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First Integrals/Invariants & Symmetries for Autonomous Difference Equations

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For the autonomous kth order difference equation $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, where $f \in C^1[U]$ with domain $U^{\text{open}} \subset \mathbb{R}^k$, a global first integral/invariant for this difference equation is a nonconstant function $H \in C^1[U]$, with $H : U \to \mathbb{R}$, which remains invariant on the forward orbit $\Gamma^+(x_0, x_1, \ldots, x_{k-1})$. If a first integral exists, then it satisfies a particular functional difference equation and a method of finding solutions is presented. Furthermore, the first integral is constant along the characteristic curves of the associated infinitesimal generator for the Lie group symmetries of the difference equation.

1 Introduction

Example 1. Consider the second order, autonomous difference equation

$$x_{n+2} = (x_{n+1})^2 + x_{n+1} - (x_n)^2 .$$

It will be useful to view this difference equation as the iteration of the mapping

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ f(x,y) \end{pmatrix},$$

where $x \leftrightarrow x_n$, $y \leftrightarrow x_{n+1}$, $z := f(x,y)$, and $f(x,y) = y^2 + y - x^2$. A first integral/invariant is a nonconstant mapping H, which remains invariant on the forward orbit, that is,

$$H(y,z) = H(x,y).$$

For this example, the function $H(x,y) := y - x^2$ is a first integral for this difference equation as is shown here

$$\begin{aligned} H(y,z) &= z - y^2 \\ &= \left(y^2 + y - x^2\right) - y^2 \\ &= H(x,y). \end{aligned}$$

In other words, a first integral is a conservation law for the difference equation.

In this paper, we discuss the existence and construction of first integrals/invariants and the relationship to the one parameter, Lie group symmetries of the difference equation.

2 Notation

Consider the autonomous, kth order ordinary difference equation

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{1}$$

where $u_n := (x_n, x_{n+1}, \ldots, x_{n+k-1}) \in U^{\text{open}} \subset \mathbb{R}^k$ and $f \in C^1[U]$. The forward orbit Γ^+ of the initial condition $u_0 = (x_0, x_1, \ldots, x_{k-1})$ is the sequence $\Gamma^+(u_0) = \{x_0, x_1, x_2, \ldots\}$. The difference equation given in equation (1) can be viewed as the iteration of the mapping $F : \mathbb{R}^k \to \mathbb{R}^k$, where

$$F : \begin{pmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+k-1} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \end{pmatrix}, \tag{2}$$

that is,

$$u_{n+1} = F(u_n). \tag{3}$$

A first integral [1,2] is a nonconstant function $H \in C^1[U]$ which remains invariant on the forward orbit, that is,

$$H(x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = H(x_n, x_{n+1}, \ldots, x_{n+k-1}), \qquad \forall\, n \in \mathbb{Z}^+,$$

or more concisely,

$$H(u_{n+1}) = H(u_n), \qquad \forall\, n \in \mathbb{Z}^+. \tag{4}$$

Let $D_j f := \partial f / \partial x_{n+j-1}$, for $j = 1, \ldots, k$, in which case the Jacobian of F is given by the companion matrix

$$F'(u_n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 \\ D_1 f(u_n) & D_2 f(u_n) & D_3 f(u_n) & D_4 f(u_n) & D_5 f(u_n) & \cdots & D_k f(u_n) \end{pmatrix}. \tag{5}$$

Additionally, we will have need for the forward shift operator Ψ, which acts on functions $g \in C^1[U]$, where $g : \mathbb{R}^k \to \mathbb{R}$ and is defined as

$$\Psi\left[g(u_n)\right] := g(u_{n+1}). \tag{6}$$

Inductively, we define

$$\Psi^{n+1} := \Psi\left[\Psi^n\right], \tag{7}$$

where $\Psi^0 := I$ and where I is the identity operator. Using this notation, a first integral is defined by the condition

$$\Psi \circ H = H. \tag{8}$$

Lastly, define X to be the infinitesimal generator of the one parameter Lie group symmetry Φ as the operator [3–5]

$$X := \xi \cdot \nabla = \sum_{j=1}^{k} \xi_j D_j. \tag{9}$$

3 Determining equation of H

Lemma 1. *Let H be a first integral for $u_{n+1} = F(u_n)$, then*

$$\nabla H(u_{n+1}) = F'(u_n)^T \cdot \nabla H(F(u_n)). \tag{10}$$

Proof. Since H is a first integral where $H \in C^1[U]$, take the exterior derivative of equation (4), and use the chain rule to obtain the desired result. ■

Theorem 1. *The function*

$$\alpha := D_k H, \tag{11}$$

is a solution to the functional equation

$$\alpha = \sum_{j=0}^{k-1} \left(\Psi^{k-j-1} D_{j+1} f\right)\left(\Psi^{k-j} \alpha\right). \tag{12}$$

Proof. Expanding the result given in Lemma 1 and examining the coefficient functions of the differentials dx_{n+i-1}, for $i = 1, \ldots, k$, we obtain the alternative form

$$D_1 H = (D_1 f)(\Psi D_k H), \tag{13}$$
$$D_j H = \Psi D_{j-1} H + (D_j f)(\Psi D_k H), \tag{14}$$

for $j = 2, \ldots, k$. These k equations can be reduced to a single equation as follows. For $j = 2$, equation (14) becomes

$$D_2 H = \Psi D_1 H + (D_2 f)(\Psi D_k H).$$

Now take the forward shift of equation (13) to get

$$\Psi D_1 H = (\Psi D_1 f)(\Psi^2 D_k H),$$

and substitute into equation (14) to get

$$D_2 H = (\Psi D_1 f)(\Psi^2 D_k H) + (D_2 f)(\Psi D_k H),$$

in which case

$$\Psi D_2 H = (\Psi^2 D_1 f)(\Psi^3 D_k H) + (\Psi D_2 f)(\Psi^2 D_k H).$$

Similarly,

$$D_3 H = (\Psi^2 D_1 f)(\Psi^3 D_k H) + (\Psi D_2 f)(\Psi^2 D_k H) + (D_3 f)(\Psi D_k H),$$
$$\Psi D_3 H = (\Psi^3 D_1 f)(\Psi^4 D_k H) + (\Psi^2 D_2 f)(\Psi^3 D_k H) + (\Psi D_3 f)(\Psi^2 D_k H).$$

In general,

$$D_j H = \sum_{i=0}^{j-1} \left(\Psi^{j-i-1} D_{i+1} f\right)\left(\Psi^{j-i} D_j H\right), \tag{15}$$

and for $j = k$, we obtain the single functional equation

$$D_k H = \sum_{j=0}^{k-1} \left(\Psi^{k-j-1} D_{j+1} f\right)\left(\Psi^{k-j} D_k H\right). \tag{16}$$

Defining $\alpha = D_k H$ and substituting into equation (16) gives the desired result. ■

Assuming the functional equation given in equation (12) can be explicitly solved, then an autonomous first integral is given by

$$H = \sum_{j=1}^{k} \int D_j H \, dx_{n+j-1}. \tag{17}$$

With this result, we now provide some examples as to how to construct first integrals by finding solutions to equation (12).

Example 2. Recall that $x_{n+2} = (x_{n+1})^2 + x_{n+1} - (x_n)^2$ has a first integral given by $H(x_n, x_{n+1}) = x_{n+1} - (x_n)^2$. The functional equation given in equation (12) reduces to

$$-2y\alpha(z,u) + (2y+1)\,\alpha(y,z) - \alpha(x,y) = 0,$$

where $x = x_n$, $y = x_{n+1}$, $z = x_{n+2}$, and $u = x_{n+3}$. By inspection, a solution is $\alpha(x,y) = c$, where c is an arbitrary constant. Using the fact that $\alpha(x,y) = D_2 H = H_y(x,y)$, then a first integral is

$$H(x,y) = cy + \varphi(x),$$

where φ is an arbitrary function of x. Forcing the condition $H(x,y) = H(y,z)$ gives

$$\varphi(x) = -cx^2,$$

in which case a first integral is

$$H(x,y) = y - x^2,$$

as was found above.

This example is unusual in that a solution can be found by inspection. In general, this is not the case, and therefore we present a technique which can sometimes find solutions to nontrivial problems.

Example 3. Consider the difference equation $x_{n+2} = x_{n+1}\,(x_n+1)\,/\,(x_{n+1}+1)$. The function $\alpha = H_y(x,y)$ is a solution of the functional equation

$$\frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y),$$

where $z = f(x,y) = y(x+1)/(y+1)$ and $u = f(y,z)$. Define the extended solution space as

$$\Omega := \left\{ \alpha \,\middle|\, \frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y) \right\},$$

and the actual solution space as

$$\tilde{\Omega} := \left\{ \alpha \,\middle|\, \frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y),\ z = f(x,y),\ u = f(y,z) \right\},$$

in which case $\tilde{\Omega} \subseteq \Omega$. We seek candidates for solutions by finding solutions in Ω and checking whether they satisfy the condition $\Psi \circ H = H$. We begin by taking $\frac{\partial}{\partial x}\big|_{y,z,u \text{ fixed}}$ which gives

$$\alpha(y,z) = (y+1)^2 \alpha_x(x,y).$$

Next, taking $\frac{\partial}{\partial z}\big|_{x,y \text{ fixed}}$ gives $\alpha_z(y,z) = 0$, followed by a backward shift $\Psi^{-1}\alpha_z(y,z) = 0$, gives $\alpha_y(x,y) = 0$, which has solution $\alpha(x,y) = \eta(x)$. Therefore, $H_y(x,y) = \eta(x)$, and after integrating we find that

$$H(x,y) = y\,\eta(x) + \varphi(x),$$

for some arbitrary functions $\eta(x)$ and $\varphi(x)$. Forcing H to be a first integral means that

$$\begin{aligned}
&H(y,z) = H(x,y),\\
&z\eta(y) + \varphi(y) = y\eta(x) + \varphi(x).
\end{aligned}$$

Replacing $z \leftrightarrow y(x+1)/(y+1)$ gives

$$\frac{y(x+1)}{y+1}\eta(y) + \varphi(y) = y\eta(x) + \varphi(x).$$

Next, take $\frac{\partial}{\partial x}\big|_{y \text{ fixed}}$ followed by $\frac{\partial^2}{\partial y^2}\Big|_{x \text{ fixed}}$, we obtain the second order differential equation in the single variable y

$$y(y+1)^2\eta''(y) + 2(y+1)\eta'(y) - 2\eta(y) = 0,$$

which has solution $\eta(y) = c_1(y+1)$. If we force H to be a first integral, then

$$\begin{aligned}
&H(y,z) = H(x,y),\\
&c_1 z(y+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\
&c_1\frac{y(x+1)}{y+1}(y+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\
&c_1 y(x+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\
&\varphi(y) = \varphi(x),
\end{aligned}$$

in which case $\varphi(x) = c_2$. It is easily verified that the function H as was found here, namely

$$H(x,y) = y(x+1)$$

is a first integral.

As this example shows, ad hoc methods such as cleverly choosing the sequence of differentiations, followed by checking to see if the condition $\Phi \circ H = H$ can be satisfied, can lead to finding a first integral. A future direction of work is to utilize symbolic packages such as *MATHEMATICA* in order to automate this procedure.

4 Symmetries & first integrals

Let us now examine the relationship between symmetries of a difference equation and first integrals. Recall that a symmetry Φ of a dynamical system F maps solutions into solutions, that is,

$$\begin{array}{ccc}
U & \xrightarrow{F} & U\\
{\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi}\\
U & \xrightarrow[F]{} & U,
\end{array}$$

in which case the commutator bracket is zero

$$[\Phi, F] = 0.$$

For autonomous ordinary difference equations, this means that

$$\begin{array}{ccc} u_n = F^n(u_0) & \xrightarrow{F} & u_{n+1} = F \circ F^n(u_0) \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ \Phi \circ F^n(u_0) & \xrightarrow[F]{} & F \circ \Phi \circ F^n(u_0) = \Phi \circ F \circ F^n(u_0). \end{array}$$

In the case where we seek continuous one–parameter symmetries, that is, Lie group symmetries, then the associated generator is given by $X := \xi \cdot \nabla = \sum_{j=1}^{k} \xi_j D_j$, in which case

$$\Phi = I + \epsilon X + \mathcal{O}\left(\epsilon^2\right).$$

Consider the second order case

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ z = f(x, y) \end{pmatrix},$$

and let

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z}$$

be the generator of the Lie group symmetry for $z = f(x, y)$, in which case

$$\xi_3 = f_x \xi_1 + f_y \xi_2. \tag{18}$$

Examining the characteristic equation of the generator X gives the relationships

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{f_x dx + f_y dy}{f_x \xi_1 + f_y \xi_2}, \tag{19}$$

which has solutions

$$\xi_2 dx - \xi_1 dy = 0. \tag{20}$$

We now show that if H exists and Φ is a Lie group symmetry of $z = f(x, y)$, then Φ is also a symmetry of H, in which case H remains constant along the characteristic curve

$$\frac{dy}{dx} = \frac{\xi_2}{\xi_1}. \tag{21}$$

Consider the one–parameter Lie group transformations

$$\begin{aligned} \hat{x} &= x + \epsilon\, \xi_1 + \mathcal{O}\big(\epsilon^2\big), \\ \hat{y} &= y + \epsilon\, \xi_2 + \mathcal{O}\big(\epsilon^2\big), \\ \hat{z} &= z + \epsilon\big(f_x \xi_1 + f_y \xi_2\big) + \mathcal{O}\big(\epsilon^2\big) \end{aligned}$$

and define

$$\Delta\mathcal{H}(x, y, z) := H(y, z) - H(x, y),$$

in which case

$$\begin{aligned}\Delta\mathcal{H}(\hat{x},\hat{y},\hat{z}) &= H(\hat{y},\hat{z}) - H(\hat{x},\hat{y}) \\ &= H(y,z) - H(x,y) + \epsilon\Big[\big(f_x(x,y)H_z(y,z) - H_x(x,y)\big)\xi_1 \\ &\quad + \big(H_y(y,z) + f_y(x,y)H_z(y,z) - H_y(x,y)\big)\xi_2\Big] + \mathcal{O}\big(\epsilon^2\big).\end{aligned}$$

Since H is a first integral, the determining equations given in equations (13) and (14) are

$$\begin{aligned}H_x(x,y) &= f_x(x,y)H_z(y,z), \\ H_y(x,y) &= H_y(y,z) + f_y(x,y)H_z(y,z),\end{aligned}$$

in which case, up to terms of $\mathcal{O}\big(\epsilon^2\big)$,

$$\Delta\mathcal{H}(\hat{x},\hat{y},\hat{z}) = 0,$$

and therefore Φ is also a symmetry of H. Since H exists, then $H(x,y) = c$, where c is a constant which only depends on the initial condition (x_0, x_1). Taking the exterior derivative gives the equation

$$H_x\,dx + H_y\,dy = 0. \tag{22}$$

Since the 1–forms dx and dy are not identically zero, equations (20) and (22) will have nontrivial solution provided the linear PDE

$$\xi_1 H_x + \xi_2 H_y = 0 \tag{23}$$

is satisfied. In other words, H remains constant along the characteristic curve given in equation (21).

Let us now discuss some implications of equation (23). If specific symmetries of $z = f(x,y)$ are desired, a first integral may be constructed by solving this PDE. In other words, the inverse problem can be studied, that is, given a set of Lie group symmetries, we want to find manifolds $z = f(x,y)$ and the associated first integral.

Example 4. Consider the group of translations defined as the one–parameter transformations

$$\begin{aligned}\hat{x} &= x + \epsilon a + \mathcal{O}\big(\epsilon^2\big), \\ \hat{y} &= y + \epsilon b + \mathcal{O}\big(\epsilon^2\big), \\ \hat{z} &= z + \epsilon(af_x + bf_y) + \mathcal{O}\big(\epsilon^2\big),\end{aligned}$$

where f unspecified. Integrating equation (23) gives

$$H(x,y) = \eta(r),$$

where η is an arbitrary smooth function and

$$r = bx - ay.$$

Consider the identity function $\eta(r) = r$, in which case $\Phi \circ H = H$ gives

$$f(x,y) = (1+\gamma)y - \gamma x,$$

where $\gamma = b/a$. In other words, the class of linear difference equations

$$x_{n+2} = (1+\gamma)x_{n+1} - \gamma x_n$$

has the above Lie group symmetries and has an associated first integral given by

$$H(x,y) = \gamma x - y.$$

Although this example is somewhat simplistic, it does illustrate how the inverse problem may be studied.

5 Summary & future directions

In this paper we have constructed a functional equation, which if solved, will give a first integral. A method of solving this equation is given. Furthermore, a first integral is constant along the characteristic curve associated with the generator of the Lie group symmetries of the difference equation.

Future work is to construct and implement algorithms using symbolic packages such as *MATHEMATICA* in order to search for solutions of the determining functional equation, and hence find first integrals. Furthermore, the solution of the determining functional equation can be thought of as a fixed point of an operator equation. For example, the difference equation $x_{n+2} = f(x_n, x_{n+1})$ has the associated functional equation

$$(\Psi D_1 f)(\Psi^2 \alpha) + (D_2 f)(\Psi \alpha) = \alpha.$$

Define the operator

$$\mathcal{L} := (\Psi D_1 f)\Psi^2 + (D_2 f)\Psi,$$

in which case the solution α to the functional equation is the fixed point of the operator equation

$$\mathcal{L}[\alpha] = \alpha.$$

A sequence of approximations to the solution can be made by using fixed point iteration, that is, defining

$$\alpha_{n+1} := \mathcal{L}[\alpha_n],$$

where α_0 is an initial guess. The obvious question is to determine what are the conditions on the function f that insure that $\mathcal{L}$ is a contraction operator, and hence guarantee that fixed point iteration converges.

[1] Hale J. and Koçak H., Dynamics and bifurcations, New York, Springer-Verlag, 1991.

[2] Kocic V. and Ladas G., Global behavior of nonlinear difference equations of higher order with applications, Boston, Kluwer Academic Publishers, 1993.

[3] Bluman G. and Kumei S., Symmetries and differential equations, New York, Springer-Verlag, 1989.

[4] Gaeta G., Nonlinear symmetries and nonlinear equations, Boston, Kluwer Academic Publishers, 1994.

[5] Olver P., Applications of Lie groups to differential equations, New York, Springer-Verlag, 1993.