Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 325–332

New Classes of Nonlinear Evolutionary Equations

Lev M. BERKOVICH

Samara State University, 1 Academician Pavlov Str., 443011 Samara, Russia
E-mail: *berk@ssu.samara.ru*

We consider new classes of nonlinear evolutionary equations (NEEs) of the n-th order depending on two arbitrary functions such that the solutions to these equations of traveling wave type satisfy the corresponding nonlinear superposition principles (NSPs). The construction is based on the methods of the factorization and exact linearization of ordinary differential equations (ODEs)

1 Introduction

In the papers [1] and [2] some higher analogs of the nonlinear evolutionary equation of Korteweg–de Vries are considered. Some NEEs of the higher order are represented in handbook [3]. In the present paper we have constructed new types of NEEs of higher order. For their construction the methods of factorization and exact linearization were used (see [4–8]).

In Section 2 the summary of results on the method of a exact linearization is given. In Section 3 the NEEs of n-th order depending on one arbitrary function and reducible to linear evolutionary equations (LEEs) by a nonlinear substitution of dependent variable are constructed. In Section 4 the NEEs of n-th order depending on two arbitrary functions are constructed. Their solutions of traveling wave type satisfying to nonlinear ODEs, linearized by transformation dependent and independent variables. In Section 5 the NEEs of n-th order depending on two arbitrary functions also are are constructed. Their stationary solutions satisfy to the nonlinear ODEs reduced to the semilinear equations.

2 On the method of exact linearization for ordinary differential equations

Lemma 1. *An autonomous second-order ODE*

$$F(y, y', y'') = 0, \qquad ()' = \frac{d}{dx} \tag{1}$$

can be reduced to the linear form

$$z''(s) + 2b_1 z'(s) + b_2 z(s) = 0, \qquad b_1, b_2 = \text{const}, \tag{2}$$

by a nonlinear transformation of the dependent and independent variables

$$y = v(y)z, \qquad ds = y dx \tag{3}$$

if and only if (1) *can be factored into noncommutative nonlinear differential operators as*

$$\left(D - \left(\frac{v^*}{v} + \frac{u^*}{u}\right) y' - r_2 u\right) \left(D - \frac{v^*}{v} y' - r_1 u\right) y = 0, \qquad D = \frac{d}{dx}, \qquad ()^* = \frac{d}{dy} \tag{4}$$

or into commutative operators as

$$\left(\frac{1}{u} D - \frac{v^*}{vu} y' - r_2\right) \left(\frac{1}{u} D - \frac{v^*}{vu} y' - r_1\right) y = 0,$$

where r_1 and r_2 are the roots of the characteristic equation

$$r^2 + 2b_1 r + b_2 = 0. \tag{5}$$

Lemma 2. *Equation* (1) *can be linearized by transformation* (3)*, if and only if it can be represented in the form*

$$y'' + fy'^2 + 2b_1\varphi y' + b_2\varphi \exp\left(-\int f dy\right)\int \varphi \exp\left(\int f dy\right) dy = 0, \tag{6}$$

where $f = f(y)$, $\varphi = \varphi(y)$, that reduces to (2) by the transformation

$$z = \beta \int \varphi \exp\left(\int f dy\right) dy, \qquad ds = \varphi(y)dx, \tag{7}$$

where $\beta = \mathrm{const} \neq 0$ is a normalizing factor.

Lemma 3. *An autonomous n-th order ODE*

$$F\big(y, y', \ldots, y^{(n)}\big) = 0 \tag{8}$$

can be reduced to a linear autonomous form

$$M_n(z) \equiv z^{(n)}(s) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(s) = 0, \qquad b_k = \mathrm{const}$$

by transformation (3) *if and only if* (8) *can be represented in the form*

$$\prod_{k=n}^{1}\left[D - \left(\frac{1}{y} - \left(\ln \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy\right)^{*} + (k-1)\frac{\varphi^*}{\varphi}\right) y' - r_k\varphi\right] y = 0, \tag{9}$$

where r_k are the roots of the characteristic equation

$$M_n(r) \equiv r^n + \sum_{k=1}^{n} \binom{n}{k} b_k r^{n-k} = 0; \tag{10}$$

the linearizing transformation (2) then has the form

$$z = \beta \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy, \qquad ds = \varphi dx, \tag{11}$$

where $\beta = \mathrm{const} \neq 0$ is a normalizing factor.

Note 1. The structure of linearizable equations. Linearizable equations depend on two arbitrary functions and n parameters serving as coefficients of the linear equations. They are algebraic with respect to the derivatives of the dependent variable which they include. The higher-order equations are constructed on the basis of recursive relations. The order of the nonlinear term is determined as the sum of the products of the orders of derivatives by their exponents. Each equation belonging to the class under examination can be represented as an algebraic sum of terms (with coefficients expressed through the dependent variable) each of which consists of nonlinear terms of the same order. The order of the term not depending on the coefficients of the transformed linear equation equals the order of the equation. All the other terms have smaller orders and contain coefficients (parameters) of the linear equation. Then, a linearizable equation can be represented in the form

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \Psi^{12\ldots n}_{k_1k_2\ldots k_n} y^{(1)k_1} y^{(2)k_2} \cdots y^{(n)k_n}$$

$$+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m\left(\sum_{l_1+2l_2+\dots+(n-m)l_{n-m}=n-m}\Psi^{12\dots n-m}_{l_1l_2\dots l_{n-m}}y^{(1)l_1}y^{(2)l_2}\dots y^{(n-m)l_{n-m}}\right)$$
$$+\,b_n\exp\left(-\int f dy\right)\int\varphi^{\frac{n^2+n-2}{2n}}\exp\left(\int f dy\right)dy=0, \tag{12}$$

where the coefficients Ψ depend on $f(y)$ and $\varphi(y)$ and

$$\Psi^{12\dots n}_{00\dots 1}=1,\qquad \Psi^{12\dots n-m}_{00\dots 1}=1,\qquad \Psi^{12\dots n-1n}_{10\dots 10}=nf(y).$$

3 Nonlinear evolutionary equations reducible to linear evolutionary equations

Proposition 1. *An NEE*

$$\frac{\partial y}{\partial t}=F\left(y,\frac{\partial y}{\partial x},\dots,\frac{\partial^n y}{\partial x^n}\right),\qquad y=y(t,x) \tag{13}$$

reduces to a linear evolutionary equation (LEE)

$$\frac{\partial z}{\partial t}=\sum_{k=0}^{n}\binom{n}{k}b_k\frac{\partial^{n-k}z}{\partial x^{n-k}}, \tag{14}$$

by substitution of the form (see (3)*)*

$$y=v(y)z, \tag{15}$$

if and only if (13) *can be factored as*

$$\left(1-\frac{v^*}{v}y\right)\frac{\partial y}{\partial t}=\prod_{k=1}^{n}\left(\frac{\partial}{\partial x}-\frac{v^*}{v}\frac{\partial y}{\partial x}-r_k\right)y, \tag{16}$$

where r_k are the roots of characteristic equation (10)*.*

Theorem 1. *NEE* (13) *reduces to LEE* (14) *by substitution* (15) *if and only if it can be represented in the form*

$$\frac{\exp\left(\int f dy\right)y}{\int\exp(\int f dy)dy}\frac{\partial y}{\partial t}=\prod_{k=1}^{n}\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\exp\left(\int f dy\right)}{\int\exp(\int f dy)dy}\right)\frac{\partial y}{\partial x}-r_k\right]y, \tag{17}$$

linearizing substitution (15) *then has an explicit form*

$$z=\beta\int\exp\left(\int f dy\right)dy,\qquad \beta=\text{const}\neq 0. \tag{18}$$

Proof. We apply Lemma 3, equation (16), and the formula

$$1-\frac{v^*}{v}y=\frac{\exp\left(\int f dy\right)y}{\int\exp\left(\int f dy\right)dy}.$$ ■

Equation (13) in form (16) satisfies the nonlinear superposition principle (NSP)

$$z=\int\exp\left(\int f dy\right)dy=\sum_{k=1}^{n}c_kz_k(t,x),$$

where c_k are arbitrary constants and $z_k(t,x)$ are linearly independent partial solutions to LEE (14).

Example 1. The equation[1]

$$y_t = \frac{h}{2} y_{xx} - \frac{1}{2} y_x^2, \qquad h = \text{const} \tag{19}$$

belongs to the class (17) and it is reduced to the LEE

$$z_t = -h z_{xx} \tag{20}$$

by the transformation $z = \exp(-y/h)$. Let $z_1(t,x)$ and $z_2(t,x)$ be its linearly independent solutions. Then the complete integral $z = c_1 z_1 + c_2 z_2$ of LEE (20), where c_1 and c_2 are arbitrary constants, is a linear superposition principle, and the formula

$$y = -h \ln \left[\exp\left(-\frac{1}{h} y_1 - \frac{\lambda_1}{h} \right) + \exp\left(-\frac{1}{h} y_2 - \frac{\lambda_2}{h} \right) \right]$$

is a NSP for (19), where λ_1 and λ_2 are arbitrary constants, $y_1(x,t)$ and $y_2(x,t)$ are particular solutions, and $y(x,t)$ is a complete integral of equation (19).

Note that equation (19) can be linearized into the equation $z_t = z_{ss}$ by the transformation $z = \exp(-y/h)$, $ds = \sqrt{2/h}dx$.

Note 2. In idempotent analysis, the correspondence principle (in the sense of Maslov) is an NSP (see for example [9]).

Example 2. The equation

$$\begin{aligned} y_t &= y^{\text{iv}} + 4fy'y''' + 3fy''^2 + 6(f^2 + f^*)y'^2 y'' + (f^3 + 3ff^* + f^{**})y'^4 \\ &\quad + 4b_1(y''' + 3fy'y'' + (f^2 + f^*)y'^3) \\ &\quad + 6b_2(y'' + fy'^2) + 4b_3 y' + b_4 \exp\left(-\int f dy\right) \int \exp\left(\int f dy\right) dy = 0 \end{aligned}$$

can be reduced by substitution (18) to the corresponding fourth-order LEE.

A nonlinear ODE linearizable by transformation (18) was constructed in [10].

4 Nonlinear evolutionary equations with linearizable right-hand sides

Proposition 2. *NEE* (13) *has a solution of the traveling wave type*

$$y(\tau) = y(x - at) \tag{21}$$

and can be reduced to a linear ODE by the substitution

$$y = v(y)z, \qquad ds = u(y)d\tau, \tag{22}$$

if and only if

$$u^{n-1}\left(1 - \frac{v^*}{v} y\right) \frac{\partial y}{\partial t} = \prod_{k=n}^{1} \left[\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u} \right) \frac{\partial u}{\partial x} - r_k u \right] y, \tag{23}$$

where r_k are the roots of characteristic equation (10).

[1]This example was adduced in works of V.P. Maslov and his coauthors.

Proof. Sufficiency. First, note that the right-hand side of equation (23) generalizes formula (4). We seek a traveling wave type solution (21) for (23):

$$-au^{n-1}\left(1-\frac{v^*}{v}y\right)y_\tau=\prod_{k=n}^{1}\left[D_\tau-\left(\frac{v^*}{v}+(k-1)\frac{u^*}{u}\right)y_\tau-r_k u\right]y,\qquad D_\tau=\frac{d}{d\tau}. \tag{24}$$

Let us rewrite the left-hand side of (24) in the form

$$-au^{n-1}\left(D_\tau-\frac{v^*}{v}y_\tau\right)y$$

and apply substitution (22). We obtain, successively,

$$-au^{n-1}vD_\tau z=v\prod_{k=n}^{1}\left[D_\tau-(k-1)\frac{u^*}{u}y_\tau-r_k u\right]z,$$

$$-au^{n-1}vuD_s z=vu^n\prod_{k=n}^{1}(D_s-r_k)z,\qquad D_s=uD_\tau, \tag{25}$$

$$\sum_{k=0}^{n}\binom{n}{k}b_k z^{(n-k)}(s)+az'(s)=0,\qquad b_0=1,\qquad ()'=\frac{d}{ds}. \tag{26}$$

The **necessity** of conditions (23) is proved on the basis of equation (25). ∎

Theorem 2. *NEE* (13) *with condition* (23) *has a traveling wave type solution* (21) *if and only if* (23) *can be represented in the form*

$$\frac{\varphi^{\frac{3n^2-3n+2}{2n}}\exp\left(\int f dy\right)y}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy}\frac{\partial y}{\partial t}$$
$$=\prod_{k=n}^{1}\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy}+(k-1)\frac{\varphi^*}{\varphi}\right)\frac{\partial y}{\partial x}-r_k\varphi\right]y, \tag{27}$$

where r_k with $k=1,2,\dots,n$ are the roots of characteristic equation (10)*, or in lexicographic form*

$$\varphi^{n-1}y_t=\sum_{k_1+2k_2+\dots+nk_n=n}\Psi^{12\dots n}_{k_1k_2\dots k_n}y^{(1)k_1}y^{(2)k_2}\dots y^{(n)k_n}$$
$$+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m\left(\sum_{l_1+2l_2+\dots+(n-m)l_{n-m}=n-m}\Psi^{12\dots n-m}_{l_1l_2\dots l_{n-m}}y^{(1)l_1}y^{(2)l_2}\dots y^{(n-m)l_{n-m}}\right)$$
$$+b_n\exp\left(-\int f dy\right)\int\varphi^{\frac{n^2+n-2}{2n}}\exp\left(\int f dy\right)dy=0, \tag{28}$$

where the coefficients Ψ depend on $f(y)$ and $\varphi(y)$ and

$$\Psi^{12\dots n}_{00\dots1}=1,\qquad\Psi^{12\dots n-m}_{00\dots1}=1,\qquad\Psi^{12\dots n-1n}_{10\dots10}=nf(y).$$

Then substitution (21) *which gives linear ODE* (25) *has the form*

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy,\qquad ds=\varphi d\tau.$$

Proof. We apply Lemma 3, Proposition 2, formulas (9), (11), (12), (14), and the formula

$$1 - \frac{v^*}{v} y = \frac{\varphi^{\frac{n^2-3n+2}{2n}} \exp\left(\int f dy\right) y}{\int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy}. \qquad \blacksquare$$

Theorem 3 (see [5]). *Let $z_1(s), z_2(s), \ldots, z_n(s)$ be linearly independent particular solutions to equation* (25)*. Then the general integral of* (25) *is*

$$z = c_1 z_1(s) + c_2 z_2(s) + \cdots + c_n z_n(s),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, and this formula is an LSP. The NSP for (27), (28) *is obtained by the formulas*

$$z = \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) = \sum_{k=1}^{n} c_k z_k(s), \qquad ds = \varphi d\tau, \qquad \tau = x - at.$$

The general form of a second-order NEE belonging to class (28) *(see also* (5) *and* (6)*) is*

$$\varphi y_t = y_{xx} + f y_x^2 + 2b_1 \varphi y_x + b_2 \varphi \exp\left(-\int f dy\right) \int \varphi \exp\left(\int f dy\right) dy.$$

Next, we have

$$-a\varphi y_\tau = y_{\tau\tau} + f y_\tau^2 + 2b_1 \varphi y_\tau + b_2 \varphi \exp\left(-\int f dy\right) \int \varphi \exp\left(\int f dy\right) dy.$$

The substitution (see formula (7)) $z = \beta \int \varphi \exp\left(\int f dy\right) dy$, $ds = \varphi d\tau$ yields a linear equation

$$z''(s) + 2b_1 z'(s) + b_2 z(s) + a z'(s) = 0.$$

Example 3. Consider the equation

$$y y_t = y_{xx} + 3 y y_x + y^3. \tag{29}$$

Seeking a solution of type (21) for equation (29) leads to the ODE $-a y y_\tau = y_{\tau\tau} + 3 y y_\tau + y^3$. Using the substitution $y^2 = z$, $ds = y d\tau$, we obtain a linear equation

$$z''(s) + 3z'(s) + 2z(s) + az'(s) = 0.$$

The general form of an NEE of order $n = 3$ belonging to a class (28), is

$$\begin{aligned} \varphi^2 y_t = {} & y_{xxx} + 3 f y_x y_{xx} + \left(\frac{1}{3}\frac{\varphi_{yy}}{\varphi} - \frac{5}{9}\frac{\varphi_y^2}{\varphi^2} - \frac{1}{3} f \frac{\varphi_y}{\varphi} + f^2 + f_y\right) y_x^3 \\ & + 3b_1 \varphi \left[y_{xx} + \left(f + \frac{1}{3}\frac{\varphi_y}{\varphi}\right) y_x^2\right] + 3b_2 \varphi^2 y_x \\ & + b_3 \varphi^{5/3} \exp\left(-\int f dy\right) \int \varphi^{4/3} \exp\left(\int f dy\right) dy. \end{aligned} \tag{30}$$

Example 4. Consider the Harry–Dym equation

$$y_t = y^3 y_{xxx}. \tag{31}$$

It belongs to class (30), namely

$$\varphi^2 y_t = y_{xxx} + \left(\frac{1}{3}\frac{\varphi_{yy}}{\varphi} - \frac{5}{9}\frac{\varphi_y^2}{\varphi^2}\right) y_x^3 + b_3\varphi^{3/5}\int \varphi^{4/5} dy$$

with $f = 0$, $b_3 = 0$, $\varphi = y^{-3/2}$. It admits the representation

$$y^{-3} y_t = \left(\partial_x + \frac{y_x}{y}\right)\left(\partial_x - \frac{1}{2}\frac{y_x}{y}\right) y_x.$$

The ordinary differential equation corresponding to a solution of (31) of type (21) has the form

$$y'''(\tau) + a y^{-3} y'(\tau) = 0. \qquad (32)$$

Equation (32) is reduced to the linear form $z'''(s) + az'(s) = 0$ by the substitution $z = 1/y$, $ds = y^{-3/2} d\tau$.

5 Nonlinear evolutionary equations with factorable right-hand sides

The new classes of the nonlinear evolutionary equations (NEE) of n-th order, depending on two arbitrary functions and $n-1$ parameters, are constructed (see also [11]):

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left[\frac{v}{v - v^* y}\prod_{k=n-1}^{1}\left(\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right)\frac{\partial y}{\partial x} - r_k u\right)\right] y, \qquad (33)$$

where $r_k = \mathrm{const}$, $k = 1, \ldots, n-1$, $v = v(y)$, $u = u(y)$, $()^* = d/dy$.

Besides solutions such as a traveling wave, stationary solutions of the equation (33) are also of interest. The corresponding ODE have the form

$$\prod_{k=n-1}^{1}\left[\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right)\frac{dy}{dx} - r_k u\right] y = C\left(1 - \frac{v^*}{v} y\right), \qquad D = \frac{d}{dx}. \qquad (34)$$

By substitution $y = v(y)z$, $ds = u(y)dx$ the equation (34) is reduced to a semilinear equation

$$\prod_{k=n-1}^{1}(D_s - r_k)z = C\left(1 - \frac{v^*}{v} y\right) v^{-1} u^{1-n}, \qquad (35)$$

where right-hand side is a function of z.

Example 5. The Korteweg–de Vries equation (KdV)

$$y_t + y_{xxx} - 6yy_x = 0$$

belongs to the class (33) and admits a representation of the form

$$y_t + \partial_x(y_{xx} - 3y^2) = 0$$

and also a factorization

$$y_t + \frac{2}{3}\partial_x\left(\partial_x - r_2 y^{1/2}\right)\left(\partial_x + \frac{1}{2}\frac{y_x}{y} - r_1 y^{1/2}\right) y = 0,$$

$$r_{1,2} = \mp\frac{3}{\sqrt{2}}, \qquad u = y^{1/2}, \qquad v = y^{-1/2}.$$

The semilinear equation corresponding to (35) is the equation

$$z''(s) - \frac{9}{2}z = Cz^{-1/3}.$$

Example 6. Consider modified KdV-equation (MKdV)

$$y_t + y_{xxx} - 6y^2 y_x = 0.$$

It admits a factorization $y_t + \partial_x(y_{xx} - 2y^3) = 0$, and the following

$$y_t + \frac{1}{2}\partial_x(\partial_x - 2y)\left(\partial_x + \frac{y_x}{y} + 2y\right)y = 0,$$

here $u = y$, $v = y^{-1}$, $r_{1,2} = \pm 2$.

Example 7. Generalized MKdV-equation

$$y_t + y_{xxx} - ay^k y_x = 0$$

admits first a representation

$$y_t + \partial_x\left(y_{xx} + \frac{a}{k+1}y^{k+1}\right) = 0,$$

and then a factorization

$$y_t + \frac{2}{k+2}\partial_x\left(\partial_x - r_2 y^{k/2}\right)\left(\partial_x + \frac{k}{2}\frac{y_x}{y} - r_1 y^{k/2}\right)y = 0,$$

where $u = y^{k/2}$, $v = y^{-k/2}$, $r_{1,2} = \pm i\sqrt{\frac{a(k+2)}{2(k+1)}}$.

Corresponding semilinear equation has the form

$$\ddot{z} + \frac{a(k+2)}{2(k+1)}z = Cz^{-\frac{k}{k+2}}.$$

[1] Lax P.D., Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.*, 1968, V.21, 467–490.

[2] Novikov S.P., Periodic problem for Korteweg–de Vries equation. I, *Funkts. Anal. Pril.*, 1974, V.8, N 3, 54–66.

[3] Polyanin A.D. and Zaitsev V.F., Handbook of nonlinear equations of mathematical physics. Exact solutions, Moscow, Fizmatlit, 2002 (in Russian).

[4] Berkovich L.M., Factorization and transformations of the differential equations. Methods and applications, Moscow, Regular and Chaotic Dynamics, 2002 (in Russian).

[5] Berkovich L.M., A new class of nonlinear evolution equations, *Doklady Mathematics*, V.67, N 3, 2003, 391–395 (translated in *Doklady Akademii Nauk*, 2003, V.390, N 5, 583–587).

[6] Berkovich L.M., The method of an exact linearization of n-order ordinary differential equations, *J. Nonlinear Math. Phys.*, 1996, V.3, N 3–4, 341–350.

[7] Berkovich L.M., Factorization of nonlinear ordinary differential equations and linearization, *Doklady Mathematics*, V.60, N 2, 1999, 278–282 (translated in *Doklady Akademii Nauk*, V.368, N 5, 1999, 604–608).

[8] Berkovich L.M., On the method of exact linearization of autonomous ordinary differential equations, Preprint 2001(73), Bonn, Max-Planck-Institut für Mathematik, 2001, 24 p.

[9] Maslov V.P. and Kolokol'tsov V.N. Idempotent analysis and its application to optimal control, Moscow, Fizmatlit, 1994 (in Russian).

[10] Berkovich L.M. and Orlova I.S., The exact linearization of some classes of ordinary differential equations for order $n > 2$, in Proceedings of Third International Conference "Symmetry in Nonlinear Mathematical Physics" (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Insitute of Mathematics, 2000, V.30, Part 1, 90–98.

[11] Berkovich L.M., Nonlinear equations: factorization, linearization, exact solutions, in book of Abstracts of Intern. Conf. "Kolmogorov and Contemporary Mathematics" (16–21 June, Moscow, 2003), Moscow, MSU, 2003, 22–23.