

An Elastodynamic State for the Solutions of Axial Symmetric Problems

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The fundamental solution is used for axial symmetric transient problems in BEM formulation. To check the formulation, a sample problem has been solved in plane strain. The strong singularity of the resulting integral equation has been reduced to weak form. New formulation provides to determine the initial velocity for a transient loading. Some differences have been introduced for the use of generalized functions.

1 Introduction

The aim of this study is to construct an elastodynamic state in an unbounded medium. It is appropriate to use for axial symmetric problems of elastodynamics in reciprocal theorem as fundamental solutions. After finding this elastodynamic state, a problem was solved. For the problem, the presented formulation leads to an integral equation. A numerical approximation is introduced for the solution of this integral equation.

2 A singular elastodynamic state for the solutions of axially symmetric elastodynamic problems

In plane elasticity, field variables are independent of the coordinate x_3 . Starting point for construction of the necessary singular solution for plane problems will be a point load of magnitude $\delta(t)$ in the e_k ($k = 1, 2$) direction, at the position $y(y_1, y_2, y_3)$. Corresponding functions \tilde{F}^k and \tilde{G}^k can be written as follows [1]:

$$\tilde{F}^k = \frac{1}{4\pi r} \left\{ H\left(t - \frac{r}{c_1}\right) \left(t - \frac{r}{c_1}\right) \right\} e_k, \tag{1}$$

$$\tilde{G}^k = \frac{1}{4\pi r} \left\{ H\left(t - \frac{r}{c_2}\right) \left(t - \frac{r}{c_2}\right) \right\} e_k, \tag{2}$$

$$\tilde{\rho}^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2, \quad r = \sqrt{\tilde{\rho}^2 + (x_3 - y_3)^2}. \tag{3}$$

It is considered that body force f is acting along a line on which y_3 varies only between the limits $-\infty$ and $+\infty$. The corresponding functions \tilde{F}^k and \tilde{G}^k to this line load is obtained between the limits [2]

$$\tilde{F}^k = \int_{-\infty}^{+\infty} \tilde{F}^k dy_3, \quad \tilde{G}^k = \int_{-\infty}^{+\infty} \tilde{G}^k dy_3. \tag{4}$$

But in equations (4), the limits of the integrals are not exactly correct. At a position defined by x , signals propagating with velocities do, however, not arrive until $c_1 t = r$ and $c_2 t = r$, respectively, which implies that the limits of integration of equations (4) will be appropriately

modified. By changing limits and performing integrals, \tilde{F}^k and \tilde{G}^k functions have been found out as follows:

$$\tilde{F}^k = \frac{1}{2\pi} \left\{ H \left(t - \frac{\bar{\rho}}{c_1} \right) \left\{ t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - \bar{\rho}^2}}{\bar{\rho}} - \frac{1}{c_1} \sqrt{c_1^2 t^2 - \bar{\rho}^2} \right\} \right\} e_k, \tag{5}$$

$$\tilde{G}^k = \frac{1}{2\pi} \left\{ H \left(t - \frac{\bar{\rho}}{c_2} \right) \left\{ t \ln \frac{c_2 t + \sqrt{c_2^2 t^2 - \bar{\rho}^2}}{\bar{\rho}} - \frac{1}{c_2} \sqrt{c_2^2 t^2 - \bar{\rho}^2} \right\} \right\} e_k. \tag{6}$$

And, corresponding displacement vector and stress tensor are

$$\tilde{u}^k = \nabla \nabla \cdot (\tilde{F}^k) - \nabla \times \nabla \times (\tilde{G}^k), \tag{7}$$

$$\tilde{\tau}_{ij}^k = \lambda \tilde{u}_{i,l}^k \delta_{ij} + \mu (\tilde{u}_{i,j}^k + \tilde{u}_{j,i}^k). \tag{8}$$

In an infinite medium, the pair of \tilde{u}^k and $\tilde{\tau}^k$ are the displacement vector and the stress tensor at a point $\tilde{x}(x_1, x_2)$ due to a body force acting at a specific point $\tilde{y}(y_1, y_2)$ in the e_k direction and of magnitude $\delta(t)$. This formulation can also be found in Achenbach’s book [2] in terms of $g(t)$ function. It is possible to construct a simpler formulation for axial symmetric problems. Here, cylindrical coordinates will be used. Starting point will be \tilde{F}^k and \tilde{G}^k . In cylindrical coordinates, the base vector e_{R1} on a point $\tilde{y}_1(R_1, \theta_1)$ can be expressed as in terms of cartesian base vectors, e_1 and e_2 as follow:

$$e_{R1} = \cos \theta_1 e_1 + \sin \theta_1 e_2. \tag{9}$$

Now, two singular body force, both having magnitude $g(t)/2R_1$, acting at the points \tilde{y}_1 and $\tilde{y}_2(R_1, \pi + \theta_1)$ in the directions of e_{R1} and e_{R2} , respectively, are considered. \tilde{F}^S and \tilde{G}^S functions at a point $\tilde{x}(R, \theta)$ due to these double forces become:

$$\begin{aligned} \tilde{F}^S = & \frac{1}{4\pi R_1} \left\{ H \left(t - \frac{\bar{\rho}_1}{c_1} \right) \left[\frac{t \ln \left(c_1 t + \sqrt{c_1^2 t^2 - \bar{\rho}_1^2} \right)}{\bar{\rho}_1} - \frac{1}{c_1} \sqrt{c_1^2 t^2 - \bar{\rho}_1^2} \right] \right\} e_{R1} \\ & + \frac{1}{4\pi R_1} \left\{ H \left(t - \frac{\bar{\rho}_2}{c_1} \right) \left[\frac{t \ln \left(c_1 t + \sqrt{c_1^2 t^2 - \bar{\rho}_2^2} \right)}{\bar{\rho}_2} - \frac{1}{c_1} \sqrt{c_1^2 t^2 - \bar{\rho}_2^2} \right] \right\} e_{R2}, \end{aligned} \tag{10}$$

$$\begin{aligned} \tilde{G}^S = & \frac{1}{4\pi R_1} \left\{ H \left(t - \frac{\bar{\rho}_1}{c_2} \right) \left[\frac{t \ln \left(c_2 t + \sqrt{c_2^2 t^2 - \bar{\rho}_1^2} \right)}{\bar{\rho}_1} - \frac{1}{c_2} \sqrt{c_2^2 t^2 - \bar{\rho}_1^2} \right] \right\} e_{R1} \\ & + \frac{1}{4\pi R_1} \left\{ H \left(t - \frac{\bar{\rho}_2}{c_2} \right) \left[\frac{t \ln \left(c_2 t + \sqrt{c_2^2 t^2 - \bar{\rho}_2^2} \right)}{\bar{\rho}_2} - \frac{1}{c_2} \sqrt{c_2^2 t^2 - \bar{\rho}_2^2} \right] \right\} e_{R2}, \end{aligned} \tag{11}$$

$$\bar{\rho}_1^2 = R^2 + R_1^2 - 2RR_1 \cos(\theta - \theta_1), \tag{12}$$

$$\bar{\rho}_2^2 = R^2 + R_1^2 + 2RR_1 \cos(\theta - \theta_1). \tag{13}$$

After dividing equations (10) and (11) by πR_1 and integrating on the half cylindrical surface whose radius R_1 from $\theta_1 = 0$ to $\theta_1 = \pi$, approaching R_1 to zero under the integral sign before performing the integration, the functions F' and G' which arise from a distributed body force in

the direction of e_r at every point on the cylindrical surface whose radius is $R_1 = 0$, are obtained as follows:

$$\begin{aligned} \tilde{F}' = \frac{1}{4\pi} \left\{ H \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1} \right) \left[\frac{\sqrt{c_1^2 t^2 - R^2}}{R} \right. \right. \\ \left. \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^2} \right) \left[c_1 t \ln \left(\frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} \right) - \sqrt{c_1^2 t^2 - R^2} \right] \right] \right\} e_R, \end{aligned} \quad (14)$$

$$\begin{aligned} \tilde{G}' = \frac{1}{4\pi} \left\{ H \left(t - \frac{R}{c_2} \right) \left(\frac{1}{c_2} \right) \left[\frac{\sqrt{c_2^2 t^2 - R^2}}{R} \right. \right. \\ \left. \left. + \delta \left(t - \frac{R}{c_2} \right) \left(\frac{1}{c_2^2} \right) \left[c_2 t \ln \left(\frac{c_2 t + \sqrt{c_2^2 t^2 - R^2}}{R} \right) - \sqrt{c_2^2 t^2 - R^2} \right] \right] \right\} e_R. \end{aligned} \quad (15)$$

And, by substituting equations (14) and (15), the expressions of potentials φ' , ψ' , nonzero components of displacement vector, strain and stress tensor, corresponding to this loading, are obtained as follows:

$$\begin{aligned} \varphi' = \frac{1}{4\pi} \left\{ H \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1} \right) \left[-\frac{1}{\sqrt{c_1^2 t^2 - R^2}} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^2} \right) \left[c_1 t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} - 3 \frac{\sqrt{c_1^2 t^2 - R^2}}{R} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^3} \right) \left[-c_1 t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} + \sqrt{c_1^2 t^2 - R^2} \right] \right\}, \end{aligned} \quad (16)$$

$$\psi' = 0, \quad (17)$$

$$\begin{aligned} u'_R = \frac{1}{4\pi} \left\{ H \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1} \right) \left[-\frac{R}{\sqrt{c_1^2 t^2 - R^2}^3} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^2} \right) \left[-\frac{c_1 t}{R^2} \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} + 2 \frac{\sqrt{c_1^2 t^2 - R^2}}{R^2} + \frac{3}{\sqrt{c_1^2 t^2 - R^2}} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^3} \right) \left[-\frac{c_1 t}{R} \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} + 4 \frac{\sqrt{c_1^2 t^2 - R^2}}{R} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^4} \right) \left[c_1 t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} - \sqrt{c_1^2 t^2 - R^2} \right] \right\}, \end{aligned} \quad (18)$$

$$\begin{aligned} \epsilon'_{RR} = \frac{1}{4\pi} \left\{ H \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1} \right) \left[-\frac{1}{\sqrt{c_1^2 t^2 - R^2}^3} - \frac{3R^2}{\sqrt{c_1^2 t^2 - R^2}^5} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^2} \right) \left[\frac{2}{R^3} c_1 t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} - 3 \frac{\sqrt{c_1^2 t^2 - R^2}}{R^3} \right. \right. \\ \left. \left. - \frac{1}{R \sqrt{c_1^2 t^2 - R^2}} + 4 \frac{R}{\sqrt{c_1^2 t^2 - R^2}^3} \right] \right. \\ \left. + \delta \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^3} \right) \left[2 \frac{c_1 t}{R^2} \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} - 5 \frac{\sqrt{c_1^2 t^2 - R^2}}{R^2} - \frac{6}{\sqrt{c_1^2 t^2 - R^2}} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \delta^{\cdot\cdot} \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^4} \right) \left[\frac{c_1 t}{R} \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} - 5 \frac{\sqrt{c_1^2 t^2 - R^2}}{R} \right] \\
 & + \delta^{\cdot\cdot\cdot} \left(t - \frac{R}{c_1} \right) \left(\frac{1}{c_1^5} \right) \left[-c_1 t \ln \frac{c_1 t + \sqrt{c_1^2 t^2 - R^2}}{R} + \sqrt{c_1^2 t^2 - R^2} \right] \Bigg\}, \tag{19}
 \end{aligned}$$

$$\epsilon'_{\theta\theta} = \frac{1}{R} u'_R, \tag{20}$$

$$\Delta' = \epsilon'_{RR} + \epsilon'_{\theta\theta}, \tag{21}$$

$$\tau'_{RR} = \lambda \Delta' + 2\mu \epsilon'_{RR}, \tag{22}$$

$$\tau'_{\theta\theta} = \lambda \Delta' + 2\mu \epsilon'_{\theta\theta}, \tag{23}$$

$$\tau'_{zz} = \lambda \Delta'. \tag{24}$$

The pair of \tilde{u}' and $\tilde{\tau}'$ forms an elastodynamic state \mathcal{S}' which is appropriate for the solution of the axial symmetric problems of elastodynamics.

3 Sample problem

An infinite medium, without body forces, having a cylindrical cavity is considered. At $t = 0$, a constant pressure p_0 is applied on the cavity and maintained. For this problem, cylindrical coordinates (R, θ, z) will be used. And, the outward normal of the boundary is $\tilde{n} = -\tilde{e}_R$. The surface tractions on the boundary for \mathcal{S} and for the elastodynamic state \mathcal{S}' are as follows:

$$T_R(a, t) = p_0 H^+(t), \tag{25}$$

$$T'_R(a, t) = \tilde{\tau}'(a, t) \tilde{n} = -\tilde{\tau}'(a, t) \tilde{e}_R = -\tau'_{RR}(a, t). \tag{26}$$

And further, for every $R \in [a, \infty)$

$$u_R(R, 0) = 0. \tag{27}$$

The expression of the dynamic reciprocal identity which is written between \mathcal{S} and \mathcal{S}' elastodynamic states will be reduced to:

$$\int_S \tilde{T}' * \tilde{u} dS = \int_S \tilde{T} * \tilde{u}' dS. \tag{28}$$

An integral equation arises substituting equations from (18) to (22) and (25) and (26) in equation (28). After this, changing the loading time of \tilde{f}' from $t = 0$ to $t = a/c_1$, this integral equation converts to:

$$\begin{aligned}
 & - \int_0^t \left\{ (\lambda + 2\mu) \left[H(t - \tau) g_0(t - \tau) + \delta(t - \tau) \frac{1}{c_1} g_1(t - \tau) \right. \right. \\
 & \quad \left. \left. + \delta^{\cdot}(t - \tau) \frac{1}{c_1^2} g_2(t - \tau) + \delta^{\cdot\cdot}(t - \tau) \frac{1}{c_1^3} g_3(t - \tau) + \delta^{\cdot\cdot\cdot}(t - \tau) \frac{1}{c_1^4} g_4(t - \tau) \right] \right. \\
 & \quad \left. + \lambda \left[H(t - \tau) f_0(t - \tau) + \delta(t - \tau) \frac{1}{c_1} f_1(t - \tau) + \delta^{\cdot}(t - \tau) \frac{1}{c_1^2} f_2(t - \tau) \right. \right. \\
 & \quad \left. \left. + \delta^{\cdot\cdot}(t - \tau) \frac{1}{c_1^3} f_3(t - \tau) \right] \right\} u_R(a, \tau) d\tau \\
 & = a p_0 \int_0^t \left\{ H(t - \tau) f_0(t - \tau) + \delta(t - \tau) \frac{1}{c_1} f_1(t - \tau) + \delta^{\cdot}(t - \tau) \frac{1}{c_1^2} f_2(t - \tau) \right. \\
 & \quad \left. + \delta^{\cdot\cdot}(t - \tau) \frac{1}{c_1^3} f_3(t - \tau) \right\} H^+(\tau) d\tau, \tag{29}
 \end{aligned}$$

where

$$\begin{aligned}
g_0(t) &= -\frac{1}{\sqrt{c_1^2(t+a/c_1)^2 - a^2}^3} - \frac{3a^2}{\sqrt{c_1^2(t+a/c_1)^2 - a^2}^5}, \\
g_1(t) &= \frac{2}{a^3}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} - 3\frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a^3} \\
&\quad - \frac{1}{a\sqrt{c_1^2(t+a/c_1)^2 - a^2}} + \frac{4a}{\sqrt{c_1^2(t+a/c_1)^2 - a^2}^3}, \\
g_2(t) &= \frac{2}{a^2}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} \\
&\quad - 5\frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a^2} - \frac{6}{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}, \\
g_3(t) &= \frac{1}{a}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} - 5\frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a}, \\
g_4(t) &= -(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} + \sqrt{c_1^2(t+a/c_1)^2 - a^2}, \\
f_0(t) &= -\frac{1}{\sqrt{c_1^2(t+a/c_1)^2 - a^2}^3}, \\
f_1(t) &= -\frac{1}{a^3}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} \\
&\quad + 2\frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a^3} + \frac{3}{a\sqrt{c_1^2(t+a/c_1)^2 - a^2}}, \\
f_2(t) &= -\frac{1}{a^2}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} + 4\frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a^2}, \\
f_3(t) &= \frac{1}{a}(c_1t+a) \ln \frac{c_1t+a + \sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a} - \frac{\sqrt{c_1^2(t+a/c_1)^2 - a^2}}{a}. \tag{30}
\end{aligned}$$

The kernel of equation (29) is strongly singular. The following equalities will be used to reduce the integral equation given in equation (29) to a simpler form

$$\delta(x)f(x) \equiv 0 \quad \text{for } f(0) = 0, \tag{31}$$

$$\delta^{(n)}(x)f(x) \equiv 0 \quad \text{for } \left. \frac{d^n f(x)}{dx^n} \right|_{x=0} = 0, \tag{32}$$

$$\begin{aligned}
\int_0^t \delta(t-\tau)f(t-\tau)u(\tau)d\tau &= -H(t-\tau)f(t-\tau)u(\tau) \Big|_{\tau=0}^{\tau=t} \\
&\quad + \int_0^t H(t-\tau)[-f'(t-\tau)u(\tau) + f(t-\tau)u'(\tau)]d\tau, \tag{33}
\end{aligned}$$

$$\begin{aligned}
\int_0^t \delta^{(n)}(t-\tau)f(t-\tau)u(\tau)d\tau &= -\delta^{(n-1)}(t-\tau)f(t-\tau)u(\tau) \Big|_{\tau=0}^{\tau=t} \\
&\quad + \int_0^t \delta^{(n-1)}(t-\tau)[-f^{(n)}(t-\tau)u(\tau) + f(t-\tau)u^{(n)}(\tau)]d\tau. \tag{34}
\end{aligned}$$

And the new form of the integral equation becomes:

$$\begin{aligned}
(\lambda + 2\mu) \left\{ \delta''(t-\tau) \frac{1}{c_1^4} u_R(\tau) [-c_1(t-\tau + a/c_1)L(t-\tau) + Q(t-\tau)] \right. \\
\left. + \delta'(t-\tau) u_R(\tau) \frac{1}{c_1^4} [-c_1(t-\tau + a/c_1)L(t-\tau) + Q(t-\tau)] \right\}
\end{aligned}$$

$$\begin{aligned}
 & + \delta(t - \tau) \frac{1}{c_1^2} \left[u_R(\tau) \frac{1}{ac_1} \{c_1(t - \tau - 2a/c_1)L(t - \tau) - 5Q(t - \tau)\} \right. \\
 & \left. + \frac{1}{c_1^2} u_R''(\tau) \{-c_1(t - \tau + a/c_1)L(t - \tau) + Q(t - \tau)\} \right] \\
 & + H(t - \tau) \frac{1}{c_1} \left[-\frac{1}{c_1} u_R(\tau) \frac{1}{Q(t - \tau)} + u_R(\tau) \left\{ -\frac{1}{aQ(t - \tau)} + \frac{c_1(t - \tau + a/c_1)}{Q^3(t - \tau)} \right\} \right] \Big|_0^t \\
 & + \int_0^t H(t - \tau) \left[\frac{1}{c_1^2} u_R''(\tau) \frac{1}{Q(t - \tau)} + \frac{1}{c_1} u_R'(\tau) \frac{1}{aQ(t - \tau)} + u_R(\tau) \frac{c_1(t - \tau)}{aQ^3(t - \tau)} \right] d\tau \Big\} \\
 & + \lambda \left\{ \left[\delta'(t - \tau) \frac{1}{ac_1^3} u_R(\tau) [c_1(t - \tau + a/c_1)L(t - \tau) - Q(t - \tau)] \right. \right. \\
 & \left. \left. + \delta(t - \tau) \frac{1}{c_1^2} \frac{1}{ac_1} u_R'(\tau) \{c_1(t - \tau + a/c_1)L(t - \tau) - Q(t - \tau)\} \right. \right. \\
 & \left. \left. + H(t - \tau) \frac{1}{c_1} \frac{1}{aQ(t - \tau)} u_R(\tau) \right] \Big|_0^t \right. \\
 & \left. - \int_0^t H(t - \tau) \left[u_R(\tau) \frac{c_1(t - \tau)}{aQ^3(t - \tau)} + u_R'(\tau) \frac{1}{c_1} \frac{1}{aQ(t - \tau)} \right] d\tau \right\} \\
 & = p_0 \left\{ \left[-\delta'(t - \tau) \frac{1}{c_1^3} [c_1(t - \tau + a/c_1)L(t - \tau) - Q(t - \tau)] \right. \right. \\
 & \left. \left. - \delta(t - \tau) \frac{1}{ac_1^2} [-c_1(t - \tau)L(t - \tau) + 4Q(t - \tau)] - H(t - \tau) \frac{1}{c_1} \frac{1}{Q(t - \tau)} \right] \Big|_0^t \right. \\
 & \left. + \int_0^t H(t - \tau) \frac{c_1(t - \tau)}{Q^3(t - \tau)} d\tau \right\}, \tag{35}
 \end{aligned}$$

where

$$Q(t - \tau) = \sqrt{c_1^2(t - \tau + a/c_1)^2 + a^2}, \tag{36}$$

$$L(t - \tau) = \ln \frac{c_1(t - \tau + a/c_1) + \sqrt{c_1^2(t - \tau + a/c_1)^2 + a^2}}{a}. \tag{37}$$

It must be emphasized that the kernels of the integrals in equation (35) are weakly singular. Using equations (27), (31) to (34) and equating to zero, the multiplier of $\delta'(t)$ in the remaining term out of the integral in equation (35), the initial velocity $u_R(a, 0)$ is found as follow:

$$u_R(a, 0) = \frac{p_0}{\lambda + 2\mu} c_1. \tag{38}$$

For convenience of notation two new dimensionless quantities are introduced as

$$t^* = \frac{c_1 t}{a}, \quad U(t^*) = \frac{2\mu}{ap_0} u_R(a, t) \tag{39}$$

the new form of the integral equation in terms of U and t^* becomes

$$\begin{aligned}
 & \frac{\lambda + 2\mu}{2\mu} \int_0^{t^*} H(t^* - \tau^*) \left\{ U \left[\frac{(t^* - \tau^*)}{\sqrt{3}} \right] + U \cdot \left[\frac{1}{\sqrt{}} \right] + U \cdot \cdot \left[\frac{1}{\sqrt{3}} \right] \right\} d\tau^* \\
 & - \frac{\lambda}{2\mu} \int_0^{t^*} H(t^* - \tau^*) \left\{ U \left[\frac{(t^* - \tau^*)}{\sqrt{3}} \right] + U \cdot \left[\frac{1}{\sqrt{}} \right] \right\} d\tau^* = \int_0^{t^*} H(t^* - \tau^*) \frac{(t^* - \tau^*)}{\sqrt{3}} d\tau^*, \tag{40}
 \end{aligned}$$

where

$$\sqrt{} = \sqrt{(t^* - \tau^* + 1)^2 - 1}. \tag{41}$$

And the dimensionless initial velocity is

$$U(0) = \frac{2\mu}{\lambda + 2\mu}. \quad (42)$$

It is known that U function and first derivative of it being continuous. Using partial integration and substituting equation (42), the last form of the integral equation can be written as follow.

$$\int_0^{t^*} U \cdot \left[\frac{\lambda + 2\mu}{2\mu} \frac{1}{\sqrt{\tau}} + \sqrt{\tau} \right] d\tau^* = \frac{t^*}{\sqrt{(t^* + 1)^2 - 1}} - \frac{2\mu}{\lambda + 2\mu} \sqrt{(t^* + 1)^2 - 1}. \quad (43)$$

Equation (43) is a Volterra integral equation of the first kind [3]. At the same time, it is also an integro-differential equation with a degenerate kernel having $1/\sqrt{t^* - \tau^*}$ singularity. This integral equation can be solved by transform techniques, but because of degenerate kernel, solution yields to an integral for every specific value of t . This integral must also be calculated numerically. Instead of this, another numerical method is introduced. The approximation function for $u_R(a, t)$ has been selected to be the same with the problem given in [1] substituting equation (39) in equation (43) the following equation is obtained between the end values of U of the j -th interval

$$\begin{aligned} & \sum_{k=1}^j \left\{ \left\{ [U_{k+1} - U_k] \frac{2}{(t_{k+1}^* - t_k^*)^2} - U_k \frac{2}{(t_{k+1}^* - t_k^*)} \right\} \right. \\ & \quad \times \left. \left\{ \int_{t_k^*}^{t_{k+1}^*} \left(\sqrt{(t_{j+1}^* - \tau^* + 1)^2 - 1} + \frac{\lambda + 2\mu}{2\mu} \frac{1}{\sqrt{(t_{j+1}^* - \tau^* + 1)^2 - 1}} \right) d\tau^* \right\} \right\} \\ & = \frac{t_{j+1}^*}{\sqrt{(t_{j+1}^* + 1)^2 - 1}} - \frac{2\mu}{\lambda + 2\mu} \sqrt{(t_{j+1}^* + 1)^2 - 1}. \end{aligned} \quad (44)$$

It is noted that:

$$t_1^* = 0, \quad U_1 = 0, \quad U_1 = U(0). \quad (45)$$

From first interval, U_2 can be calculated because U_1 and U_1 are known. Then writing $t^* = t_{k+1}^*$ [1], U_2 are calculated. With these values, U_3 becomes the only unknown of equation (44) for $j = 2$. The obtained results (Poisson's ratio = 0.3) are given in Figs. 1 and 2.

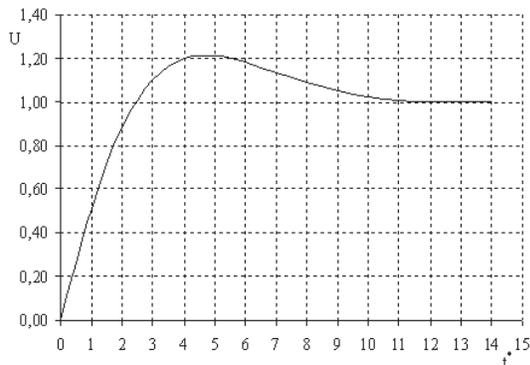


Figure 1. Variation of U on the surface of cylindrical cavity versus t^* .

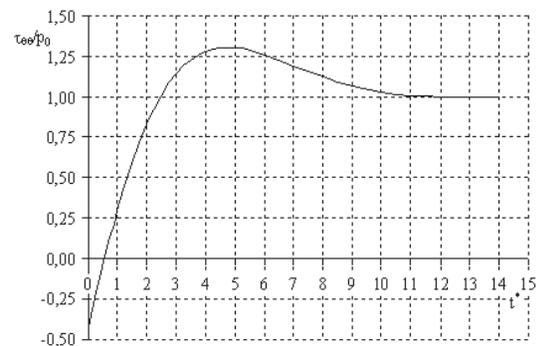


Figure 2. Variation of $\tau_{\theta\theta}/p_0$ on the surface of cylindrical cavity versus t^* .

4 Conclusions

A fundamental elastodynamic state has been derived. An integral equation is obtained by writing the reciprocal identity between one fundamental state and another state which represents the problem. The elastodynamic state can be used for the solutions of the problems having axial symmetry. And, the reciprocal identity which is written for axial symmetric problems is reduced to a Volterra integral equation whose kernel is strongly singular. The resulting integral equation of this problem has been converted to another form whose kernel is weakly singular and has been solved using a simple numerical technique. The most interesting part of the formulation is that it gives the exact value of the initial velocity.

- [1] Kadioglu N. and Ataoglu S., A simplification of the boundary element method for spherically symmetric problems, in Proceedings of the Second European Boundary Element Symposium "Boundary Element Research in Europe" (May 1998, Southampton), Editor C.A. Brebbia, Southampton, WIT, 1998, 179–188.
- [2] Achenbach J.D., Wave propagation in elastic solids, Amsterdam, North-Holland, 1973.
- [3] Srivastava H.M. and Buschman R.G., Theory and applications of convolution integral equations, Dordrecht, Kluwer Academic Publisher, 1992.