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Angular Momentum and Killing–Yano Tensors

Dumitru BALEANU

Cankaya University, Faculty of Arts and Science,
Department of Mathematics and Computer Science, 06530 Balgat, Ankara, Turkey
E-mail: *dumitru@cankaya.edu.tr*

Institute of Space Sciences, P.O. Box, MG-23, R 77125 Magurele-Bucharest, Romania

Quadratic Lagrangians are obtained by adding surface terms involving the angular momentum to a free particle Lagrangian. Killing–Yano tensors corresponding to the manifold structure induced by the quadratic Lagrangians were calculated and the associated non-generic symmetries were presented.

1 Introduction

Symmetries of a Riemannian space are associated with conservation laws of geodesic motion in this space, the investigation of Killing vectors and Killing tensors are very important in classifying and understanding a space-time.

Killing tensors of rank two or higher cannot in general be interpreted as symmetries of a metric alone. But on the other hand they can be interpreted as symmetries on the phase space for single particles.

The Killing–Yano (KY) tensors of rank two or higher order [1–7] play an important role for new non-generic symmetries [8, 9] as well as the generalized KY tensors [10], which are deeply connected to generalized Runge–Lenz symmetry. Also recently, it was discovered that there is a connection between Lax tensors [11] and KY tensors of order three. KY tensors can be understood as an object generating non-generic supersymmetry, i.e. supersymmetry belonging to some specific space-times.

Within the same context of the Killing and KY tensors, the notion of geometric duality was defined and applied on Taub-NUT and Kerr–Newman space-times [12]. The direct way to construct the dual metrics is to calculate the Killing tensors [13, 14]. An alternative way is to find the KY tensor and contract two of them to find Killing tensors [15]. However, finding an explicit space-time with a physical significance admitting higher rank KY tensors is not an easy task, not only because of the complexity of the calculations, but also earlier it was found that not all metrics admit KY tensors of higher order [16].

In the last years a huge effort was devoted to analyzing of the importance of KY tensors [17–24] in several areas.

There were some attempts to find new geometries admitting KY tensors of order two or three. For example, the three- particle open Toda lattice was geometrized by a suitable canonical transformation, and it was realized as the geodesic system of a certain Riemannian geometry [25]. Adding a time-like dimension, a four-dimensional space-times admitting two Killing vector fields were found [25].

Motivated by the above results we decided to construct geometries admitting (KY) and Killing tensors by adding a surface term [26, 27] to a known free Lagrangian.

The main aim of this paper is to add, using suitable Lagrangian multipliers, the components of the angular momenta to a free Lagrangian in two or three dimensions and to study the non-generic symmetries of the induced manifolds.

2 Killing–Yano tensors and non-generic supercharges

An n^{th} rank KY tensor $f_{\nu_1\nu_2\cdots\nu_n}$, is an antisymmetric tensor fulfilling the following equations:

$$f_{\nu_1\nu_2\cdots(\nu_n;\lambda)} = 0, \tag{1}$$

where semicolon denotes the covariant derivative.

The spinning particle model was constructed to be supersymmetric [8, 28–30], therefore independent of the form of the metric there is always a conserved supercharge

$$Q_0 = \Pi_\mu \psi^\mu.$$

Here, Π_μ is the covariant momenta and ψ^μ are odd Grassmann variables. The existence of Killing–Yano tensors of valence n are related to non-generic supersymmetries described by the supercharge

$$Q_f = f_{\nu_1\nu_2\cdots\nu_r}\Pi^{\nu_1}\psi^{\nu_2}\cdots\psi^{\nu_r} \tag{2}$$

which is a superinvariant

$$\{Q_0, Q_f\} = 0.$$

The KY equation and the Jacobi identities guarantee that it is also a constant of motion

$$\{Q_f, H\} = 0 \qquad \text{with} \quad H = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu$$

and with the appropriate definitions of the brackets.

3 Killing tensors and dual space-times

A Killing tensor of valence two is defined through the equation

$$K_{(\mu\nu;\alpha)} = 0. \tag{3}$$

Killing–Yano tensors of any valence can be considered as the square root of the Killing tensors of valence two in the sense that, their appropriate contractions yield

$$K_{\mu\nu} = g^{\alpha\beta}f_{\mu\alpha}f_{\beta\nu} \tag{4}$$

or for valence three it can be written as

$$K_{\mu\nu} = g^{\alpha\delta}g^{\beta\gamma}f_{\mu\alpha\beta}f_{\gamma\delta\nu}.$$

It has been shown in detail in reference [12] that $K^{\mu\nu}$ and $g^{\mu\nu}$ are reciprocally the contravariant components of the Killing tensors with respect to each other. If $K^{\mu\nu}$ is non-degenerate, then through the relation

$$K^{\mu\alpha}k_{\alpha\nu} = \delta^\mu{}_\nu,$$

the second rank non-degenerate tensor $k_{\mu\nu}$, can be viewed as the metric on the "dual" space. Furthermore, the notion of geometric duality extends to that of phase space. The constant of motion

$$K = \frac{1}{2}K^{\mu\nu}p_\mu p_\nu,$$

generates symmetry transformations on the phase space linear in momentum

$$\{x^\mu, K\} = K^{\mu\nu} p_\nu.$$

The Poisson brackets satisfy

$$\{H, K\} = 0,$$

where

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu.$$

Thus, in the phase space there is a reciprocal model with constant of motion H and the Hamiltonian K.

4 Extended Lagrangians and their corresponding geometries

Let us assume that a given free Lagrangian $L(\dot{q}^i, q^i)$ admits a set of constants of motion denoted by L_i, $i = 1, \ldots, 3$. If we add the components of the angular momentum corresponding to L, the extended Lagrangian [31]

$$L' = L + \dot{\lambda}^i L_i, \qquad i = 1, \ldots, 3$$

can be rewritten as

$$L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j.$$

Since a_{ij} is symmetric by construction, the issue is to find a way to construct induced manifolds. In other words we are looking to find whether a_{ij} is singular or not. If the matrix a_{ij} is singular, L' corresponds to a singular system [31]. Assuming that a_{ij} is a singular $n \times n$ matrix of rank $n-1$, we obtain non-singular symmetric matrices of order $(n-1) \times (n-1)$, where n will be 3, 5 and 6. The final step is to consider the obtained matrices as metrics on the extended space and to investigate their generic and non-generic symmetries.

4.1 The nonsingular case

As a starting point let us consider the following Lagrangian

$$L' = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \dot{\lambda}_3 (x\dot{y} - y\dot{x}). \tag{5}$$

From (5) we obtain $L^{'} = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$, where a_{ij} is given by

$$a_{ij} = \left(\begin{array}{ccc} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{array} \right). \tag{6}$$

If a_{ij} is considered as a metric of a manifold, by direct calculations we obtain that the corresponding Killing vector [2] of (6) is given by

$$V = (y, -x, 0).$$

From (1) a KY tensor of order two is defined as

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0. \tag{7}$$

Solving (7) corresponding to (6) we obtained the following KY tensor

$$f_{12} = 0, \qquad f_{23} = -Cx\sqrt{x^2+y^2}, \qquad f_{13} = Cy\sqrt{x^2+y^2}, \tag{8}$$

where C is a constant.

If a (KY) tensor exists, then a Killing tensor of order two is generated as in (4). Taking into account (4) and (8) a Killing tensor is constructed as

$$K_{ij} = \begin{pmatrix} y^2 & -xy & -y\left(y^2+x^2\right) \\ -xy & x^2 & x\left(x^2+y^2\right) \\ -y\left(y^2+x^2\right) & x\left(x^2+y^2\right) & 0 \end{pmatrix}.$$

5 The motion on a sphere and its induced geometries

It was proved in [32] that the motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum. In the following using the surface term we will generate four-dimensional manifolds. In this case the Lagrangian is given by

$$\begin{aligned} L' &= \frac{1}{2}\left(1+\frac{x^2}{u}\right)\dot{x}^2 + \frac{1}{2}\left(1+\frac{y^2}{u}\right)\dot{y}^2 + \frac{xy}{u}\dot{x}\dot{y} - \frac{xy}{\sqrt{u}}\dot{\lambda}_1\dot{x} + \left(\frac{x^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda}_2\dot{x} \\ &\quad - \left(\frac{y^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda}_1\dot{y} + \frac{xy}{\sqrt{u}}\dot{\lambda}_2\dot{y} + x\dot{\lambda}_3\dot{y} - y\dot{\lambda}_3\dot{x}, \end{aligned} \tag{9}$$

where $u = 1 - x^2 - y^2$. From (9)we identify the singular matrix a_{ij} as

$$a_{ij} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Because (10) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we treat these minors as metrics, we observe that they are not conformaly flat but their scalar curvatures are zero. The first metric is given by

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & \sqrt{u}+\frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u}+\frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix}. \tag{11}$$

The Killing vectors of (11) have the following forms

$$\begin{aligned} V_1 &= (y, -x, 0, 0), \qquad V_2 = \left(\sqrt{1-x^2-y^2} + \frac{x^2}{1-x^2-y^2}, \frac{xy}{1-x^2-y^2}, 0, 0\right), \\ V_3 &= \left(-\frac{xy}{1-x^2-y^2}, -\sqrt{1-x^2-y^2} - \frac{y^2}{1-x^2-y^2}, 0, 0\right). \end{aligned}$$

The next step is to investigate its KY tensors. Solving (7) we obtain the following set of solutions:

One-solution is given by

$$f_{21} = \frac{C_1}{\sqrt{1 - x^2 - y^2}}, \tag{12}$$

others zero.

Two-by-two solution has the form

$$f_{31} = f_{42} = C. \tag{13}$$

Finally, three by three solution is

$$f_{21} = \frac{C_1}{\sqrt{-1 + x^2 + y^2}}, \qquad f_{31} = f_{42} = C, \tag{14}$$

where C and C_1 are constants.

Using (2), (12), (13) and (14) we can obtain the form of the non-generic supercharge Q_0.

From (10) two more metrics can be identified as

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{15}$$

and

$$g^{(3)}_{\mu\rho} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 \\ \frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix} \tag{16}$$

respectively. By direct calculations we obtained that the metrics (15) and (16) admit the same Killing vector as in (11). Solving the equation (7) corresponding to (15) and (16) we find one non-zero component of KY tensor as $f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}$.

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