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# Integration of Bi-Hamiltonian Systems by Using the Dressing Method

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The integration by the dressing method of integrable by Lax systems from the $\mathcal{D}$HcmKP hierarchy is considered. The applicability of this method to construction of exact solutions of the nonlinear bi-Hamiltonian systems, also with nonstandart (recursive) Lax representations, is shown.

## 1 Introduction

The paper [1] introduces the so-called scalar $\mathcal{D}$-Hermitian constrained modified Kadomtsev–Petviashvili ($\mathcal{D}$HcmKP) hierarchy. The integrable systems of this hierarchy contain already known nonlinear models of the soliton theory and their new modifications and vector (multi-component) generalizations. The unified form of the Lax operator (4) allows to construct a general method of the Lax flows investigation describing a group of transformation operators (group of $\mathcal{D}$-unital Volterra operators) which corresponds to the Lie algebra of the integro-differential symbols of $\mathcal{D}$-skew-Hermitian operators.

This paper continues integration of integrable systems from $\mathcal{D}$HcmKP by using the method of dressing transformations. For $n = 2$, under additional reduction, Lax operator (4) becomes the generating operator for the modified Korteweg–de Vries equation (mKdV) in a real case. The vector generalization of the mKdV equation can be reduced to common Korteweg–de Vries equation (KdV).

In Section 2 we submit basic definitions in the $\mathcal{D}$HcmKP hierarchy and reductions to well-known dynamical systems.

In Sections 3 we propose the method of construction of exact solutions for the KdV equation and the mKdV equation.

## 2 $\mathcal{D}$HcmKP hierarchy and its reductions

Let us consider the algebra $\zeta$ of the micro-differential operators [2],

$$\zeta := \left\{ L = \sum_{i=-\infty}^{n(L)} a_i \mathcal{D}^i : \ a_i = a_i(x, y, t_m); \ i, n(L) \in \mathbb{Z} \right\}.$$

The coefficients $a_i$ are, in general, smooth $(N \times N)$-matrix-valued functions of $x \in \mathbb{R}$ and of finite quantity of the evolution parameters $t_m \in \mathbb{R}$, $t_2 := y$, $t_3 := t$. The micro-differential operator $L \in \zeta$ satisfies additional constraints. The Hermitian-conjugated operator $L^* := \sum_{i=-\infty}^{n(L)} (-1)^i \mathcal{D}^i a_i^*$, where $a_i^* = \bar{a}^\top$, $(\alpha\partial_y)^* := -\bar{\alpha}\partial_y$, $(\beta\partial_{t_m})^* := -\bar{\beta}\partial_{t_m}$.

**Definition 1.** We say that an operator $L \in \zeta$ is $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) if $L^* = \mathcal{D}L\mathcal{D}^{-1}$ ($L^* = -\mathcal{D}L\mathcal{D}^{-1}$).

**Definition 2.** We say that an integral operator $W \in \zeta_{<1} := \left\{ L_{<1} := \sum_{i=-\infty}^{0} u_i \mathcal{D}^i \right\}$ is $\mathcal{D}$-unital if $W^{-1} = \mathcal{D}^{-1} W^* \mathcal{D}$.

**Lemma 1 ([1]).** *Let $L$ be a $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) operator and $W$ is a $\mathcal{D}$-unital operator. Then $\hat{L} := WLW^{-1}$ is a $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) operator.*

Let $\varphi = \varphi(x, y, t_m)$ be a matrix $N \times K$ function, $\Omega := C + \int_{-\infty}^{x} \varphi^* \varphi_x dx$ be a non-degenerate $K \times K$ function such that the improper integral $\int_{-\infty}^{x} \varphi^* \varphi_x dx$ converges absolutely, $C$ be a constant complex $K \times K$-matrix.

**Theorem 1 ([1]).** 1. *Let $C^* = -C = \text{const} \in \text{Mat}_{K\times K}(\mathbb{C})$ and $w_0 := I_N - \varphi \Omega^{-1} \varphi^*$ (where $I_N$ is a unitary $(N \times N)$-matrix. Then $w_0^{-1} = w_0^* = I_N - \varphi \Omega^{*-1} \varphi^*$, where $\Omega^* = \varphi^* \varphi - \Omega$.*

2. *Operator $W := w_0 + \varphi \Omega^{-1} \mathcal{D}^{-1} \varphi_x^*$ is a $\mathcal{D}$-unital and the inverse operator is defined by the formula $W^{-1} := w_0^{-1} + \varphi \mathcal{D}^{-1} (\Omega^{*-1} \varphi^*)_x$.*

**Lemma 2 ([3]).** *The following property holds true:*

$$\det w_0 = (-1)^K \frac{\det \Omega^*}{\det \Omega}.$$

**Remark 1.** Let $\varphi \in \text{Mat}_{N\times K}(\mathbb{R})$ and $C^\top = -C$. Then $\det w_0 = \det(I_N - \varphi \Omega^{-1} \varphi^\top) = (-1)^K$.

Let us consider the modified Korteweg–de Vries equation (mKdV)

$$u_t = u_{xxx} + a u^2 u_x,$$

where $u = u(x,t) \in C^{(\infty)}(\mathbb{R}^2, \mathbb{R})$.

$$u_t = \mathcal{L}(\nabla H_1) = \mathcal{M}(\nabla H_2),$$

where $\mathcal{L} = -\mathcal{D}$, $\mathcal{M} = \mathcal{D}^3 + \frac{2}{3} a u \mathcal{D} u \mathcal{D}^{-1} u \mathcal{D}$ is a Hamiltonian pair and the functionals

$$H_1 = \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2 - \frac{1}{12} a u^4 \right) dx, \qquad H_2 = \int_{\mathbb{R}} \frac{1}{2} u^2 dx$$

are first integrals of the mKdV equation.

The generation operators $\Lambda_{\text{mKdV}} = \mathcal{L}^{-1}\mathcal{M}$ and $\Lambda^\tau_{\text{mKdV}} = \mathcal{M}\mathcal{L}^{-1}$ have the form

$$\Lambda = -\mathcal{D}^2 - \frac{2}{3} a u \mathcal{D}^{-1} u \mathcal{D}, \qquad \Lambda^\tau = -\mathcal{D}^2 - \frac{2}{3} a \mathcal{D} u \mathcal{D}^{-1} u \tag{1}$$

and satisfy the equation of Lax type $\Lambda_{t_m} = [\Lambda, K'^\tau]$, where $K'^\tau = -\mathcal{D}^3 - a u \mathcal{D}$.

Let us consider also the KdV equation

$$u_t = u_{xxx} + a u u_x = K[u]. \tag{2}$$

Similarly to the previous equation, the operators will have the form

$$\begin{aligned} &\Lambda_{\text{KdV}} = \mathcal{D}^2 + \frac{2}{3} a u - \frac{1}{3} a \mathcal{D}^{-1} u_x, \qquad \Lambda^\tau_{\text{KdV}} = \mathcal{D}^2 + \frac{2}{3} a u + \frac{1}{3} a u_x \mathcal{D}^{-1}, \\ &K' = \mathcal{D}^3 + a u \mathcal{D} + a u_x, \qquad K'^\tau = -\mathcal{D}^3 - a u \mathcal{D}. \end{aligned} \tag{3}$$

The operators $\Lambda$, $K'^\tau$ are $\mathcal{D}$-Hermitian and $\mathcal{D}$-skew-Hermitian respectively. These operators are partial cases of the $\mathcal{D}$-Hermitian constrained modified Kadomtsev–Petviashvili ($\mathcal{D}$HcmKP) hierarchy introduced in the paper [1] and determined in the following form: let

$$\zeta \ni L_{\mathcal{D}\text{HcmKP}} := L_n = \mathcal{D}^n + u_{n-1} \mathcal{D}^{n-1} + \cdots + u_1 \mathcal{D} - \boldsymbol{q} \mathcal{M} \mathcal{D}^{-1} \boldsymbol{q}^* \mathcal{D}, \tag{4}$$

$\mathcal{M}^* = (-1)^n\mathcal{M}$ is a complex constant $(l \times l)$-matrix, $\boldsymbol{q}(x,t_m) = (q_1,\dots,q_l)$, $k,l \in \mathbb{N}$, and an additional reduction for operator $L_n$: $L_n^* = \mu\mathcal{D}L_n\mathcal{D}^{-1}$, $\mu = \pm 1$.

We consider the evolution equations

$$\alpha_m L_{n_{t_m}} = [B_m, L_n], \tag{5}$$

where $L_n := L_{\mathcal{D}\mathrm{HcmKP}}$, and $B_m$ are fractional powers $m/n$ of $L_n$; $n,m \in \mathbb{N}$.

Let $n = 2$. For $L_2 = \mathcal{D}^2 + iu\mathcal{D} - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D}$ we obtain that $B_2 = (L_2)_{>0} = \mathcal{D}^2 + iu\mathcal{D}$, $B_3 = \mathcal{D}^3 + \frac{3}{2}iu\mathcal{D}^2 - \left(\frac{3}{8}u^2 - \frac{3i}{4}u_x + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*\right)\mathcal{D}$, $\mathcal{M}^* = \mathcal{M}$. For $\alpha_2 = i$, $\alpha_3 = 1$ the following systems of equations are consequences of (5)

$$i\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} + iu\boldsymbol{q}_x, \qquad u_{t_2} = 2(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*)_x \tag{6}$$

and

$$\begin{aligned}\boldsymbol{q}_{t_3} &= \boldsymbol{q}_{xxx} + \frac{3}{2}iu\boldsymbol{q}_{xx} - \left(\frac{3}{8}u^2 + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^* - \frac{3}{4}iu_x\right)\boldsymbol{q}_x,\\ u_{t_3} &= \frac{1}{4}u_{xxx} + \frac{3}{8}u^2u_x - \frac{3}{2}(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*u)_x + \frac{3}{2}i(\boldsymbol{q}_x\mathcal{M}\boldsymbol{q}^* - \boldsymbol{q}\mathcal{M}\boldsymbol{q}^*_x)_x.\end{aligned} \tag{7}$$

System (6) is a multi-component modification of the integrable Yajima–Oikawa model [4], which describes the interaction of the Laengmur wave packets in the physics of plasma. System (2) is a modification of a higher Yajima–Oikawa model. Note that equation (2) admits some interesting reductions on invariant sub-manifolds, where evolution is introduced by known dynamical systems. So, for $\boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D} \equiv 0$ we have the scalar mKdV equation $u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}u^2u_x$. For $u \equiv 0$ the complex multi-component mKdV equation obtained

$$\boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*\boldsymbol{q}_x \tag{8}$$

with the differential condition

$$(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*_x - \boldsymbol{q}_x\mathcal{M}\boldsymbol{q}^*)_x = 0, \tag{9}$$

and for $\bar{\boldsymbol{q}} \equiv \boldsymbol{q}$, $\mathcal{M} \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ its real version is

$$\boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\boldsymbol{q}_x. \tag{10}$$

In this case, the differential constraint is satisfied.

**Remark 2.** The vector generalization of the complex mKdV equation can be obtained from system (8)–(9), if we satisfy condition (9) in such a way: let us $\boldsymbol{q} = (\vec{\boldsymbol{q}}, \bar{\vec{\boldsymbol{q}}})$, $\vec{\boldsymbol{q}} = (q_1,\dots,q_k)$ be a $k$-component vector-function ($l = 2k$) and the matrix $\mathcal{M}$ have a block form

$$\mathcal{M} = \begin{pmatrix} B & A \\ \bar{A} & \bar{B}\end{pmatrix}, \qquad A^\top = A, \qquad B^* = B. \tag{11}$$

Condition (9) is satisfied similarly.

## 3 Integration of some bi-Hamiltonian systems

**Theorem 2.** *Let:* 1) $\varphi(x,t_3)$ *be a real* $K$*-component solution of the system of equations*

$$\varphi_{t_3} = \varphi_{xxx}, \qquad \varphi_{xx} = \varphi\Lambda. \tag{12}$$

2) $C^\top = -C$ *is a real skew-symmetric matrix.*

3) *Matrix function* $\Omega := C + \int_{-\infty}^x \varphi^\top\varphi_x dx$ *is non-degenerate on the left semi-axis.*

*Then the function* $\boldsymbol{q}(x,t_3) := \varphi\Omega^{-1}$ *satisfies mKdV equation* (10)*, where* $\mathcal{M} = C\Lambda - \Lambda^\top C$.

**Proof.** Let $L_0 := \mathcal{D}^2$, then using Theorem 1

$$L := W\mathcal{D}^2W^{-1} = \mathcal{D}^2 - 2w_{0x}w_0^{-1}\mathcal{D} - \left[\varphi_{xx} - \varphi\Omega^{-1}\int_{-\infty}^{x}\varphi^*\varphi_{xxx}dx\right]\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}$$
$$- \varphi\Omega^{-1}\mathcal{D}^{-1}\left[\varphi^*_{xx} - \int_{-\infty}^{x}\varphi^*_{xxx}\varphi dx\Omega^{*-1}\varphi^*\right]\mathcal{D}.$$

Under conditions a)–b) we obtain that

$$L = \mathcal{D}^2 - 2w_{0x}w_0^{-1}\mathcal{D} - \varphi\Omega^{-1}(C\Lambda - \Lambda^*C)\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}.$$

Let $M_0 := \partial_{t_3} - \mathcal{D}^3$, similarly we proceed to the operator

$$M := WM_0W^{-1} = \partial_{t_3} - \mathcal{D}^3 + 3w_{0x}w_0^{-1}\mathcal{D}^2$$
$$- \big[w_0(w_0^{-1})_{xx} - 2w_0\varphi_{xx}\Omega^{*-1}\varphi^* - w_0\varphi_x(\Omega^{*-1}\varphi^*)_x$$
$$+ \varphi\Omega^{-1}\varphi^*_x(w_0^{-1})_x - \varphi\Omega^{-1}\varphi^*_{xx}w_0^{-1} - \varphi\Omega^{-1}\varphi^*_x\varphi_x\Omega^{*-1}\varphi^*\big]\mathcal{D}$$
$$- \left[\varphi_{t_3} - \varphi_{xxx} - \varphi\Omega^{-1}\int_{-\infty}^{x}\varphi^*(\varphi_{t_3} - \varphi_{xxx})_x dx\right]\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}$$
$$+ \varphi\Omega^{-1}\mathcal{D}^{-1}\left[\varphi^*_{t_3} - \varphi^*_{xxx} - \int_{-\infty}^{x}(\varphi^*_{t_3} - \varphi^*_{xxx})_x\varphi dx\Omega^{*-1}\varphi^*\right]\mathcal{D}.$$

Under condition a) the integral parts are equal to zero. With Remark 1 the formulas for the "dressed" operators $L$ and $M$ are:

$$L = \mathcal{D}^2 - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^\top\mathcal{D}, \qquad M := \partial_{t_3} - \mathcal{D}^3 + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\mathcal{D}. \tag{13}$$

For $l = 1$ (that is $\boldsymbol{q} = q$ is a scalar function) the operators $L$ and $M$ (13) constitute the recursive Lax pair for the mKdV equation (1). ■

Let us consider Lax operators (13) for the $K$-component real mKdV equation (10),

$$[L, M] = 0 \;\Leftrightarrow\; \boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\boldsymbol{q}_x = K[\boldsymbol{q}]. \tag{14}$$

The operator $L$ is a $K$-generalization [5] of the generating operator $\Lambda_{\text{mKdV}}$ (1) for the scalar mKdV equation and $M^\tau$ is its linearization ($M^\tau = \partial_{t_3} - K'[\boldsymbol{q}]$).

Let $K = 2k$, $\boldsymbol{q} = (\vec{\alpha}, \vec{\boldsymbol{q}})$, $\vec{\alpha} \in \mathbb{R}^k$, $\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(x, t_3)$,

$$\mathcal{M} = \begin{pmatrix} 0 & -\Lambda^\top \\ -\Lambda & 0 \end{pmatrix}, \tag{15}$$

where $\Lambda$ is a constant real $(k \times k)$-matrix. In this case, operators (13) reduce to the form

$$L = \mathcal{D}^2 - (\vec{\alpha}, \vec{\boldsymbol{q}})\mathcal{M}(\vec{\alpha}, \vec{\boldsymbol{q}})^\top + (\vec{\alpha}, \vec{\boldsymbol{q}})\mathcal{M}\mathcal{D}^{-1}(\vec{\alpha}, \vec{\boldsymbol{q}})^\top_x$$
$$= \mathcal{D}^2 + 2\vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top - \mathcal{D}^{-1}(\vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top)_x = \mathcal{D}^2 + 2u - \mathcal{D}^{-1}u_x =: \Lambda_{KdV}, \tag{16}$$
$$M = \partial_{t_3} - \mathcal{D}^3 - 3u\mathcal{D}, \tag{17}$$

where $u = \vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top$, and constitute a recursive Lax pair for KdV equation (see (2)–(3) for $a = 3$)

$$[\partial_{t_3} + K'^\tau[u], \Lambda_{KdV}] = 0 \;\Leftrightarrow\; \frac{\partial}{\partial t_3}\Lambda_{KdV} = [\Lambda_{KdV}, K'^\tau]$$
$$\Leftrightarrow u_{t_3} = K[u] = u_{xxx} + 3uu_x. \tag{18}$$

Described process of reduction of the real version of the vector mKdV equation (14) to scalar KdV equation (18) allows to formulate the following statement.

**Theorem 3.** *Let:* 1) $\varphi(x,t_3) = (\vec{\varphi}, \vec{\alpha})$, *where* $\vec{\varphi} = \vec{\varphi}(x,t_3)$ *is a $k$-component real field,* $\mathbb{R}^k \ni \vec{\alpha}$ *is a $k$-component real vector.*

2) $\vec{\varphi}$ *is a solution of the linear system*

$$\vec{\varphi}_{t_3} = \vec{\varphi}_{xxx}, \qquad \vec{\varphi}_{xx} = \vec{\varphi}\Lambda, \qquad \Lambda \in \mathrm{Mat}_{k\times k}(\mathbb{R}).$$

3) $\Omega := C + \int_{-\infty}^{x} \varphi^\top \varphi_x dx$ *is a non-degenerate on the left semi-axis* $(2k)\times(2k)$*-matrix function, where* $C = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

*Then the function*

$$u(x,t_3) := \vec{\alpha}\Lambda^\top (\vec{\varphi}^\top \vec{\alpha} - I_k)^{-1} \left( \vec{\varphi}^\top + \int_{-\infty}^{x} \vec{\varphi}_x^\top \vec{\varphi}\, dx \vec{\alpha}^\top \right)$$

*is a solution of KdV equation* (18).

**Proof.** We consider

$$\Omega := C + \int_{-\infty}^{x} \varphi^\top \varphi_x\, dx = \begin{pmatrix} \displaystyle\int_{-\infty}^{x} \vec{\varphi}^\top \vec{\varphi}_x\, dx & I_k \\ \vec{\alpha}^\top \vec{\varphi} - I_k & 0 \end{pmatrix}.$$

In order to prove the Theorem we use the known formula for the matrix $A^{-1}$ that is inverse the matrix for the block $(2k)\times(2k)$-matrix $A$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \ \rightarrow \ A^{-1} = \begin{pmatrix} A_{11}^{-1}(I + A_{12}T^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}T^{-1} \\ -T^{-1}A_{21}A_{11}^{-1} & T^{-1} \end{pmatrix},$$

where $T = A_{22} - A_{21}A_{11}^{-1}A_{12}$. In our case $A_{12} = I_k$, $A_{22} = 0 := 0_k$, $T = -A_{21}A_{11}^{-1} \Rightarrow T^{-1} = -A_{11}A_{21}^{-1}$, whence we derive the simple formula for the matrix $\Omega^{-1}$

$$\Omega^{-1} = \begin{pmatrix} 0 & A_{21}^{-1} \\ I_k & -A_{11}A_{21}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & (\vec{\alpha}^\top\vec{\varphi} - I_k)^{-1} \\ I_k & -\displaystyle\int_{-\infty}^{x} \vec{\varphi}^\top \vec{\varphi}_x\, dx\, (\vec{\alpha}^\top \vec{\varphi} - I_k)^{-1} \end{pmatrix}.$$

Thus $\boldsymbol{q} := \varphi\Omega^{-1} = (\vec{\alpha}, \vec{\boldsymbol{q}})$, where

$$\vec{\boldsymbol{q}} = \left( \vec{\varphi} + \vec{\alpha} \int_{-\infty}^{x} \vec{\varphi}^\top \vec{\varphi}_x\, dx \right) (\vec{\alpha}^\top \vec{\varphi} - I_k)^{-1}. \tag{19}$$

The matrix $\mathcal{M} = C\hat{\Lambda} - \hat{\Lambda}^\top C$ is of the form $\mathcal{M} = \begin{pmatrix} 0 & -\Lambda^\top \\ -\Lambda & 0 \end{pmatrix}$, as $\hat{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}$. For the completion of the proof it is sufficient to refer to formulas (15)–(19) and to the corresponding results of Theorem 3 for the mKdV equation (14). ■

The possibility of application of other reductions in $\mathcal{D}$HcmKP hierarchy requires further research.

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[1] Sidorenko Yu., Transformation operators for integrable hierarchies with additional reductions, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 1, 352–357.

[2] Dickey L.A., Soliton equations and Hamiltonian systems, *Advanced Series in Mathematical Physics*, Vol. 12, 1991.

[3] Sidorenko Yu.M. and Berkela Yu.Yu., Integration of nonlinear space-two-dimensional Heisenberg equations, *Matematychni Studii*, 2002, V.18, N 1, 57–68.

[4] Yajima N. and Oikawa M., Formation and interaction of Sonic-Langmuir solitons-inverse scattering method, *Progress Theor. Phys.*, 1976, V.56, N 6, 1719–1739.

[5] Sidorenko Yu.M., KP-hierachy and $(1 + 1)$-dimensional multicomponent integrable systems, *Ukr. Math. J.*, 1993, V.45, N 1, 91–104.