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Conformal Maps and Integrable Hamiltonian Structures

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A discussion of integrable structures of complex analysis is presented. It is shown that a conformal map $z(w)$ of the exterior of the unit disc to the exterior of a simply-connected domain and a map $\overline{z}(w^{-1})$ are conjugate variables with respect to some Poisson structure. Deformations of the functions $z(w)$, $\overline{z}(w^{-1})$ with respect to the harmonic moments of the domain are described by the Hamiltonian equations. These equations are known as the Lax–Sato equations.

1 Introduction

Basic notions of the complex analysis and the physics have deep inter-connections. It is well known that an analytic function of complex variable may be considered as a complex potential of plane vector field and this fact leads to construction of the conformal field theory. But it is less known that a conformal map may be considered as an integrable Hamiltonian system, although this connection between complex analysis and mechanics was established some years ago. In our report we present a pedagogical introduction to this problem.

2 Moments of a curve (domain). Generating function of moments

We designate a curve on complex plane z as a closed analytic curve if it is an analytic image of circle, i.e. if it is parametrized by a function $z(w)$ that is analytic in a ring $a < |w| < b$ containing a circle $|w| = 1$.

Let us consider on a complex plane z a closed analytic curve Γ and designate by D_+ and D_- internal and external domains with respect to this curve. We shall assume that the point $z = 0$ belongs to the domain D_+.

For the curve Γ (or domains D_+ and D_-) we introduce two sets of the harmonic moments, namely, the external moments $\{t_0, t_1, \ldots\}$,

$$t_0 = \frac{1}{\pi}\int_{D_-} \mathrm{d}^2 z, \qquad t_k = -\frac{1}{\pi k}\int_{D_-} z^{-k}\mathrm{d}^2 z, \qquad k \geq 1,$$

and the internal moments $\{v_0, v_1, \ldots\}$,

$$v_0 = \frac{2}{\pi}\int_{D_+} \log|z|\mathrm{d}^2 z, \qquad v_k = \frac{1}{\pi}\int_{D_+} z^k \mathrm{d}^2 z, \qquad k \geq 1.$$

Moments t_0, v_0 are real numbers, other moments are complex numbers.

Let us introduce a generating function of moments F and τ-function

$$F = \log\tau = -\frac{1}{\pi}\int_{D_+} \mathrm{d}^2 z \int_{D_+} \mathrm{d}^2 z' \log\left|\frac{1}{z} - \frac{1}{z'}\right|.$$

The following theorem is valid.

Theorem 1. *A differential of the generating function of moments F (or the logarithm of τ-function) is of a form*

$$\mathrm{d}F = \mathrm{d}\log\tau = v_0\mathrm{d}t_0 + \sum_{k=1}^{\infty}\left(v_k\mathrm{d}t_k + \overline{v}_k\mathrm{d}\overline{t}_k\right),$$

and it allows to express the internal moments in terms of external ones,

$$v_0 = \frac{\partial F}{\partial t_0} = \frac{\partial\log\tau}{\partial t_0}, \qquad v_k = \frac{\partial F}{\partial t_k} = \frac{\partial\log\tau}{\partial t_k}, \qquad \overline{v}_k = \frac{\partial F}{\partial\overline{t}_k} = \frac{\partial\log\tau}{\partial\overline{t}_k}.$$

Thus, the sets of external and internal moments are not independent, there is connection between them. Any of these sets of moments defines uniquely the curve Γ (or domains D_+ and D_-) [1–3].

3 The Schwarz function $S(z)$ and its potential $\Omega(z)$

Let us present the equation $g(x,y)=0$ of an analytic curve Γ in complex coordinates, $g(\frac{z+\overline{z}}{2},\frac{z-\overline{z}}{2i})$ $=0$, and resolve the latter one with respect to the variable z. We obtain the equation of the curve Γ in the following form

$$\overline{z} = S(z).$$

The function $S(z)$ is called the Schwarz function of the curve Γ. This function is analytic in a strip that contains the curve [4].

The Schwarz function maps a point z into the conjugate one $S(z)$ with respect to the curve Γ. On the curve itself the Schwarz function coincides obviously with the complex conjugation.

According to definition the Schwarz function cannot be an arbitrary one, it must satisfy the so called unitary condition

$$z = \overline{S}(S(z)),$$

i.e. the inverse function must coincide with with the complex conjugated one. We remark here that if a function $f(z)$ is defined with the Laurent series $f(z)=\sum_j f_j z^j$, then $\overline{f}(z)=\sum_j \overline{f}_j z^j$.

Theorem 2. *The Laurent series for the Schwarz function $S(z)$ of a curve Γ is of the form*

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1} = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1},$$

i.e. the Laurent series coefficients of the Schwarz function are expressed in terms of the moments of the curve.

Proof. Let us assume that a function $f(z)$ is analytic in the domain $D_\pm$. Then for the function $g(z)=f(z)\overline{z}=f(z)S(z)$ the Green–Ostrogradskii formula

$$\int_{D_\pm}\frac{\partial g(z)}{\partial\overline{z}}\mathrm{d}^2z = \pm\frac{1}{2i}\int_\Gamma g(z)\mathrm{d}z,$$

leads to the relation

$$\int_{D_\pm} f(z)\mathrm{d}^2z = \pm\frac{1}{2i}\int_\Gamma f(z)S(z)\mathrm{d}z.$$

Using this formula in the case $f(z) = z^k$, $k \in \mathbb{Z}$, we can express the moments of curve Γ in terms of the Schwarz function $S(z)$,

$$t_k = \frac{1}{2\pi i k} \int_\Gamma z^{-k} S(z) \mathrm{d}z, \qquad v_k = \frac{1}{2\pi i} \int_\Gamma z^k S(z) \mathrm{d}z.$$

Substituting in integrals the Schwarz function in the form of its Laurent series,

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1}$$

and calculating integrals by means of the orthogonality relation

$$\frac{1}{2\pi i} \int_\Gamma z^j \mathrm{d}z = \delta_{j,-1},$$

we obtain

$$s_0 = t_0, \qquad s_k = k t_k, \qquad s_{-k} = v_k, \qquad k \geq 1.$$

Thus the moments of the curve Γ are the coefficients of the Laurent series for the Schwarz function $S(z)$,

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1} = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}. \qquad \blacksquare$$

Now we introduce the function $\Omega(z)$ that is a potential for the Schwarz function,

$$S(z) = \partial_z \Omega(z).$$

According to definition we can present the function $\Omega(z)$ in a form of the Laurent series,

$$\Omega(z) = \sum_{k=1}^{\infty} t_k z^k - \frac{1}{2} v_0 + t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k},$$

and hence as a sum of the following functions,

$$\Omega(z) = \Omega^{(+)}(z) + \Omega^{(-)}(z) - \frac{1}{2} v_0,$$

where the functions $\Omega^{(\pm)}(z)$ are analytic in domains $D_\pm$ appropriately. Here we have used the notations

$$\Omega^{(+)}(z) = \sum_{k=1}^{\infty} t_k z^k = \frac{1}{\pi} \int_{D_-} \log\left(1 - \frac{z}{z'}\right) \mathrm{d}^2 z',$$

$$\Omega^{(-)}(z) = t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k} = \frac{1}{\pi} \int_{D_+} \log\left(z - z'\right) \mathrm{d}^2 z'.$$

At the curve Γ we can write down the potential $\Omega(z)$ in the form

$$\Omega(z) = \frac{1}{2} |z|^2 + i2A(z), \qquad z \in \Gamma,$$

where $|z|$ is the distance from the origin of coordinates to the point z, and $A(z)$ is the area of the domain D_+, bounded with a ray $\phi = \arg z$ and a real line.

The following theorem is valid.

Theorem 3. *The differential of the function $\Omega(z)$ is of the form*

$$\mathrm{d}\Omega = S\mathrm{d}z + \log w\,\mathrm{d}t_0 + \sum_{k=1}^{\infty}\left(H_k\,\mathrm{d}t_k - \overline{H}_k\,\mathrm{d}\bar{t}_k\right),$$

where

$$H_k(z) = \partial_{t_k}\Omega(z) = \left(z^k(w)\right)_+ + \frac{1}{2}\left(z^k(w)\right)_0,$$

$$\overline{H}_k(z) = -\partial_{\bar{t}_k}\Omega(z) = -\left(\overline{z}^k(w)\right)_- - \frac{1}{2}\left(\overline{z}^k(w)\right)_0.$$

Here the symbol $(f(w))_\pm$ designates the part of the Laurent series that contains summands only with positive (negative) powers w, and $(f(w))_0$ designates the summand with w^0.

4 Conformal maps as Hamiltonian systems with respect to moments

Let us consider the disc of unit radius $U(0,1)$ on a plane w, and the domain D on a plane z. Let us consider the univalent analytic function $z(w)$, that maps conformally a complement of the unit disc to a complement of the domain D. In order to emphasize a dependence of this function on the domain D it would be reasonable to write down the function in the form $z(w,D)$ or $z(w,t_0,t_1,\ldots)$ since moments $(t_0,t_1,\ldots)$ of the domain D define it uniquely. We shall use the latter designation.

For arbitrary functions $f(w,t_0)$, $g(w,t_0)$ let us define the Poisson brackets as follows

$$\{f,g\} = \frac{\partial f}{\partial \log w}\frac{\partial g}{\partial t_0} - \frac{\partial g}{\partial \log w}\frac{\partial f}{\partial t_0} = w\left(\frac{\partial f}{\partial w}\frac{\partial g}{\partial t_0} - \frac{\partial g}{\partial w}\frac{\partial f}{\partial t_0}\right).$$

It is easy to verify that according to definition these Poisson brackets have the following properties:

1)Antisymmetry

$$\{f,g\} = -\{g,f\},$$

2) Linearity

$$\{f_1+f_2,g\} = \{f_1,g\} + \{f_2,g\},$$

3) The Jacobi identity

$$\{f,\{g,h\}\} = \{g,\{h,f\}\} = \{h,\{f,g\}\}.$$

We can present the functions

$$z(w) = \overline{S}\left(\overline{z}\left(w^{-1}\right)\right), \qquad \overline{z}\left(w^{-1}\right) = S(z(w)),$$

by means of series

$$z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}, \qquad \overline{z}\left(w^{-1}\right) = rw^{-1} + \sum_{j=0}^{\infty}\overline{u}_j w^j,$$

where r is the so called conformal radius.

Using the latter expressions we can prove the following theorem.

Theorem 4 ([5]). *A pair of variables* $(\log w, t_0)$ *and* $\big(z(w,t_0), \overline{z}(w^{-1},t_0)\big)$ *are canonical variables with respect to the Poisson brackets defined above,*

$$\left\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\right\} = 1.$$

Proof. According to definition of the Poisson bracket

$$\left\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\right\} = w\left(\frac{\partial z(w,t_0)}{\partial w}\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial t_0} - \frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w}\frac{\partial z(w,t_0)}{\partial t_0}\right).$$

Now using the formula $\overline{z}\big(w^{-1},t_0\big) = S(z(w,t_0))$ we get

$$\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial t_0} = \frac{\partial S(z,t_0)}{\partial t_0} + \frac{\partial S(z,t_0)}{\partial z}\frac{\partial z(w,t_0)}{\partial t_0}, \qquad \frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w} = \frac{\partial S(z,t_0)}{\partial z}\frac{\partial z(w,t_0)}{\partial w},$$

and substituting these expressions in the Poisson bracket we obtain the equality

$$\left\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\right\} = w\frac{\partial S(z,t_0)}{\partial t_0}\frac{\partial z(w,t_0)}{\partial w}.$$

The r.h.s. of this equality due to

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}, \qquad \text{and} \qquad z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j},$$

is 1 plus positive powers in w.

Similarly using the formula $z(w) = \overline{S}\big(\overline{z}(w^{-1})\big)$, we prove that

$$\left\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\right\} = -w\frac{\partial \overline{S}(\overline{z},t_0)}{\partial t_0}\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w}.$$

The r.h.s. of this equality due to

$$\overline{S}(z) = \sum_{k=1}^{\infty} k \overline{t}_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} \overline{v}_k z^{-k-1}, \qquad \text{and} \qquad \overline{z}\left(w^{-1}\right) = rw^{-1} + \sum_{j=0}^{\infty} \overline{u}_j w^{j},$$

is 1 plus negative powers of w.

Comparing two expressions for the Poisson bracket we prove the theorem. ■

Thus we established the symplectic structure of the conformal map.

Theorem 5 ([5]). *The deformation of the conformal map* $z(w)$ *as a result of variation of moments* $\{t_k\}_{k=0}^{\infty}$ *is described with the so called Lax–Sato equations*

$$\frac{\partial z(w)}{\partial t_k} = \{H_k, z(w)\}, \qquad \frac{\partial \overline{z}\left(w^{-1}\right)}{\partial t_k} = \left\{H_k, \overline{z}\left(w^{-1}\right)\right\}.$$

Self-consistency conditions of the Lax–Sato equations are

$$\partial_{t_k} H_l - \partial_{t_l} H_k + \{H_k, H_l\} = 0.$$

5 An example: ellipse

Let us consider an example when the curve Γ is ellipse and is characterized with three external moments t_0, t_1, t_2, all other moments are equal to zero, $t_3 = t_4 = \cdots = 0$.

An ellipse with center at a point $z = 0$ and half-axes a, b along coordinate axes x, y is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

Three non-zero external moments of the ellipse in terms of half-axes a, b looks as follows,

$$t_0 = ab, \qquad t_1 = 0, \qquad t_2 = \frac{1}{2}\frac{a-b}{a+b}.$$

All internal moments of the ellipse are non-zero and the first two of them are of the following form,

$$v_1 = 0, \qquad v_2 = \frac{2t_0^2 t_2}{1-4t_2^2}.$$

The Schwarz function of ellipse is

$$S(z) = \frac{a^2+b^2}{a^2-b^2}z - \frac{2ab}{a^2-b^2}\sqrt{z^2-(a^2-b^2)},$$

and its Laurent series looks as follows,

$$S(z) = 2t_2 z + \frac{t_0}{z} + \frac{v_2}{z^3} + \cdots .$$

A logarithm of τ-function is

$$\log\tau = \frac{1}{2}t_0^2\left(\log\frac{t_0}{1-t_2^2} - \frac{3}{2}\right).$$

The conformal map of a complement of unit disc to a complement of ellipse is the function

$$z(w) = rw + u_1 w^{-1}$$

with coefficients

$$r^2 = \frac{t_0}{1-4t_2^2} = \frac{1}{4}(a+b)^2, \qquad u_1^2 = \frac{4t_0t_2^2}{1-4t_2^2} = \frac{1}{4}(a-b)^2.$$

The first two Hamiltonians are

$$H_1 = rw, \qquad H_2 = r^2w^2 + u_1 r,$$

and the appropriate Hamiltonian flows do not change the form of disc. The Lax–Sato equations with the Hamiltonians $H_3, H_4, \ldots$ describe a deformation of the disc into the ellipse.

6 The Dirichlet boundary problem

The Dirichlet boundary problem

$$\Delta u(z) = 4\partial_z \partial_{\overline{z}} u(z) = f(z), \quad z \in D; \qquad u(z)|_{\partial D} = g(z),$$

has a solution

$$u(z) = \frac{1}{2\pi} \iint_D f(\zeta) G(z,\zeta) \mathrm{d}\zeta \mathrm{d}\overline{\zeta} + \frac{1}{2\pi} \int_{\partial D} g(\zeta) \frac{\partial}{\partial n} G(z,\zeta) |\mathrm{d}\zeta|,$$

where $G(z,\zeta)$ is the Green function which is uniquely defined by the following properties:

1) $G(z,\zeta) - \log|z-\zeta|$ is a symmetric, bounded and harmonic function in both arguments everywhere in the domain D,

2) $G(z,\zeta) = 0$ if any one of their arguments is at the boundary of the domain D.

We can present the Green function by the expression

$$G(z,\zeta) = |w(z,\zeta)|,$$

where $w(z,\zeta)$ is the analytic function which describe a conformal map of the domain D to the unit disc U.

According to J. Hadamard [6, 7] variations of the Green function under small deformations of the domain looks as follows

$$\delta G(z,\zeta) = \frac{1}{2\pi} \int_{\partial D} \partial_n G(z,\eta) \partial_n G(\zeta,\eta) \delta h(\eta) |\mathrm{d}\eta|,$$

where $\delta h(\eta)$ is a shift from the curve ∂D to the deformed curve along the normal n at the point $\eta \in \partial D$.

It is possible to write down the deformations $\delta h(\eta)$ of the domain boundary in terms of harmonic moments of the domain and as a result of that to express the variations of the Green function in these terms.

7 Quantization

Now let us quantize the above Hamiltonian structure and introduce respectively basic quantities of the quantized theory.

The Lax operators are

$$L = r(t_0) \exp\left(\hbar \frac{\partial}{\partial t_0}\right) + \sum_{k=0}^{\infty} u_k(t_0) \exp\left(-k\hbar \frac{\partial}{\partial t_0}\right),$$

$$\overline{L} = \exp\left(-\hbar \frac{\partial}{\partial t_0}\right) r(t_0) + \sum_{k=0}^{\infty} \exp\left(k\hbar \frac{\partial}{\partial t_0}\right) \overline{u}_k(t_0).$$

The Hamiltonians are

$$H_k = \left(L^k\right)_+ + \frac{1}{2}\left(L^k\right)_0, \qquad \overline{H}_k = \left(\overline{L}^k\right)_- + \frac{1}{2}\left(\overline{L}^k\right)_0,$$

where the symbol $\left(L^k\right)_\pm$ means positive (negative) parts of series for the shift operator $\exp\left(\hbar\frac{\partial}{\partial t_0}\right)$.

The Lax–Sato equations attain now the following form

$$\hbar \frac{\partial L}{\partial t_k} = [H_k, L], \qquad \hbar \frac{\partial L}{\partial \overline{t}_k} = [\overline{H}_k, L].$$

The spectrum of Lax operator is defined by the equation

$$L\Psi = z\Psi,$$

and the appropriate wave function can be presented as follows,

$$\Psi(z, t_0, t_1, \ldots) = z^{\hbar^{-1}t_0}\tau_{\hbar}^{-1}(t_0, t_1, \ldots)\exp\left(\frac{1}{\hbar}\sum_{k>0} t_k z^k\right)\exp\left(\hbar\sum_{k>0}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k}\right)\tau_{\hbar}(t_0, t_1, \ldots).$$

By means of the last expression we introduce the function

$$\tau_{\hbar}(t_0, t_1, \ldots).$$

The commutator

$$[L, \overline{L}] = \hbar$$

is called the selection rule or the string equation

In the semiclassical limit $\hbar \to 0$ we get the following correspondence with the notions considered above:

$$W = \exp\left(\hbar\frac{\partial}{\partial t_0}\right) \to w, \qquad L \to z(w),$$
$$\Psi \to \exp\left(\frac{\Omega}{\hbar}\right), \qquad \tau_{\hbar} \to \exp\left(\log\frac{\tau}{\hbar^2}\right),$$
$$\frac{1}{\hbar}[L, \overline{L}] = 1 \to \left\{z(w), \overline{z}\left(w^{-1}\right)\right\} = 1.$$

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