sigma(x_{-1}), \sigma^{(2)}(x_{-1}), \ldots) = p$. So mapping ϕ is bijection. ∎

Leaving the description of P-partition construction in general case for further investigations we will concentrate on the partitions of interval.

Consider a unimodal dynamical system $(f, [c, d])$ such that

$$f(x) = \begin{cases} f_1(x), & x \in [c, b], \\ f_2(x), & x \in (b, d], \end{cases} \tag{2}$$

$c \leq f(c) < d$, $f_1(b) = f_2(b) = d$, $f(d) = c$.

To simplify our work we rewrite the definition of P-partition in more convenient and simplified form just by choosing the partition of preimage.

Definition 2. Let (f, I) be a dynamical system with mapping f defined in (2). We assume that $f_i^{-1}(x)$ is undefined for some $x \in I$, when there does not exist $x' \in I$ such that $f(x') \stackrel{\text{def}}{=} f_i(x') = x$, $i = 1, 2$.

We say that dynamical system (f, I) possesses a P-partition whenever $I = \bigcup\limits_{j} S_j$, $j \in T$, $S_i \cap S_j = \varnothing$, $i \neq j$, where index set T generally is not countable, and the following conditions are being held.

1. $\forall\, j \in T$ $f_1^{-1}(x)$ and $f_2^{-1}(x)$ are either undefined for all $x \in S_j$ or defined for all $x \in S_j$.

2. a) $\forall\, j \in T$ whenever mappings are defined $f_1^{-1}(S_j) \subseteq S_s$ and $f_2^{-1}(S_j) \subseteq S_t$ for some $s, t \in T$, b) $s \neq t$.

Denote $\delta_{(x)}$ unilateral orbit $(x, f(x), f^{(2)}(x), \dots)$ of the point x.

Definition 3. We say that the partition of the interval on the disjoint elements *is defined by the closed set S*, whenever
1) any point at S is the element of the partition,
2) other elements of the partition are opened intervals.

Proposition 1. *Let (f, I) be a dynamical system with mapping f defined in* (2). *Then any closed set S, such that $f(S) \subseteq S$ and $\delta_{(b)} \subseteq S$ defines a P-partition.*

Proof. First condition in definition of P-partition is fulfilled because $f^{(n)}(d) \in S$, $n \geqslant 0$.

Let's check the second condition. For one-point elements of partition it is clear. Let $I_{ij} = (x_i, x_j)$, $x_i, x_j \in S$, $i \neq j$ be a certain interval of partition. Let suppose that $f_1^{-1}(I_{ij}) \not\subseteq I_{kl}$ for some k and l, hence there exists a point $x \in f_1^{-1}(I_{ij})$, $x \in S$. Since $f(S) \subset S$ then there exists a point $f(x) \in I_{ij}$, $f(x) \in S$. Thus we came to contradiction. For f_2^{-1} is similar. f_1^{-1}, f_2^{-1} can't map different intervals into one because $b \in S$. ■

3 Anti-Fock representation set description

The following theorem (see [6]) connects representations of $*$-algebra A_f with certain orbits of dynamical system $(f, \mathbb{R}_+)$.

Theorem 2 (V. Ostrovskyi V. and Yu. Samoilenko [6]). *There exists the following correspondence between positive orbits of dynamical system $(f, \mathbb{R}_+)$ and irreducible representations of $*$-algebra $\mathcal{A}_f = \mathbb{C}\langle X, X^* \,|\, XX^* = f(X^*X)\rangle$.*

1. There is an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition that corresponds to every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$.

2. There is an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$ that corresponds to positive non-cyclic Fock-orbit $\omega = (x_k)_{k\in N}$.

3. There is an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$ that corresponds to positive non-cyclic anti-Fock-orbit $\omega = (x_{-k})_{k\in N}$.

4. There is a family of m-dimensional irreducible representations $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \dots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \dots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$ that corresponds to cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m.

Note that in the case when mapping f is not polynomial the $*$-algebra $\mathcal{A}_f$ is undefined. Instead of it we may consider $C^*(\mathcal{A}_f)$, a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with sub-norm $\|b\| = \sup\limits_{\pi} \|\pi(b)\|$, where supremum is taken over all $\pi \in \operatorname{Rep}(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure.

Definition 4. We call the element S_i of P-partition positive or respectively negative if $S_i \subset [0, +\infty)$ or respectively $S_i \subset (-\infty, 0)$. Let N be a set of vertices of the graph Γ_P that correspond to negative elements of the partition.

Theorem 3. *Let (f, I) be a dynamical system with mapping f defined in (2), and let Γ_P be a graph associated with P-partition defined by the set $S = \{\delta_{(b)} \cup \delta_{(0)} \cup \omega(\delta_{(b)}) \cup \omega(\delta_{(0)})\}$. Then there exists one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to point 0 and do not pass through this vertex and vertices of the set N.*

Proof. By Proposition 1 the set S defines a P-partition. By Theorem 1 there exists one-to-one correspondence between the set of anti-Fock orbits and the set of infinite paths on the graph Γ_P starting from the vertex that corresponds to the element of the partition containing point 0. Since point 0 belongs to S, all elements of the partition are either positive or negative. Therefore anti-Fock orbits correspond to paths that do not pass through vertex corresponding to 0 and vertices of the set N. ■

Remark 1. We can reduce the number of elements in the partition mentioned in Theorem 3 just by splitting elements $f^n(0)$ with appropriate intervals. For example, 0 with interval $(0, x)$.

Corollary 1. *Let (f, I) be a dynamical system with mapping f defined in (2) and let Γ_P be a graph associated with P-partition defined by the set $S = \{\delta_{(b)} \cup \omega(\delta_{(b)})\}$. Let Q be an element of P-partition such that $0 \in Q$. And let any preimage of zero do not belong to Q or equivalently no closed path starting from corresponding vertex. Then there exists one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to element Q and do not pass through vertices of the set N.*

4 Linear-fractional mappings

Many important examples of $*$-algebras, C^*-algebras and their representations arising in physical models are connected with dynamical systems. In particular, the two-parameter unit quantum disk algebra [4]

$$\mathbb{C}\langle z, z^* \mid qzz^* - z^*z = q - 1 + \mu(1 - zz^*)(1 - z^*z)\rangle,$$
$$0 \le \mu \le 1, \qquad 0 \le q \le 1, \qquad (\mu, q) \ne (0, 1),$$

that can be rewritten [6] in the form $\mathcal{A}_f = \mathbb{C}\langle X,\, X^* \mid XX^* = f(X^*X)\rangle$, where

$$F(\lambda) = \frac{(q+\mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.$$

In present paper we investigate "unimodal deformation" of the relation written above and describe anti-Fock representations of a corresponding $*$-algebra.

Consider a continuous unimodal piecewise linear-fractional map $f : \mathbb{R} \to \mathbb{R}$ that consists of two hyperbolae:

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, & x \leqslant b, \\ f_2(x) = \dfrac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}, & x > b. \end{cases} \tag{3}$$

According to paper [3] we restrict ourselves to the following conditions

$$f_1(x) \text{ is increasing}, \quad f_2(x) \text{ is decreasing}, \quad b > 0, \quad f(b) > b, \quad f(f(b)) < b,$$
$$0 \geqslant f^{(2)}(b), \quad f^{(3)}(b) \geqslant f^{(2)}(b). \tag{4}$$

Under these conditions anti-Fock orbit sets of dynamical systems $(f,\mathbb{R})$ and $(f,[f(f(b)),f(b)])$ coincide. Hence we will consider dynamical systems as defined in (2).

Considering mapping (3) up to the topological conjugacy [8, 1] under condition $(\alpha_1-\delta_1)^2+4\gamma_1\beta_1\geqslant 0$ we may assume $\gamma_1=0$, $\alpha_2=1$.

Thus we restrict ourselves to the following set of mappings (see Fig. 1): $f=f_{l,s,t}:[0,1]\to[0,1]$, $l>0$,

$$f_{l,s,t}(x)=\begin{cases} f_1(x)=lx+a, & x\in[0,b],\\ f_2(x)=\dfrac{x-1}{sx+t}, & x\in(b,1],\end{cases} \tag{5}$$

Conditions $f_1(b)=f_2(b)=1$, $f_2(1)=0$ give us formulae for a and b via l,s,t:

$$a=1-l\frac{t+1}{1-s},\quad b=\frac{t+1}{1-s}.$$

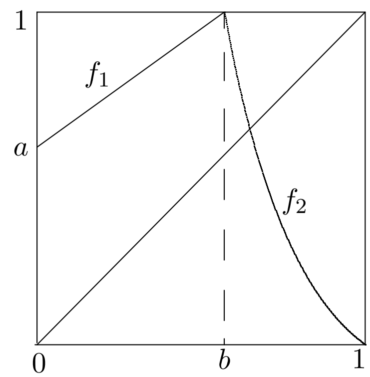


Figure 1

In paper [1] the conditions of existence of stable cycle $\gamma_n=\{x_1,\dots,x_n\}$ such that

$$x_i<x_{i+1},\qquad f(x_i)=x_{i+1},\qquad i=1,\dots,n-1,\qquad f(x_n)=x_1$$

were obtained.

Let us display some necessary results from paper [1].

Conditions $b\in(0,1)$, $f_2'(x)<0$ and $a\in[0,1)$ give us following restrictions:

$$s<1,\qquad -1<t<-s,\qquad t\leqslant\frac{1-s}{l}-1.$$

So we will assume that parameters l, s, t lie in the area

$$\Pi=\left\{(l,s,t):\ l>0,\ s<1,\ t\in\left(-1,\min\left\{-s,\frac{1-s}{l}-1\right\}\right)\right\}.$$

Denote

$$L_n=1+l+l^2+\cdots+l^n=\frac{1-l^{n+1}}{1-l}.$$

Theorem 4. *Mapping f has the cycle γ_n iff*

$$t\geqslant-\frac{l^{n-2}+sL_{n-3}}{L_{n-2}}\quad \text{and it is attracting iff}\quad \begin{cases} s<\dfrac{1}{L_{n-1}},\\ t<\dfrac{l^{n-1}+s(1-2l^{n-1})-s^2L_{n-2}}{sL_{n-1}-1}.\end{cases}$$

We denote the corresponding areas of parameters by Π_n

$$\Pi_n = \left\{ (l,s,t) : \ \frac{l^{n-1} + s(1-2l^{n-1}) - s^2 L_{n-2}}{sL_{n-1} - 1} > t \geqslant \frac{-l^{n-2} - sL_{n-3}}{L_{n-2}}, \frac{1}{L_{n-1}} > s \right\}.$$

The lower border of each area Π_n is defined by the surface of existence, we denote it by E_n. The upper border is defined by surface of stability, we denote it by S_n.

On the Fig. 2 we give section of these areas for $s = -2$.

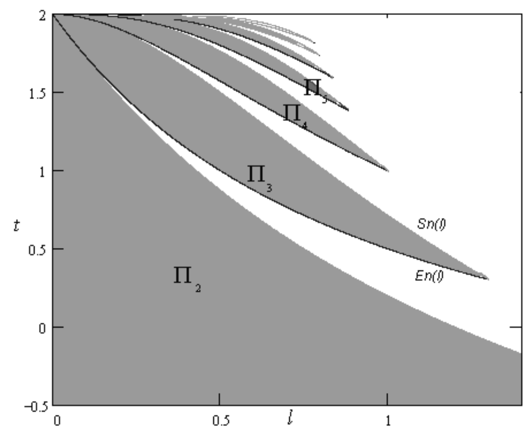


Figure 2.

It is well-known [2], [7] that the structure of anti-Fock representation set as well as the structure of $C^*(\mathcal{A}_f)$ heavily depends on the dynamics of (f, I). We will show that anti-Fock representation set has a good description whenever the corresponding dynamical system has an attracting cycle.

The following theorem from [8] helps us to describe possible attracting cycles of dynamical system generated by piecewise linear mappings ($s = 0$). In Theorem 6 we will give such a description.

Theorem 5. *Let $f : I \to I$ be a unimodal mapping, $f \in C^3(I \backslash \{c\})$ and $Sf(x) = 0$, $x \in I\backslash\{c\}$ ($Sf(x)$ is Schwarzian derivative of $f(x)$, $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$) then the set of all not repellent cycles is either empty or consists of one attracting or semiattracting cycle or represents the cycle of closed intervals $B = \{J_0, J_1, \ldots, J_{n-1}\}$ of some period $n \geqslant 1$ such that $f^{(n)}(x) = x$, $x \in J_0$ and the point c is one of the ends of interval J_0.*

Theorem 6. *Let $f : I \to I$, $I = [c, d]$ be a continuous piecewise linear unimodal mapping such that ($f(b) = \max_{x \in I} f(x) = d$, $f(d) = c$, $f(c) \geq c$). Whenever corresponding dynamical system has a stable cycle then this cycle is of type γ_n.*

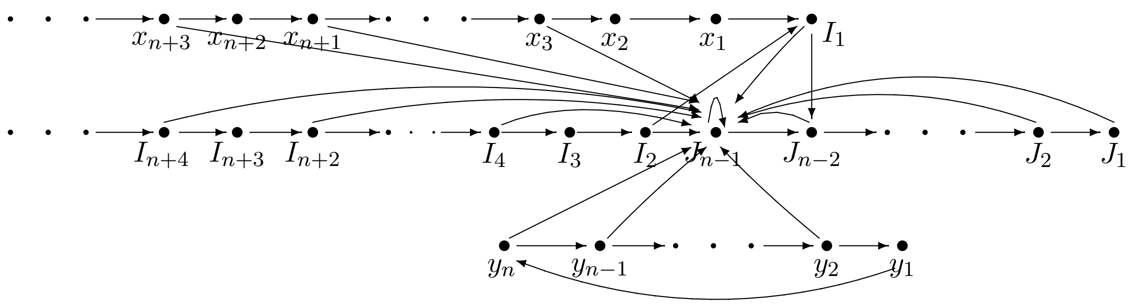
Proof. First of all we recall that cycle γ_n exists iff $s-2$ iterations of point cv are smaller than b. Let the stable cycle $\beta = \{x_1, x_2, \ldots x_k\}$, $k > 2$ exist. It is easy to check that there exists a sequence of points $x_j, x_{j+1}, \ldots, x_{j+s}$ such that $x_i < b$, $i = j, \ldots, j+s-1$, $x_{j+s} > b$, $s \geq 1$, $1 \leq j \leq k-s$ and product of derivatives at these points equals to u, $|u| < 1$. Also $s-2$ iterations of point c are smaller than b. Therefore the cycle γ_s exists. And product of derivatives at points of this cycle also equals u. Hence cycle γ_s is stable. According to Theorem 5 there exists only one stable cycle. Therefore $\beta = \gamma_s$. ■

The following proposition gives description of anti-Fock representation set of $C^*(\mathcal{A}_{f_{l,s,t}})$ under condition of existing of stable cycle.

Proposition 2. *Let point (l,s,t) be an inner point of area P_n. Then there exists countable P-partition of the following form:*

Let $I_1=(f^{(2n-1)}(1),b]$, $I_2=(f^{(2n)}(1),1)$ and $I_j=f^{(j-2)}(I_2)$, $j\geqslant 3$. Then $I=\underset{j}{\cup} I_j\cup (f^{(n+1)}(1),f^{(2)}(1))\cup(f^{(n+2)}(1),f^{(3)}(1))\cup\cdots\cup(f^{(2n-2)}(1),f^{(n-1)}(1))\cup(b,f^{(n)}(1))\cup S=\underset{j}{\cup} I_j\cup J_1\cup J_2\cup\cdots\cup J_{n-1}\cup S$, where set S consists of the points of unilateral orbit $\delta_{(1)}=(f^{(k)}(1),\ k\geqslant 0)=(x_i)_{i\in\mathbb{N}}$ and points of the cycle $\gamma_n=\{y_1,y_2,\ldots,y_n\}$.

Corresponding transition graph has the following form:



There exists one-to-one correspondence between the paths on the graph that start at point x_2, and do not pass this point, and irreducible anti-Fock representations of $C^*(\mathcal{A}_f)$.

Proof. Structure of P-partition and corresponding transition graph follows from $\omega(\delta_{(1)})=\gamma_n$ and can be checked by the direct calculations.

From $f_1^{-1}(x)=\varnothing$, $x\in[0,f(0))$ it follows that: $f_1^{-1}(I_{nk+3})=\varnothing$, $f_1^{-1}(x_{nk+2})=\varnothing$, $k\geqslant 0$, and $f_1^{-1}(J_1)=\varnothing$, $f_1^{-1}(y_1)=\varnothing$.

We also have:

$$
\begin{aligned}
&f_1^{-1}(I_t)=I_{t-1},\quad t\neq nk+3,\quad t\geqslant 2,\quad k\geqslant 0,\\
&f_1^{-1}(x_t)=x_{t-1},\quad t\geqslant 2,\quad t\neq nk+2,\quad k\geqslant 0,\\
&f_1^{-1}(J_k)=J_{k-1},\quad 2\leqslant k\leqslant n-1,\\
&f_1^{-1}(y_k)=y_{k-1},\quad k\neq 1, f_2^{-1}(I_{nk+3})=I_{nk+2},\quad f_2^{-1}(x_{nk+2})=x_{nk+1},\quad k\geqslant 0,\\
&f_2^{-1}(I_1)\in J_{n-1},\quad f_1^{-1}(I_1)\in J_{n-2},\\
&f_2^{-1}(x_1)=\varnothing,\quad f_1^{-1}(x_1)=b\in I_1,\\
&f_2^{-1}(y_1)=y_n,\quad f_2^{-1}(1)=\varnothing.
\end{aligned}
$$

From $f_2^{-1}(x)\in(b,f^{(n)}(1))$ when $x\in(f^{(n+1)}(1),1)$ it follows that:

$$
\begin{aligned}
&f_2^{-1}(I_t)\in J_{n-1},\quad t\neq nk+3,\quad k\geqslant 1,\\
&f_2^{-1}(x_t)\in J_{n-1},\quad t\neq 1,\quad t\neq nk+2,\quad k\geqslant 1,\\
&f_2^{-1}(J_t)\in J_{n-1},\quad 1\leqslant t\leqslant n-1,\\
&f_2^{-1}(y_k)\in J_{n-1},\quad k\neq 1.
\end{aligned}
$$

One-to-one correspondence between paths and representations follows from Theorem 3. ■

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