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Monopole Equations on 8-Manifolds with Spin(7) Holonomy: An Overview of the Energy Integral

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On 8-manifolds with Spin(7) holonomy, we have written an action coupling the curvature of a $U(1)$ bundle to a negative spinor field determined by the Bonan 4-form. The minimizers of this action are the inhomogeneous counterparts of the monopole equations of Corrigan et al. [Corrigan E., Devchand C., Fairlie D. and Nuyts J., *Nucl. Phys. B*, 1983, V.214, 452–464].

1 Introduction

The setup for gauge theories is vector bundles over differentiable manifolds, which are equipped with a fiber metric and a connection compatible with it. The curvature of this connection is a 2-form taking values in the Lie algebra of the structure group of the vector bundle. The connection and the curvature of the vector bundle are usually interpreted as potentials and fields, respectively. Whenever globally meaningful, one can work with additional structures on the base manifold such as other vector bundles, whose sections are also interpreted as physical fields. An "action integral" is a functional of the fields involved, and the crucial point is to write it globally on the manifold. The field equations can be derived either using variational techniques or by determining conditions to attain "topological lower bounds". These topological lower bounds are the integrals of the characteristic classes of the vector bundle and, although related intrinsically to the curvature of the connection, they are determined by topology of the vector bundle [1].

The prototype of these structures is the Yang–Mills theory on four manifolds with an $SU(2)$ vector bundle: The action integral is the L_2 norm of the curvature, which is minimized when the curvature 2-form is self-dual in the Hodge sense, it reaches its topological lower bound and the field equations form a first order elliptic system for the connection. Besides its importance in physics, Yang–Mills theory is extensively used as a tool in low dimensional topology, to compute differentiable invariants of 4-manifolds from the properties of the solution set of Yang–Mills equations [2]. Later Seiberg and Witten [3] proposed to work with a $U(1)$ bundle and an associated spinor bundle, to get a set of inhomogeneous equations. These equations were minimizers of a certain action involving a coupling of the spinor field to the curvature. It turned out that the properties of the solution set of these new equations also yielded the differentiable invariants of the base manifold, but with much ease. A concise review of the Seiberg–Witten theory can be found in [4].

In a series of papers we studied the problem of extending these structures to higher dimensional manifolds [5–7]. The starting point was the definition of "strong self-duality" of 2-forms, as one cannot talk of self-duality of 2-forms in dimensions other than four. Strong self-duality of a 2-form is defined as the equality of the absolute values of the eigenvalues of the corresponding matrix [5] and we have shown that (i) a 2-form ω in $2n$ dimensions is strongly self-dual if and only if ω^{n-1} is proportional to the Hodge dual of ω, and (ii) a 2-form ω in $4n$ dimensions is strongly self-dual if ω^n is self-dual in the Hodge sense [8]. The first condition was used as a definition of self-duality by Trautman [9] and the second one was used in the work of Grossman et al. [10].

Although this nonlinear criterion encompasses the definitions used in the literature and realizes topological lower bounds in many situations, it is not easily workable, hence a linear notion of self-duality is preferable. The linear subspaces of the strong self-dual forms are closely related to the representations of Clifford algebras and provide such a setting [11]. The dimension of the maximal linear subspaces of strong self-dual 2-forms on R^{2n} is equal to the number of linearly independent vector fields on the sphere S^{2n-1}, known as the Radon–Hurwitz number. These structures however are quite restricted, for example the Radon–Hurwitz number in dimensions $N = 2(2a + 1)$ is just 1. It turns out that $N = 8$ is the most interesting dimension, and the linear subspaces of strong self-dual 2-forms coincide with the linear self-duality definition given first in [12] and [13]. The choice of a maximal linear subspace of strong self-dual 2-forms in 8 dimensions gives rise to a splitting of the 28 dimensional linear space of 2-forms into 7 and 21 dimensional subspaces. The 7 dimensional subspace, consisting of strong self-dual 2-forms, are building blocks of the instanton solution of Grossman et al. [10], while the 21 dimensional subspace leads to the equations given in Corrigan et al. [12]. The first set is overdetermined, but the second one is shown to be an elliptic system for the connection [14]. The question is now whether one can choose these subpaces consistently on the manifold, or whether the equations given in local coordinates are globally meaningful. This question was addressed in Corrigan et al. [12] by requiring that the fourth rank tensor defining self-duality be invariant under a subgroup of the structure group of the tangent bundle of the base manifold. It turns out that when the manifold has Spin(7) holonomy, there is a globally defined 4-form, called the Bonan 4-form [15] or the "calibration form" [16] that gives rise to the self-duality equations. The Bonan 4-forms appears also in a topological lower bound for the Yang–Mills action on 8-manifolds [17].

The structures discussed so far involved a single usually non-Abelian vector bundle over a base manifold. As mentioned above, the Seiberg–Witten theory involves a manifold equipped with a $U(1)$ vector bundle and a spinor bundle. A direct generalization of the Seiberg–Witten theory to 8-manifolds was considered in [18], but leads to trivial results. An interpretation of the Seiberg–Witten equations as projections resulted in a set of monopole equations for a coupling to positive spinors [6]. Finally, in [7] we obtained a set of equations for a coupling to negative spinors as minimizers of a quadratic action integral. In the present paper we outline the construction of the action integral with an emphasis on local expressions.

2 Preliminaries

We start by introducing our notation. We define a Hermitian inner product on $n \times m$ matrices by $(A, B) = \frac{1}{n}\operatorname{tr}(\bar{A}^t B)$. It follows that if X is an m-vector and A is skew-symmetric matrix, then $(AX, X) = \frac{n}{2}(A, X\bar{X}^t - \bar{X}X^t)$. The point-wise norm of a real differential form is defined as $|\omega|^2 = (\omega, \omega) = *(\omega \wedge *\omega)$, where $*$ denotes Hodge dual.

A spin^c structure on a $2n$-dimensional real inner-product space V is a pair (W, Γ), where W is a 2^n dimensional Hermitian vector space, and $\Gamma : V \to \operatorname{End}(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \qquad \Gamma(v)^2 = -|v|^2$$

for $v \in V$. Globalization of this construction on each fiber to the tangent bundle of a $2n$-dimensional oriented manifold M will be possible if and only if the second Stiefel–Whitney class $w_2(M)$ has an integral lift, a condition which always holds for manifolds with $Spin(7)$ holonomy.

Given a spin^c structure, the matrix $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ has eigenvalues $\pm i^n$, and its eigenspaces determine a splitting $W = W^+ \oplus W^-$. The elements of W^+ and W^- are called respectively "positive" and "negative" spinors. It follows that one can write the representation in block form as

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix} \tag{1}$$

with Γ as given in (1), ρ is block diagonal and one can define the maps

$$\rho^{\pm}(\eta) = \rho(\eta)\mid_{W^\pm}, \tag{2}$$

for $2k$-forms η.

The matrices $\gamma(e_i) = \gamma_i$ are characterized by

$$\gamma_1 = I, \qquad \gamma_j^2 + I = 0, \qquad \gamma_j\gamma_k + \gamma_k\gamma_j = 0, \qquad \text{for} \quad j \geq 2, \quad j \neq k. \tag{3}$$

In 8-dimensions, the γ_j's are real, skew-symmetric matrices and the set

$$\{\gamma_2, \gamma_3, \ldots, \gamma_8, \gamma_2\gamma_3, \ldots, \gamma_7\gamma_8\} \tag{4}$$

is orthonormal. The effects of the $\rho^{\pm}$ maps are given by

$$\begin{aligned} &\rho^{\pm}(e_1^* \wedge e_j^*) = \pm\gamma_j, \qquad \rho^{\pm}(e_j^* \wedge e_k^*) = \gamma_j\gamma_k, \\ &\rho^{\pm}(e_1^* \wedge e_j^* \wedge e_k^* \wedge e_l^*) = \pm\gamma_j\gamma_k\gamma_l, \qquad \rho^{\pm}(e_j^* \wedge e_k^* \wedge e_l^* \wedge e_m^*) = \gamma_j\gamma_k\gamma_l\gamma_m. \end{aligned} \tag{5}$$

The γ_j matrices satisfy the relations $\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8 = -I$, which allow the duality identifications

$$\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 = \gamma_7\gamma_8, \qquad \gamma_2\gamma_3\gamma_4\gamma_5 = -\gamma_6\gamma_7\gamma_8. \tag{6}$$

A Hermitian connection on W, compatible with the Levi-Civita connection of the manifold induces an imaginary valued connection on a certain associated line bundle. The corresponding connection 1-form is denoted by $\mathcal{A}$, its curvature 2-form is F and the Dirac operators $D^{\pm}$ corresponding to $\mathcal{A}$ are differential operators mapping positive spinors to negative spinors and vice versa. The Weitzenböck formula, which is the key in writing of the action integral for the Seiberg–Witten equations, gives

$$\begin{aligned} D^-D^+\phi^+ &= \nabla^*\nabla\phi^+ + \frac{1}{4}s\phi^+ + \rho^+(F)\phi^+, \\ D^+D^-\phi^- &= \nabla^*\nabla\phi^- + \frac{1}{4}s\phi^- + \rho^-(F)\phi^-, \end{aligned} \tag{7}$$

where s is the scalar curvature of M and ∇^* is the L_2-adjoint of ∇. Taking the inner product of (7) with $\phi^{\pm}$ and integrating over M, we obtain

$$\int_M |D^{\pm}\phi^{\pm}|^2 \,\mathrm{dvol} = \int_M \left[|\nabla\phi^{\pm}|^2 + \frac{1}{4}s|\phi^{\pm}|^2 + (\rho^{\pm}(F)\phi^{\pm}, \phi^{\pm})\right] \mathrm{dvol}. \tag{8}$$

Hence if $D^-\phi^- = 0$, from (8) we have

$$\int \left[|\nabla\phi^-|^2 + \frac{1}{4}s|\phi^-|^2\right] \mathrm{dvol} = -\int (\rho^-(F)\phi^-, \phi^-)\,\mathrm{dvol}. \tag{9}$$

We have thus expressed the spinor coupling in terms of the ρ map. Now we will compute the topological term.

The expression of the *Bonan 4-form* Φ in an orthonormal basis $\{e_i\}$, $i = 1, \ldots, 8$, is

$$\begin{aligned}\Phi = [&e_{1234} - e_{1256} - e_{1278} - e_{1357} + e_{1368} - e_{1458} - e_{1467} \\ &+ e_{5678} - e_{3478} - e_{3456} - e_{2468} + e_{2457} - e_{2367} - e_{2358}]. \end{aligned} \tag{10}$$

where $e_{ijkl} = e_i^* \wedge e_j^* \wedge e_k^* \wedge e_l^*$. The form Φ is $Spin(7)$ invariant, hence it can be extended to the manifold. Φ is self-dual and it defines a linear map on 2-forms as

$$\omega \to *(\Phi \wedge \omega)$$

with eigenvalues 3 and -1. This map has 7 and 21 dimensional eigenspaces, and a basis consisting of the corresponding eigenvectors are given below. The eigenvectors corresponding to the eigenvalue 3 are

$$\begin{aligned} \omega_{12} &= e_{15} + e_{26} + e_{37} + e_{48}, \\ \omega_{13} &= e_{12} + e_{34} - e_{56} - e_{78}, \\ \omega_{14} &= e_{16} - e_{25} - e_{38} + e_{47}, \\ \omega_{15} &= e_{13} - e_{24} - e_{57} + e_{68}, \\ \omega_{16} &= e_{17} + e_{28} - e_{35} - e_{46}, \\ \omega_{17} &= e_{14} + e_{23} - e_{58} - e_{67}, \\ \omega_{18} &= e_{18} - e_{27} + e_{36} - e_{45}. \end{aligned} \tag{11}$$

The ones corresponding to the eigenvalue -1 can be obtained by changing signs as given in [7]. The eigenvectors $\{\omega_{1j}\}$ and $\{\omega_{jk}\}$ for $j, k = 2, \ldots, 8$ are a basis for local sections of 2-form fields, and the curvature 2-form F has the splitting $F = F^{(7)} + F^{(21)}$, where $F^{(7)} = \sum i a_j \omega_{1j}$, $F^{(21)} = \sum i a_{kl} \omega_{kl}$. It can be seen that

$$F^{(7)} \wedge F^{(21)} \wedge \Phi = 0,$$

hence

$$F \wedge F \wedge \Phi = F^{(7)} \wedge F^{(7)} \wedge \Phi + F^{(21)} \wedge F^{(21)} \wedge \Phi.$$

It follows that one can obtain

$$|F|^2 = \frac{4}{3}|F^{(21)}|^2 - \frac{2}{3} * (F \wedge F \wedge \Phi) \tag{12}$$

that relates the norm of the curvature to the topological term.

3 Seiberg–Witten theory on 4-manifolds

In Seiberg–Witten theory over a 4-manifold, a $U(1)$ connection is coupled to a spinor field. The Seiberg–Witten equations are the minimizers of an action involving the curvature 2-form F and a positive Dirac spinor ϕ^+

$$I(A, \phi^+) = \int_M \left[|F|^2 + |\nabla \phi^+|^2 + \frac{1}{4} s |\phi^+|^2 + \frac{1}{4} |\phi^+|^4 \right] \mathrm{dvol} \geq \int_M \operatorname{tr} F^2, \tag{13}$$

where s is the scalar curvature of the 4-manifold M and A and F are respectively the connection and curvature of a line bundle associated with the spin^c structure on M [4]. The Weitzenböck formula relates the covariant derivatives $\nabla\phi^{\pm}$ to $D^{\pm}\phi^{\pm}$, compensating for the scalar curvature term and bringing in the coupling $(\rho^{\pm}(F)\phi^{\pm},\phi^{\pm})$.

In 4 dimensions, $\rho^{+}(F)$ is a 2×2 traceless, Hermitian matrix such that $\rho^{+}(F^{-})$ is identically zero. If $D^{+}\phi^{+}=0$, the action (13) can be written as the sum of a topological term and an integrand involving $2|F|^2+(\rho^{+}(F^{+})\phi^{+},\phi^{+})+\frac{1}{4}|\phi^{+}|^4$. It turns out that the integrand vanishes when

$$D^{+}\phi^{+}=0, \qquad \rho^{+}(F^{+})=[\phi^{+}(\bar{\phi}^{+})^{t}]_{o}, \tag{14}$$

where the subscript o denotes the trace-free part of a matrix. From the details of the computation as given in [7] it follows that the term $\frac{1}{4}|\phi^{+}|^4$ is added just to complete the square. Choosing $\rho^{+}(F^{+})$ as in (14) reduces the action to its topological lower bound. In some sense, the spinors $\phi^{\pm}$ are chosen in such a way that the coupling is independent of the part of F, which is expected to be free. This was our idea in looking for Seiberg–Witten type equations on 8-manifolds.

The coupling of a spinor field to the Yang–Mills action in 8-dimensions may follow different paths. A direct generalization of the Seiberg–Witten theory in 4-dimensions has led to trivial results [10]. In our previous paper [11], we defined a generalization by interpreting the right hand sides of the Seiberg–Witten equations as a projection onto a subspace determined by the map ρ^{+} and coupled positive spinors to the curvature. However, it was not possible to express these equations as absolute minimizers of an action.

In [7], we defined monopole equations via a projection using the map ρ^{-}, coupling the curvature to negative spinors. These new set of equations are absolute minimizers of an action, provided that the real and imaginary parts of the negative spinor belongs to certain complemenatary subspaces of negative spinors determined by the image of the Bonan 4-form under the ρ^{-} map, to be described below.

4 The action integral

We will now obtain monopole equations for $F^{(7)}$ and ϕ^{-} derived from an action principle, when ϕ^{-} belongs to a certain subspace. The determination of this subspace in local coordinates is almost trivial, as it imposes itself from the local expression of the ρ. The subtle point is the invariant description of this subspace in terms of the Bonan 4-form, independently of a specific choice of the spin^c structure. We refer to [7] for the complete proof of Proposition 1, below.

Proposition 1. *Let M be an 8-manifold with $Spin(7)$ holonomy, with scalar curvature s and Bonan 4-form Φ, (Γ, W) be any spin^c structure and W^{-} and ρ^{-} be defined in terms of Γ as in (2). Define a section ϕ of W^{-} by*

$$\phi=(1-P)U+iPU$$

where U is a real section of W^{-} and

$$P=\frac{1}{8}-\frac{1}{16}\rho^{-}(\Phi).$$

Then the monopole equations

$$D^{-}\phi=0, \qquad \rho^{-}(F^{(7)})=\frac{1}{2}(\phi\bar{\phi}^{t}-\bar{\phi}\phi^{t}), \qquad \operatorname{div} A=0$$

are absolute minimizers of the action

$$I(A,U)=\int_M\left[|F|^2+|\nabla\phi|^2+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^{t}-\bar{\phi}\phi^{t}|^2\right]\mathrm{dvol}$$

and $I(A,U)\geq\frac{2}{3}\int_M F\wedge F\wedge\Phi$.

Proof. We give here only a brief outline of the proof, concentrating on local computations which are more illustrative.

Using (9), (12), we can express the action as

$$I(A,U)=\frac{2}{3}\int_M F\wedge F\wedge\Phi+\int_M\left[4|F^{(7)}|^2-(\rho^-(F)\phi,\phi)+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^t-\bar{\phi}\phi^t|^2\right]\mathrm{dvol}.$$

At this stage it is useful to have the expression of the $\rho^-(F)$ for a specific choice of spin^c structure that we used in our computations. Other choices lead to similar expressions, and the whole computation can be done in an abstract Clifford algebra setting using the relations (2)–(6). The effect of ρ^+ can be written similarly, but the resulting matrices are nonsingular. This is the main reason for working with negative spinors.

$$\begin{aligned}
\rho^-(\omega_{12})&=-\gamma_5+\gamma_{26}+\gamma_{37}+\gamma_{48}=-4e_{68},\\
\rho^-(\omega_{13})&=-\gamma_2+\gamma_{34}-\gamma_{56}-\gamma_{78}=-4e_{26},\\
\rho^-(\omega_{14})&=-\gamma_6-\gamma_{25}-\gamma_{38}+\gamma_{47}=4e_{46},\\
\rho^-(\omega_{15})&=-\gamma_3-\gamma_{24}-\gamma_{57}+\gamma_{68}=4e_{56},\\
\rho^-(\omega_{16})&=-\gamma_7+\gamma_{28}-\gamma_{35}-\gamma_{46}=4e_{67},\\
\rho^-(\omega_{17})&=-\gamma_4+\gamma_{23}-\gamma_{58}-\gamma_{67}=-4e_{16},\\
\rho^-(\omega_{18})&=-\gamma_8-\gamma_{27}+\gamma_{36}-\gamma_{45}=-4e_{36}.
\end{aligned}$$

It is then possible to check that $|F^{(7)}|^2=|\rho^-(F^{(7)})|^2$. The orthonormality of the set (4) implies that $|\rho^-(\omega_{1j})|^2=4$. On the other hand, from (11) $|\omega_{1j}|^2=4$, hence the ρ^- map is an isometry. It follows that

$$\begin{aligned}
I(A,U)=&\frac{2}{3}\int_M F\wedge F\wedge\Phi\\
&+\int_M\left[4|\rho^-(F^{(7)})|^2-(\rho^-(F)\phi,\phi)+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^t-\bar{\phi}\phi^t|^2\right]\mathrm{dvol}.
\end{aligned}$$

The main difficulty is to show that $(\rho^-(F^{(21)})\phi,\phi)=0$ for ϕ given as in the hypothesis of the theorem. If we use the specific spin^c structure corresponding to $\gamma_j=\omega_{1j}$, P is the diagonal matrix with the only nonzero entry, $P_{66}=1$. It can also be seen that the 6th row and the 6th column of $\rho^-(\omega_{jk})$'s are zero, while in $\rho^-(\omega_{1j})$'s non-zero elements are only in the 6th row and the 6th column. Thus, in a local formulation it can be seen that

$$\rho^-(F^{(21)})P=0,\qquad \rho^-(F^{(7)})P+P\rho^-(F^{(7)})=\rho^-(F^{(7)}).$$

As F is pure imaginary and the representation is real, $\rho^-(F)$ is Hermitian but skew-symmetric. Now using $\rho^-(F^{(21)})P=0$, we have

$$(\rho^-(F^{(21)})\phi,\phi)=(\rho^-(F^{(21)})U,(1-P)U+iPU)=(U,\rho^-(F^{(21)})U)=0,$$

where the second equality follows from hermiticity and the third from skew-symmetry. Then

$$(\rho^-(F^{(7)})\phi,\phi)=4(\rho^-(F^{(7)}),\phi\bar{\phi}^t-\bar{\phi}\phi^t),$$

and it can be seen that the action reduces to

$$I(A,U)=\frac{2}{3}\int_M F\wedge F\wedge\Phi+\int_M|2\rho^-(F^{(7)})-(\phi\bar{\phi}^t-\bar{\phi}\phi^t)|^2\mathrm{dvol}$$

which gives the monopole equations. As the left hand side of the matrix equation

$$\rho^-\big(F^{(7)}\big) = \frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t)$$

belongs to a 7-dimensional subspace, we need to check its compatibility. For this we use the orthogonal decomposition of any skew-symmetric matrix A in the form

$$A = (PA + AP) + (I - P)A(I - P).$$

and write

$$\frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t) = \frac{1}{2}[-i(I-P)UU^tP + iPUU^t(I-P)] = i[-UU^tP + PUU^t].$$

Using the properties of the projection operator P, it can be seen that both sides of the equation belong to the same linear subspace. ■

5 The monopole equations

In [6], we have given monopole equations using a projection with $\rho^+(F^{(7)})$, where we denoted $F^{(7)}$ as F^+, and ω_{1j} by f_{j-1}. The same formula (equation (26) in [6]) can be applied as well using a projection with ρ^-. Hence we can write

$$\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} \langle \rho^\pm(\omega_{1j}), \phi^\pm(\phi^\pm)^*\rangle |\rho^\pm(\omega_{1j})|^{-2}\rho^\pm(\omega_{1j}),$$

where ω_{1j}'s are an orthonormal basis for the subspace $F^{(7)}$. We may choose the ω_{1j}'s as above, and with respect to this basis we can write $F^{(7)} = \sum_{j=2}^{8} ia_{1j}\omega_{1j}$, then $\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} ia_{1j}\rho^\pm(\omega_{1j})$, the monopole equations are

$$ia_{1j} = |\rho^\pm(\omega_{1j})|^{-2}\langle \rho^\pm(\omega_{1j}), \phi^\pm(\bar{\phi}^\pm)^t\rangle, \qquad j = 2, \ldots, 8.$$

We give below the monopole equations for the coupling to a negative spinor

$$\begin{aligned}
4a_{12} &= F_{15} + F_{26} + F_{37} + F_{48} = (1/2)(-\phi_6\bar{\phi}_8 + \phi_8\bar{\phi}_6),\\
4a_{13} &= F_{12} + F_{34} - F_{56} - F_{78} = (1/2)(-\phi_2\bar{\phi}_6 + \phi_6\bar{\phi}_2),\\
4a_{14} &= F_{16} - F_{25} - F_{38} + F_{47} = (1/2)(\phi_4\bar{\phi}_6 - \phi_6\bar{\phi}_4),\\
4a_{15} &= F_{13} - F_{24} - F_{57} + F_{68} = (1/2)(\phi_5\bar{\phi}_6 - \phi_6\bar{\phi}_5),\\
4a_{16} &= F_{17} + F_{28} - F_{35} - F_{46} = (1/2)(\phi_6\bar{\phi}_7 - \phi_7\bar{\phi}_6),\\
4a_{17} &= F_{14} + F_{23} - F_{58} - F_{67} = (1/2)(-\phi_1\bar{\phi}_6 + \phi_6\bar{\phi}_1),\\
4a_{18} &= F_{18} - F_{27} + F_{36} - F_{45} = (1/2)(-\phi_3\bar{\phi}_6 + \phi_6\bar{\phi}_3).
\end{aligned}$$

When the spinor belongs to the subspace determined by the Bonan 4-form, we put

$$\bar{\phi}_6 = -\phi_6, \qquad \bar{\phi}_j = \phi_j, \qquad \text{for} \quad j \neq 6.$$

The final step is to show that the resulting equations are elliptic. To check ellipticity of a system of first order partial differential equations in the independent variables $x_1, \ldots, x_k$ and dependent variables $u_1, \ldots, u_n$, we replace $\frac{\partial u_i}{\partial x_j}$ by $\xi_j u_i$, where ξ_j's are indeterminates, and obtain a linear homogeneous system of equations for the u_i's. The characteristic determinant of the system is the determinant of the coefficient matrix of the corresponding linear system. The system is called *elliptic* if its characteristic determinant has no real roots. As this property is independent of the inmogeneous part, the elipticity of the new set of equations follows from the ellipticity of the equations of Corrigan et al. [12] after checking a nondegeneracy condition arising from the projection of the spinor field to a subspace.

[1] Milnor J.W. and Statsheff J.D., Characteristic classes, Princeton, NJ, Princeton Univ. Press, 1974.

[2] Donaldson S. and Kronheimer P., The geometry of four manifolds, Oxford, Clarendon, 1990.

[3] Seiberg N. ans Witten E., Electromagnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang–Mills theory, *Nucl. Phys. B*, 1994, V.426, 19–52.

[4] Morgan J.W., The Seiberg–Witten equations and applications to the topology of smooth four-manifolds, Princeton, NJ, Princeton Univ. Press, 1996.

[5] Bilge A.H., Dereli T. and Kocak S., Self-dual Yang–Mills fields in eight dimensions, *Lett. Math. Phys.*, 1996, V.36, N 3, 301–309.

[6] Bilge A.H., Dereli T. and Kocak S., Monopole equations on 8-manifolds with $Spin(7)$ holonomy, *Commun. Math. Phys.*, 1999, V.203, 21–30.

[7] Bilge A.H., Dereli T. and Kocak S., Seiberg–Witten type monopole equations on 8-manifolds with $Spin(7)$ holonomy, as minimizers of a quadratic action, *J. High. Ener. Phys.*, 2003, V.4, art.no. 003; hep-th/0303098.

[8] Ozdemir F. and Bilge A.H., Self-duality in dimensions $2n \geq 4$, equivalence of various definitions and the derivation of the octonionic instanton solution, *Bulletin of Istanbul Technical University, ARI*, 1999, V.51, 247–253.

[9] Trautman A., Solutions of the Maxwell and Yang–Mills equations associated with Hopf fiberings, *Int. J. Theor. Phys.*, 1977, V.16, 561–565.

[10] Grossman B., Kephart T.W. and Statsheff J.D., Solutions to Yang–Mills field equations in eight dimensions and the last Hopf map, *Comm. Math. Phys.*, 1984, V.96, 431–437.

[11] Bilge A.H., Dereli T. and Kocak S., The geometry of self-dual two-forms, *J. Math. Phys.*, 1997, V.38, N 8, 4804–4814.

[12] Corrigan E., Devchand C., Fairlie D. and Nuyts J., First order equations for gauge fields in spaces of dimensions greater than four, *Nucl. Phys. B*, 1983, V.214, 452–464.

[13] Dundarer R., Gürsey F. and Tze C.-H., Generalized vector products,duality and octonionic identities in $D = 8$ geometry, *J. Math. Phys.*, 1984, V.25, 1496–1506.

[14] Bilge A.H., Solution and ellipticity properties of the self-duality equations of Corrigan et al. in eight dimensions, *Int. J. Theor. Phys.*, 1996, V.35, 2507–2515.

[15] Bonan E., Sur les varieties riemanniennes a groupe d'holonomie G_2 ou $Spin(7)$, *C.R. Acad. Sci. Paris*, 1966, V.262, 127–129.

[16] Harvey R. and Lawson H.B.Jr, Calibrated geometries, *Acta. Math.*, 1982, V.148, 47–157.

[17] Acharya B.S., O'Loughlin M. and Spence B., Higher dimensional analogues of Donaldson-Witten theory, *Nucl. Phys. B*, 1997, V.503, N 3, 657–674; hep-th/970513;
Acharya B.S. and O'Loughlin M., Self-duality in $D \leq 8$ dimensional Euclidean gravity, *Phys. Rev. D*, 1997, V.55, N 8, R4521–R4524.

[18] Bilge A.H., Dereli T. and Kocak S., Seiberg–Witten equations on R^8, in Proceedings of 5th Gökova Geometry-Topology Conference (1997, TUBITAK, Ankara), Editors S. Akbulut, T. Önder and R. Stern, *Turkish J. Math.*, 1997, V.21, N 1, 87–92.