Systematic Construction of Separable Systems with Quadratic in Momenta First Integrals

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Liouville integrable separable systems with quadratic in momenta first integrals are considered. Particular attention is paid to the systems generated by so-called special conformal Killing tensors, i.e. Benenti systems. Then infinitely many new classes of separable systems are constructed by appropriate deformations of Benenti class systems.

1 Introduction

The separation of variables for solving by quadratures of the Hamilton–Jacobi (HJ) equations of related Liouville integrable dynamic systems with quadratic in momenta first integrals has a long history as a part of analytical mechanics. There are some milestones of that theory. First, in 1891 Stäckel initiated a program of classification of separable systems presenting conditions for separability of the HJ equations in orthogonal coordinates [1–3]. Then, in 1904 Levi-Civita found a test for the separability of a Hamiltonian dynamics in a given system of canonical coordinates [4]. The next was Eisenhart [5–7], who in 1934 inserted a separability theory in the context of Riemannian geometry, making it coordinate free and introducing the crucial objects of the theory, i.e. Killing tensors. This approach was then developed by Woodhouse [8], Klanins [9, 10] and others. Finally, in 1992, Benenti [11–13] constructed a particular, but very important subclass of separable systems, based on the so called special conformal Killing tensors.

The first constructive theory of separated coordinates for dynamical systems was made by Sklyanin [15]. He adopted the method of soliton theory, i.e. the Lax representation, to systematic derivation of separated coordinates. In that approach involutive functions appear as coefficients of characteristic equation (spectral curve) of Lax matrix. The method was successfully aplied to separation of variables for many integrable systems [15–20].

Recently, a modern geometric theory of separability on bi-Poisson manifolds was constructed [21–26], related to the so-called Gel'fand–Zakharevich (GZ) bi-Hamiltonian systems [27, 28]. Obviously, it contains as a special case Liouville integrable systems with all constants of motion being quadratic in momenta functions. Indeed, Ibort et.al. [29] proved that the Benenti class of systems can be lifted to the GZ bi-Hamiltonian form.

In the following paper we construct in a systematic way all Liouville integrable systems on Riemannian spaces, which are of the GZ type, including as a special case the Benenti class of systems. What is important, infinitely many classes of separable systems are constructed from appropriate deformations of the Benenti class of systems. In that sense we demonstrate the crucial role of this particular class of systems in the separability theory of dynamic systems with quadratic in momenta first integrals.

2 Separable dynamics on Riemannian spaces

Let (Q, g) be a Riemann (pseudo-Riemann) manifold with covariant metric tensor g and local coordinates $q^1, \ldots, q^n$. Moreover, let $G := g^{-1}$ be a contravariant metric tensor satisfying

$\sum_{j=1}^{n} g_{ij}G^{jk} = \delta_i^k$. The equations

$$q_{tt}^i + \Gamma_{jk}^i q_t^j q_t^k = G^{ik}\partial_k V(q), \quad i = 1, \ldots, n, \qquad q_t \equiv \frac{dq}{dt} \tag{1}$$

describe the motion of a particle in the Riemannian space with the metric g, where Γ_{jk}^i are the Levi-Civita connection components. Equations (1) can be obtained by varying the Lagrangian

$$\mathcal{L}(q, q_t) = \frac{1}{2}\sum_{i,j} g_{ij} q_t^i q_t^j - V(q) \tag{2}$$

and are called Euler–Lagrange equations. Obviously, for $G = I$ equations (1) reduce to Newton equations of motion.

One can pass in a standard way to the Hamiltonian description of dynamics, where the Hamiltonian function takes the form

$$H(q,p) = \sum_{i=1}^{n} q_t^i \frac{\partial \mathcal{L}}{\partial q_t^i} - \mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} p_i p_j + V(q), \qquad p_i := \frac{\partial \mathcal{L}}{\partial q_t^i} = \sum_j g_{ij} q_t^j$$

and equations of motion are

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dH = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = X_H \iff q_t^i = \frac{\partial H}{\partial p_i}, \qquad p_{it} = -\frac{\partial H}{\partial q^i}.$$

X_H denotes the Hamiltonian vector field with respect to a canonical Poisson tensor θ_0 and the whole dynamics takes place on the phase space $M = T^*Q$ in local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$.

Of special importance is the geodesic motion $V(q) \equiv 0$, with Euler–Lagrange equations and Hamiltonian representation in the form

$$q_{tt}^i + \Gamma_{jk}^i q_t^j q_t^k = 0, \quad i = 1, \ldots, n \quad \iff \quad \begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dE = X_E, \quad E = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} p_i p_j.$$

Functionally independent Hamiltonian functions H_i, $i = 1, \ldots, n$ are said to be separable in the canonical coordinates (λ, μ) if there are n relations, called the separation conditions (Sklyanin [15]), of the form

$$\varphi_i(\lambda^i, \mu_i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \qquad \det\left[\frac{\partial \varphi_i}{\partial H_j}\right] \neq 0, \tag{3}$$

which guarantee the solvability of the appropriate Hamilton–Jacobi equations and involutivity of H_i. A special case, when all separation relations (3) are affine in H_i, is given by the set of equations

$$\sum_{k=1}^{n} \phi_i^k(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n, \tag{4}$$

where ϕ and ψ are arbitrary smooth functions of their arguments, is called the *Stäckel separation conditions* and the related dynamic systems are called Stäckel separable.

We are going to present a subclass of one-particle dynamics, containing Liouville integrable and separable systems with n quadratic in momenta constants of motion. Stäckel [1–3] was the first who gave the characterization of equations of motion integrable by separation of variables.

He proved that if in a system of orthogonal coordinates (λ, μ) there exists a non-singular matrix $\varphi = (\varphi_k^l(\lambda_k))$, called a *Stäckel matrix* such that the Hamiltonians H_r are of the form

$$H_r = \frac{1}{2}\sum_{i=1}^{n}(\varphi^{-1})_r^i(\mu_i^2 + \sigma_i(\lambda_i)), \tag{5}$$

then H_r are functionally independent, pairwise commute with respect to the canonical Poisson bracket and the Hamilton–Jacobi equation associated to H_1 is separable. Indeed, for quadratic in momenta constants of motion, the Stäckel separation conditions (4) take the general form

$$\sum_{k=1}^{n}\phi_i^k(\lambda^i)H_k = \frac{1}{2}f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i), \qquad i = 1, \ldots, n, \tag{6}$$

where f_i, γ_i, ϕ_i^k are arbitrary smooth functions of its argument and the normalization $\phi_i^n = 1$, $i = 1, \ldots, n$ is assumed. To get the explicit form of $H_k = H_k(\lambda, \mu)$ one has to solve the system of linear equations (6). The results are the following

$$\varphi = \begin{pmatrix} \frac{\phi_1^1(\lambda^1)}{f_1(\lambda^1)} & \frac{\phi_1^2(\lambda^1)}{f_1(\lambda^1)} & \cdots & \frac{\phi_1^{n-1}(\lambda^1)}{f_1(\lambda^1)} & \frac{1}{f_1(\lambda^1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\phi_n^1(\lambda^n)}{f_n(\lambda^n)} & \frac{\phi_n^2(\lambda^n)}{f_n(\lambda^n)} & \cdots & \frac{\phi_n^{n-1}(\lambda^n)}{f_n(\lambda^n)} & \frac{1}{f_n(\lambda^n)} \end{pmatrix}, \qquad \sigma_i(\lambda_i) = \gamma_i(\lambda^i)/f_i(\lambda^i).$$

Eisenhart considered Stäckel separable systems in the frame of one-particle dynamics on Riemannian (pseudo-Riemannian) space (Q, g). He gave a coordinate-free representation for Stäckel geodesic motion introducing a special family of *Killing tensors* [5–7]. As known, a $(1,1)$-type tensor $K = (K_j^i)$ (or a $(2,0)$-type tensor $KG = A = (A^{ij})$) is called a Killing tensor with respect to g if

$$\left\{\sum A^{ij}p_ip_j, \sum G^{ij}p_ip_j\right\}_{\theta_0} = 0,$$

where $\{\cdot,\cdot\}_{\theta_0}$ means a canonical Poisson bracket. He proved [5–7] that the geodesic Hamiltonians can be transformed into the Stäckel form (5) if the contravariant metric tensor $G = g^{-1}$ has $(n-1)$ commuting independent contravariant Killing tensors A_r of a second order such that

$$E_r = \frac{1}{2}\sum_{i,j}A_r^{ij}p_ip_j,$$

admitting a common system of closed eigenforms α_i

$$(A_r^* - v_r^iG)\alpha_i = 0, \qquad d\alpha_i = 0, \qquad i = 1, \ldots, n,$$

where v_r^i are eigenvalues of $(1,1)$ Killing tensor $K_r = A_rg$ $(K_r^* = gA_r^*)$.

For n degrees of freedom, let us consider n Stäckel Hamiltonian functions in separated coordinates in the following form

$$H_r = \frac{1}{2}\sum_{i=1}^{n}v_r^iG^{ii}\mu_i^2 + V_r(\lambda) = \frac{1}{2}\mu^TK_rG\mu + V_r(\lambda), \quad r = 1, \ldots, n, \tag{7}$$

where $\mu = (\mu_1, \ldots, \mu_n)^T$ and $V_r(\lambda)$ are appropriate potentials separable in (λ, μ) coordinates. For the integrable system (7) calculated from (6)

$$G^{ii} = (-1)^{i+1}\frac{f_i(\lambda^i)\det W^{i1}}{\det W}, \qquad v_r^i = (-1)^{r+1}\frac{\det W^{ir}}{\det W^{i1}}, \qquad V_r = \sum_{i=1}^{n}(-1)^{i+r}\gamma_i(\lambda^i)\frac{\det W^{ir}}{\det W},$$

where

$$W = \begin{pmatrix} \phi_1^1(\lambda^1) & \phi_1^2(\lambda^1) & \cdots & \phi_1^{n-1}(\lambda^1) & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_n^1(\lambda^n) & \phi_n^2(\lambda^n) & \cdots & \phi_n^{n-1}(\lambda^n) & 1 \end{pmatrix}$$

and W^{ik} is the $(n-1)\times(n-1)$ matrix obtained from W after we cancel its ith row and kth column.

In our further considerations we restrict to the so-called GZ case, when $\phi_i^k(\lambda_i)$ are monomials, $f_i = f$ and $\gamma_i = \gamma$. Then, the Stäckel separation conditions are n copies of a so-called separation curve

$$H_1\xi^{m_1} + \cdots + H_n\xi^{m_n} = \frac{1}{2}f(\xi)\mu^2 + \gamma(\xi), \qquad m_n = 0 < m_{n-1} < \cdots < m_1 \in \mathbb{N}, \tag{8}$$

with $(\xi, \mu) = (\lambda^i, \mu_i)$, $i = 1, \ldots, n$.

In (λ, μ) coordinates n related Hamilton–Jacobi (HJ) equations

$$H_r\left(\lambda, \frac{\partial W}{\partial \lambda}\right) = a_r, \qquad r = 1, \ldots, n,$$

for a generation function $W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda_i, a)$, decouple into n ordinary differential equations

$$\frac{1}{2}f(\lambda^i)\left(\frac{dW_i}{d\lambda^i}\right)^2 + \gamma(\lambda^i) = a_1(\lambda^i)^{m_1} + a_2(\lambda^i)^{m_2} + \cdots + a_n \equiv a(\lambda^i)$$

and hence, the implicit solution of dynamical system with Hamiltonian function H_r is given by

$$\sum_{k=1}^{n}\int^{\lambda^k} \frac{\xi^{m_i}}{\sqrt{\psi(\xi)}}d\xi = t_r\delta_{ri} + \mathrm{const}_i, \qquad i = 1, \ldots, n,$$

where $2f(\xi)[a(\xi) - \gamma(\xi)] \equiv \psi(\xi)$, called the *inverse Jacobi problem.*

In this context, a question about classification and construction in natural coordinates of all separable systems on Riemannian spaces, with n quadratic in momenta constants of motion, arises. The classification can be made with respect to the admissible forms of Stäckel separability conditions. The right hand side of the conditions (8) is always the same for the class of systems considered

$$\text{r.h.s.} = \frac{1}{2}f(\lambda^i)\mu_i^2 + \gamma(\lambda^i) = \psi(\lambda^i, \mu_i), \tag{9}$$

so different classes of separable systems are described by different forms of the l.h.s. of Stäckel conditions, while systems from the same class are described by different f and γ in (9).

3 Separable systems in natural coordinates

Among all Stäckel systems a particularly important subclass consists of these considered by Benenti [11–13] and constructed with the help of the so-called *conformal Killing tensor.* Let $L = (L_j^i)$ be a second order mixed type tensor on Q and let $\overline{L} : M \to \mathbb{R}$ be a function on M defined as $\overline{L} = \frac{1}{2}\sum_{i,j=1}^{n}(LG)^{ij}p_ip_j$. If

$$\{\overline{L}, E\}_{\theta_0} = \kappa E, \qquad \text{where} \quad \kappa = \{\varepsilon, E\}_{\theta_0}, \quad \varepsilon = \mathrm{Tr}\,(L),$$

then L is called a conformal Killing tensor with the associated potential $\varepsilon = \mathrm{Tr}\,(L)$. If we assume additionally that L has simple eigenvalues and its Nijenhuis torsion vanishes, then L is called a special conformal Killing tensor [14].

For the Riemannian space (Q, g, L), geodesic flow has n constants of motion of the form

$$E_r = \frac{1}{2}\sum_{i,j=1}^{n} A_r^{ij} p_i p_j = \frac{1}{2}\sum_{i,j=1}^{n} (K_r G)^{ij} p_i p_j, \qquad r = 1, \ldots, n, \tag{10}$$

where A_r and K_r are Killing tensors of type $(2, 0)$ and $(1, 1)$, respectively. Moreover, as was shown by Benenti [11, 12], all the Killing tensors K_r with a common set of eigenvectors, are constructed from L by

$$K_{r+1} = \sum_{k=0}^{r} \rho_k L^{r-k},$$

where ρ_r are coefficients of the characteristic polynomial of L

$$\det(\xi I - L) = \xi^n + \rho_1 \xi^{n-1} + \cdots + \rho_n, \qquad \rho_0 = 1, \tag{11}$$

or equivalently by the following 'cofactor' formula [30–32]

$$\mathrm{cof}\,(\xi I - L) = \sum_{i=0}^{n-1} K_{n-i} \xi^i,$$

where $\mathrm{cof}\,(A)$ stands for the matrix of cofactors, so that $\mathrm{cof}\,(A)A = (\det A)I$. So, for a given metric tensor g, the existence of a special conformal Killing tensor L is a sufficient condition for the geodesic flow on Q to be a Liouville integrable Hamiltonian system with all constants of motion quadratic in momenta. Moreover, the basic separable potentials $V_r^{(m)}$ are given by the following recursion relation [30–32]

$$V_r^{(m+1)} = \rho_r V_1^{(m)} - V_{r+1}^{(m)}, \tag{12}$$

where the first nontrivial potentials are $V_r^{(0)} = -\rho_r$.

It turns out that we can (generically) associate with the tensor L a coordinate system on Q in which the geodesic flows associated with all the functions E_r separate. Namely, let $(\lambda^1(q), \ldots, \lambda^n(q))$ be n distinct, functionally independent eigenvalues of L, i.e. solutions of the characteristic equation $\det(\xi I - L) = 0$. Solving these relations with respect to q we get the transformation $\lambda \to q : \;\; q_i = \alpha_i(\lambda)$. The remaining part of the transformation to the separation coordinates can be reconstructed from the generating function $W(p, \lambda) = \sum_i p_i \alpha_i(\lambda)$. In the (λ, μ) coordinates the Stäckel separation conditions (8) for Benenti Hamiltonian functions $H_r = E_r + V_r^{(j)}$ are given by the separation curve of the form

$$H_1 \xi^{n-1} + H_2 \xi^{n-2} + \cdots + H_n = \frac{1}{2} f(\xi) \mu^2 + \xi^{n+j}, \qquad j = 0, 1, 2, \ldots. \tag{13}$$

It is important to notice that all Liouville integrable systems of classical mechanics, with quadratic in momenta first integrals, that was separated in XIX and XX centuries, belong to the Benenti class. The reason is that only the Benenti class contains dynamic systems on flat and constant curvature Riemannian spaces. Another important feature of Benenti systems is that all of them can be lifted to one Casimir bi-Hamiltonian form [29].

Now we present the way to construct all remaining classes of separable systems by an appropriate deformations of the Benenti class. Let us start from the separability condition (8) for n Hamiltonian functions in the following form

$$\widetilde{H}_1\xi^{(n+k)-1}+\widetilde{H}_2\xi^{(n+k)-2}+\cdots+\widetilde{H}_{n+k}=\frac{1}{2}f(\xi)\mu^2+\xi^{n+j},\qquad k\in\mathbb{N},\qquad j\geq k,\tag{14}$$

where $\widetilde{H}_{n_1}=\widetilde{H}_{n_2}=\cdots=\widetilde{H}_{n_k}=0$, $1<n_1<\cdots<n_k<n+k$, and the separability condition for Benenti systems with the same separation coordinates (λ,μ)

$$H_1\xi^{n-1}+H_2\xi^{n-2}+\cdots+H_n=\frac{1}{2}f(\xi)\mu^2+\xi^{n+j}.\tag{15}$$

Theorem 1. *Deformation of the Benenti system with separation curve* (15) *to the new system with separation curve* (14) *is given by a following determinant form*

$$\widetilde{H}_r=\begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k}\\ H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k}\\ \cdots & \cdots & \cdots & \cdots\\ H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k}\end{vmatrix}\Bigg/\begin{vmatrix}\rho_{n_1-1} & \cdots & \rho_{n_1-k}\\ \cdots & \cdots & \cdots\\ \rho_{n_k-1} & \cdots & \rho_{n_k-k}\end{vmatrix},\tag{16}$$

where ρ_i, $i=0,\dots,n$ are coefficients of the characteristic polynomial of the conformal Killing tensor L (11) *related to the Benenti system.*

The formula (16) applies separately to the geodesic and the potential parts. The geodesic part $\widetilde{E}_r$ can be presented in the following form

$$\widetilde{E}_r=\frac{1}{2}p^T\widetilde{K}_r\widetilde{G}p,\qquad r=1,\dots,n+k,\tag{17}$$

where metric tensor $\widetilde{G}$ and Killing tensors $\widetilde{K}_r$ are

$$\widetilde{G}=(-1)^k\varphi^{-1}D_0G,\qquad \widetilde{K}_r=K_r-K_{r-1}D_1D_0^{-1}+\cdots+(-1)^kK_{r-k}D_kD_0^{-1},$$

$$\varphi=\begin{vmatrix}\rho_{n_1-1} & \cdots & \rho_{n_1-k}\\ \cdots & \cdots & \cdots\\ \rho_{n_k-1} & \cdots & \rho_{n_k-k}\end{vmatrix},\qquad D_0=\begin{vmatrix}K_{n_1-1} & \cdots & K_{n_1-k}\\ \cdots & \cdots & \cdots\\ K_{n_k-1} & \cdots & K_{n_k-k}\end{vmatrix},$$

$$D_i=\begin{vmatrix}K_{n_1} & \cdots & K_{n_1-i+1} & K_{n_1-i-1} & \cdots & K_{n_1-k+1}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ K_{n_k} & \cdots & K_{n_k-i+1} & K_{n_k-i-1} & \cdots & K_{n_k-k+1}\end{vmatrix},\qquad i=1,\dots,k$$

and K_m in determinant calculations are treated as symbols, not as matrices. The proof of the theorem as well as other details can be found in [32].

Systems constructed in such way although obtained through the deformation procedure on the level of Hamiltonian functions, are far from being trivial generalizations of Benenti systems. There is no obvious relations between solutions of a given Benenti system and all its deformations. In each case we have a different inverse Jacobi problem to solve. Notice that the common feature of appropriate deformed systems is the same set of separated coordinates determined by the related Benenti system. Moreover, all of them can be lifted to a multi-Casimir bi-Hamiltonian form [32].

4 Example

Consider the case $n=2$. Let Q be a two dimensional flat space parametrized by canonical coordinates $q=(q^1,q^2)$ with the contravariant metric tensor G and related special conformal Killing tensor L of the form

$$G=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad L=\begin{pmatrix}q^1&\frac{1}{2}q^2\\ \frac{1}{2}q^2&0\end{pmatrix}\quad\Longrightarrow\quad K_2=\begin{pmatrix}0&\frac{1}{2}q^2\\ \frac{1}{2}q^2&-q^1\end{pmatrix}.$$

Two geodesic Hamiltonians E_1 and E_2, according to (10) are

$$E_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2, \qquad E_2 = \frac{1}{2}q^2 p_1 p_2 - \frac{1}{2}q^1 p_2^2.$$

Let us choose the potential $V^{(2)}$. As $\rho_1 = -q^1$ and $\rho_2 = -\frac{1}{4}(q^2)^2$ hence

$$V_1^{(2)} = (q^1)^3 + \frac{1}{2}q^1(q^2)^2, \qquad V_2^{(2)} = \frac{1}{16}(q^2)^4 + \frac{1}{4}(q^1)^2(q^2)^2.$$

It is one of the integrable cases of the Henon–Heiles system with Hamiltonian function $H_1 = E_1 + V_1^{(3)}$, second constant of motion $H_2 = E_2 + V_2^{(3)}$ and Newton equations

$$(q^1)_{tt} = -3(q^1)^2 - \frac{1}{2}(q^2)^2, \qquad (q^2)_{tt} = -q^1 q^2.$$

The transformation to separated coordinates (λ, μ) takes the form

$$q^1 = \lambda^1 + \lambda^2, \qquad q^2 = 2\sqrt{-\lambda^1\lambda^2},$$
$$p_1 = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \qquad p_2 = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right),$$

and the separation curve is

$$H_1\xi + H_2 = \frac{1}{2}\xi\mu^2 + \xi^4.$$

Now, let us consider the simplest deformation given by $k = 1$, $n_1 = 2$. Then,

$$\widetilde{G} = -\frac{1}{\rho_1}G = \begin{pmatrix} \frac{1}{q^1} & 0 \\ 0 & \frac{1}{q^1} \end{pmatrix}, \quad \widetilde{K}_2 = 0, \quad \widetilde{K}_3 = -K_2^2 = \begin{pmatrix} -\frac{1}{4}(q^2)^2 & \frac{1}{2}q^1q^2 \\ \frac{1}{2}q^1q^2 & -\frac{1}{4}(q^2)^2 - (q^1)^2 \end{pmatrix},$$
$$\widetilde{V}_1 = -\frac{1}{\rho_1}V_1, \quad \widetilde{V}_2 = 0, \quad \widetilde{V}_3 = V_2 - \frac{\rho_2}{\rho_1}V_1,$$

hence

$$\widetilde{H}_1 = \frac{1}{2}\frac{1}{q^1}p_1^2 + \frac{1}{2}\frac{1}{q^1}p_2^2 + (q^1)^2 + \frac{1}{2}(q^2)^2,$$
$$\widetilde{H}_3 = -\frac{1}{8}\frac{q_2^2}{q^1}p_1^2 + \frac{1}{2}q^2 p_1 p_2 - \frac{1}{8}\frac{q_2^2}{q^1}p_2^2 - \frac{1}{16}(q^2)^4.$$

$\widetilde{H}_1$ and $\widetilde{H}_2$ are in involution and are separated in the same coordinates (λ, μ) as the Henon–Heiles system. The appropriate separation curve takes the form

$$\widetilde{H}_1\xi^2 + \widetilde{H}_3 = \frac{1}{2}\xi\mu^2 + \xi^4.$$

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