Regularity Properties of Infinite-Dimensional Evolutions, Related with Anharmonic Lattice Systems

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In this paper we discuss the C^∞ properties of evolution for infinite system of interacting particles. Such evolution is described in terms of semigroup generated by second order differential operator of infinite number of variables. For anharmonic systems this operator has unbounded non-Lipschitz coefficients and thus corresponding semigroup is not strongly continuous in time in the space of continuously differentiable functions. This circumstance makes inapplicable the standard technique of analysis and semigroup theory. Using the connection between the semigroup theory and stochastic differential equations and observation about nonlinear symmetry of high order variational calculus, we state smoothing properties of such infinite-dimensional evolution in the space of continuously differentiable functions. The main attention is devoted to the influence of the nonlinearity on the topology of regularity.

1 Introduction

The appearance of different type of nonlinearities can to a great extent influence the behaviour of a system and usually requires relevant techniques adopted to the description of arising phenomena. Consider the infinite-dimensional system of anharmonic oscillators described by the Gibbs measure with finite range interaction of radius r_0:

$$d\mu = \frac{1}{Z}\exp\left\{-\lambda\sum_{|k-j|\le r_0} b_{k-j}x_kx_j\right\}\prod_{k\in\mathbb{Z}^d}\exp\{-\mathcal{U}(x_k)\}dx_k.$$

It exists and is unique for small $\lambda > 0$, and can be correctly constructed in terms of thermodynamic limit $\Lambda \to \mathbb{Z}^d$ [1, 2]. The associated Hamiltonian, defined on cylinder functions of polynomial behaviour $u, v \in \mathcal{P}_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{Z}^d})$, is given by

$$H_\mu = \sum_{k\in\mathbb{Z}^d}\left\{-\frac{1}{2}\frac{\partial^2}{\partial x_k^2} + (F(x_k) + (Bx)_k)\frac{\partial}{\partial x_k}\right\},$$

where $F(x_k) = \mathcal{U}'(x_k)$ and $(Bx)_k = \sum\limits_{j:|j-k|\le r_0} b_{k-j}x_j$.

We should remark that semigroup of this operator e^{-tH_μ} is well-defined, strongly continuous and even preserves the Sobolev-type spaces of differentiable functions $\mathcal{W}_\Theta$ [3]. This in particular implies that operator H_μ is essentially self-adjoint in $L_2(\mathbb{R}^{\mathbb{Z}^d}, \mu)$ and corresponding dynamics is well-defined. However, spaces $\mathcal{W}_\Theta$ are of very special structure and actually do not give precise information about the *smooth* properties of dynamics $P_t = e^{-tH_\mu}$. Thus, it would be interesting to investigate the properties of semigroup P_t in some spaces with *sup*-topology, like C_b^n (continuously differentiable functions). But in C_b^n semigroup P_t is not strongly continuous on t, and the standard methods of analysis and semigroup theory are not applicable. Nevertheless, such semigroups can be described in the terms of stochastic differential equation in $\mathbb{R}^{\mathbb{Z}^d}$:

$$d\xi_k^0(t, x^0) = dW_k(t) - \{F(\xi_k^0(t, x^0)) + (B\xi^0(t, x^0))_k\}dt, \qquad \xi_k^0(0, x^0) = x_k^0, \quad k \in \mathbb{Z}^d \tag{1}$$

then

$$(P_t f)(x^0) = \mathbf{E}\, f(\xi^0(t, x^0)), \tag{2}$$

where the cylinder Wiener process $W = \{W_k(t)\}_{k\in\mathbb{Z}^d}$ with values in $\ell_2(a) = \ell_2(a, \mathbb{Z}^d)$, $\sum a_k = 1$ is canonically realized on measurable space $(\Omega = C_0([0, T], \ell_2(a)), \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with canonical filtration $\mathcal{F}_t = \sigma\{W(s)|0 \le s \le t\}$ and cylinder Wiener measure $\mathbf{P}$. Processes W_k, $k \in \mathbb{Z}^d$ are independent $\mathbb{R}^1$-valued Wiener processes and $\mathbf{E}$ denotes the expectation with respect to measure $\mathbf{P}$. Further the set of all vectors $a = \{a_k\}_{k\in\mathbb{Z}^d}$ such that $\delta_a = \sup\limits_{|k-j|=1} |a_k/a_j| < \infty$ we denote by $\mathbb{P}$. On the coefficients of H_μ we impose the following conditions: the nonlinear map $F : \mathbb{R}^{\mathbb{Z}^d} \ni x \longrightarrow F(x) = \{F(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ is generated by the C^∞ monotone function F, $F(0) = 0$, which has polynomial growth on the infinity

$$\exists\, \mathbf{k} \ge -1 \ \ \forall\, i \ge 1 \quad |F^{(i)}(x) - F^{(i)}(y)| \le C_i|x - y|(1 + |x| + |y|)^{\mathbf{k}}. \tag{3}$$

Note that for $\mathbf{k} > -1$ the map $F : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is not locally Lipschitz in any space $\ell_p(c, \mathbb{Z}^d)$.

Due to (2) the investigation of regular properties of semigroup leads to the problem of smooth dependence with respect to the initial data for solutions of stochastic differential equations (1).

In the case of Lipschitz coefficients with bounded derivatives the question of C^∞-smoothness of the stochastic flow and corresponding semigroup has already been solved using the standard fixed point arguments and implicit function technique (see for example [4, 5]). In contrary, equation (1) has coefficients, for which the Lipschitz property breaks even locally on balls in $\ell_p(a)$ and these methods fail to work.

Basing on a simple observation about the symmetry of the system in variations we develop technique, adopted to the investigation of C^∞ smoothness of semigroups and associated stochastic flows in essentially non-Lipschitz case. Our aim is to construct such spaces $C_{(\Theta)^m}$ with sup-topology, that semigroup $P_t : C_{(\Theta)^m} \mapsto C_{(\Theta)^{m+1}}$ and the following estimate is true:

$$\|P_t f\|_{C_{(\Theta)^m}} \le \frac{1}{t^{m/2}} K_{\Theta,m} e M_{\Theta,m} t \|f\|_{C_{(\Theta)^0}}.$$

2 Main definitions, stochastic methods and nonlinear quasi-contractive estimate

To study the smooth properties of semigroup, first of all we need the representation for derivatives of semigroup $\partial_\tau P_t f$, where $\tau = \{k_1, \dots, k_m\}$ is the set of directions, in which differentiation is done and $\partial_\tau = \partial^{|\tau|}/\partial x_{k_1} \cdots \partial x_{k_m}$. Formula (2) leads to

$$\partial_\tau (P_t f)(x^0) = \sum_{s=1}^{m} \sum_{\gamma_1\cup\dots\cup\gamma_s=\tau} \mathbf{E}\, \langle \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes \dots \otimes \xi_{\gamma_s} \rangle. \tag{4}$$

$\partial^{(s)} f = \{\partial_\gamma f\}_{|\gamma|=s}$ denotes above the set of s^{th}-order partial derivatives of function and we used notation

$$\langle \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes \dots \otimes \xi_{\gamma_s} \rangle = \sum_{j_1,\dots,j_s\in\mathbb{Z}^d} (\partial_{\{j_1,\dots,j_s\}} f)(\xi^0) \xi_{j_1,\gamma_1} \cdots \xi_{j_s,\gamma_s}.$$

Vector $\xi_\tau = \{\xi_{k,\tau}\}_{k\in\mathbb{Z}^d}$ is interpreted as a derivative of ξ^0 with respect to the initial data $x^0 = \{x^0_k\}_{k\in\mathbb{Z}^d}$ and $\xi_{k,\tau} = \frac{\partial^{|\tau|}\xi^0_k(t,x^0)}{\partial x^0_{j_n}\cdots\partial x^0_{j_1}}$ is called below a τ^{th} *variation* of ξ^0. The equation on ξ_τ is derived by the formal successive differentiation of (1) with respect to x^0:

$$\frac{d\xi_{k,\tau}}{dt} = -F'(\xi^0_k)\xi_{k,\tau} - \sum_{j:\,|j-k|\le r_0} b_{k-j}\xi_{j,\tau} - \varphi_{k,\tau}, \qquad \xi_{k,\tau}(0) = x_{k,\tau} \tag{5}$$

where $\varphi_{k,\tau} = \varphi_{k,\tau}(\xi^0, \xi_{\cdot,\gamma}, \gamma \subset \tau, \gamma \neq \tau)$

$$\varphi_{k,\tau} = \sum_{\gamma_1 \cup \cdots \cup \gamma_s = \tau,\ s \geq 2} F^{(s)}(\xi^0) \xi_{k,\gamma_1} \cdots \xi_{k,\gamma_s} \tag{6}$$

In (6) the summation $\sum\limits_{\gamma_1 \cup \cdots \cup \gamma_s = \tau,\, s \geq 2}$ runs on all possible subdivisions of the set $\tau = \{j_1, .., j_n\}$, $j_i \in \mathbb{Z}^d$ on the nonintersecting subsets $\gamma_1, \ldots, \gamma_s \subset \tau$, with $|\gamma_1| + \cdots + |\gamma_s| = |\tau|$, $s \geq 2$, $|\gamma_i| \geq 1$.

It is well-known that for initial data $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ there is a unique *strong solution* to equation (1), i.e. $\ell_2(a)$-continuous $\mathcal{F}_t$-adapted process $\xi^0(t, x^0) \in \mathcal{D}_{\ell_2(a)}(F)$ that fulfills $\mathbf{P}$ a.e. equation (1) in $\ell_2(a)$. For $x^0 \in \ell_2(a)$ the *generalized solution* is obtained as a uniform on $[0, T]$ $\mathbf{P}$ a.e. limit of strong solutions [6–8].

The solution to (5) is understood as the $\mathcal{F}_t$-adapted process $\xi_\gamma(t, x^0)$ that fulfills (5) in $\ell_{m_\gamma}(c_\gamma)$ a.e. on $[0, T]$ and such that for $\mathbf{P}$ a.e. $\omega \in \Omega$ the map $[0, T] \ni t \to \xi_\gamma(t, x^0) \in \ell_{m_\gamma}(c_\gamma)$ is Lipschitz continuous, $\xi_{k,\gamma}(0, x^0) = x_{k,\gamma}$, $k \in \mathbb{Z}^d$, $\xi_\gamma(t, x^0) \in \mathrm{Dom}_{\ell_{m_\gamma}(c_\gamma)}(F'(\xi^0(t, x^0)) + B)$, for a.e. $t \in [0, T]$, and there is a strong $\ell_{m_\gamma}(c_\gamma)$-derivative $d\xi_\gamma(t, x^0)/dt$ a.e. on $[0, T]$.

Now we are going to obtain the representation for semigroup derivatives, which immediately leads to the raise of smoothness for semigroup. To do this we need to recall some definitions of analysis on Wiener space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ (or Malliavin calculus).

Denote by $\mathcal{J}_{cyl}$ the set of $\mathcal{F}_t$-adapted continuous *integrable cylinder-valued* processes $u_t = \{u_{t,k}\}_{k \in \mathbb{Z}^d}$ for which there exists such set $\Lambda_u \subset \mathbb{Z}^d$, $|\Lambda_u| < \infty$ (support of cylindricity) that for all points $k \notin \Lambda_u$ $u_{t,k} \equiv 0$, $t \in [0, T]$ and

$$\forall\, k \in \Lambda_u \;\; \forall\, p \geq 1 \qquad \mathbf{E} \int_0^T |u_{t,k}|^p dt < \infty.$$

Definition 1. Measurable function G on Ω is *stochastically differentiable in direction* $u \in \mathcal{J}_{\mathrm{cyl}}$ and has directional derivative $D_u G$ if $\exists\, \varepsilon_0 > 0\ \forall |\varepsilon| \leq \varepsilon_0$ function $G(\omega_\bullet + \varepsilon \int_0^\bullet u_s ds)$ belongs to $\bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P})$ and there is a measurable function $D_u G \in \bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P})$ such that

$$\forall\, p \geq 1 \quad \lim_{|\varepsilon| \to 0} \mathbf{E} \,| \frac{G(\omega_\bullet + \varepsilon \int_0^\bullet u_s ds) - G(\omega)}{\varepsilon} - D_u G(\omega)|^p = 0.$$

Definition 2. We say that $G \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ (*locally stochastically differentiable*) iff $\forall\, j \in \mathbb{Z}^d$ there is a map $\mathbf{D}_j G \in \bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P}, \mathcal{H})$ and $\forall\, u \in \mathcal{J}_{\mathrm{cyl}}\ \exists\, D_u G$ that has representation

$$D_u G = \sum_{j \in \Lambda_u} \langle \mathbf{D}_j G, \int_0^\bullet u_{s,j} ds \rangle_{\mathcal{H}}.$$

Above $\mathcal{H}$ denotes the Cameron–Martin space of absolutely continuous functions $\gamma : [0, T] \to \mathbb{R}^1$, $\gamma(0) = 0$, equipped with the scalar product $\langle \gamma, \gamma \rangle_{\mathcal{H}} = \int_0^T |\dot{\gamma}(s)|^2 ds$.

Derivative D_u has the following properties [9–11].

1. For smooth function $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^1)$ of polynomial with all derivatives behaviour at infinity and $G_1, \ldots, G_n \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ we have $f(G_1, \ldots, G_n) \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ and

$$D_u f(G_1, ..., G_n) = \sum_{i=1}^n [\partial_i f \circ (G_1, \ldots, G_n)]\, D_u G_i, \qquad u \in \mathcal{J}_{\mathrm{cyl}}. \tag{7}$$

2. For real-valued $\mathcal{F}_t$-adapted continuous processes $H_t \in \mathcal{D}_{\mathrm{loc}}(\Omega)$, $t \in [0, T]$ such that $\mathbf{E} \int_0^T |H_s|^p ds < \infty$ and $\mathbf{E} \int_0^T \|\mathbf{D}_j H_s\|_{\mathcal{H}}^p ds < \infty$, for all $p \geq 1$, $j \in \mathbb{Z}^d$ we have

$$\forall\, t \in [0, T]\ \forall\, k \in \mathbb{Z}^d \quad \left\{ \int_0^t H_s ds,\ \int_0^t H_s dW_k(s) \right\} \in \mathcal{D}_{\mathrm{loc}}(\Omega).$$

3. $D_u \int_0^t H_s\,ds = \int_0^t D_u H_s\,ds,$

$$D_u \int_0^t H_s\,dW_k(s) = \int_0^t H_s u_{s,k}\,ds + \int_0^t D_u H_s\,dW_k(s), \qquad u \in \mathcal{J}_{\mathrm{cyl}}.$$

Theorem 1. *Let $\xi^0(t,x^0)$ be a generalized solution to (1), $x^0 \in \ell_2(a)$. For vector $v \in \mathbb{R}^{\mathbb{Z}^d}$ with finite number of nonzero coordinates introduce process*

$$\Gamma_t v = [Id + t(F'(\xi^0(t,x^0)) + B)]v \in \mathcal{J}_{\mathrm{cyl}}.$$

Then the derivative in direction $u_t = \Gamma_t v$ gives

$$D_{\Gamma v}\xi^0(t,x^0) = tv. \tag{8}$$

Moreover, the **integration by parts formula** *holds*

$$\mathbf{E}\,\langle \partial f(\xi_t^0), v\rangle_{\ell_2(1)}\Psi = \frac{1}{t}\mathbf{E}\ f(\xi_t^0)\{\Psi \int_0^t \langle \Gamma_s v, dW(s)\rangle_{\ell_2(1)} - D_{\Gamma v}\Psi\} \tag{9}$$

for all $\mathcal{F}_t$-measurable $\Psi \in \mathcal{D}_{\mathrm{loc}}(\Omega)$, $f \in \mathcal{P}^\infty_{\mathrm{cyl}}(\ell_2(a))$, $t > 0$.

The proof is given in [12]. Using this theorem we rewrite the representation of partial derivatives $\partial_\tau P_t f$ (4) in the terms of stochastic derivatives. Introduce notation $\mathbb{D}^k = D_{\Gamma e_k}$ where $e_k = (\ldots, 0, 1_k, 0, \ldots) \in \mathbb{R}^{\mathbb{Z}^d}$ is k^{th} unit vector. Formula (9) gives for $\mathcal{F}_t$-measurable $\Psi \in \mathcal{D}_{\mathrm{loc}}(\Omega)$

$$\forall\, f \in \mathcal{P}^\infty_{\mathrm{cyl}}(\ell_2(a)) \quad \mathbf{E}\ \partial_k f(\xi_t^0)\Psi = \frac{1}{t}\,\mathbf{E}\, f(\xi_t^0)\mathbb{D}_k^*\Psi \tag{10}$$

with

$$\mathbb{D}_k^*\Psi = \Psi \int_0^t \langle \Gamma_s e_k, dW(s)\rangle_{\ell_2(1)} - \mathbb{D}^k\Psi.$$

Therefore, the partial derivatives of semigroup permit representation

$$\partial_\tau P_t f(x) = \sum_{\ell=1}^{|\tau|} \sum_{\gamma_1\cup\cdots\cup\gamma_\ell=\tau} \sum_{j_1,\ldots,j_\ell\in\mathbb{Z}^d} \mathbf{E}\, f(\xi_t^0)\,\frac{\mathbb{D}_{j_1}^*\cdots\mathbb{D}_{j_\ell}^*(\xi_{j_1,\gamma_1}\cdots\xi_{j_\ell,\gamma_\ell})}{t^\ell}. \tag{11}$$

To obtain the raise of smoothness under the action of semigroup P_t we have to investigate the behaviour of derivatives $\mathbb{D}^\beta\xi_\tau = \mathbb{D}^{j_1}\cdots\mathbb{D}^{j_\ell}\xi_\tau$, $\beta = \{j_1,\ldots,j_\ell\}$.

Theorem 2. *For initial data $x^0 \in \ell_2(a)$ the coordinates of variations are locally stochastically differentiable $\xi_{k,\tau}(t,x^0) \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ and $\mathbb{D}^\beta\xi_{k,\tau} \in \mathcal{D}_{\mathrm{loc}}(\Omega)$, $\beta \subset \mathbb{Z}^d$, $|\beta| \geq 1$. The stochastic derivatives $\mathbb{D}^\beta\xi_{k,\tau}$ are represented as a strong solutions in $\ell_{m_\tau}(c_{\tau,\beta})$, $m_\tau = m_1/|\tau|$, to system*

$$\mathbb{D}^\beta\xi_{k,\tau}(t) = \widetilde{x}_{k;\tau,\beta} - \int_0^t [(F'(\xi^0) + B)\mathbb{D}^\beta\xi_\tau]_k ds - \int_0^t \varphi_{k;\tau,\beta}(s)ds, \tag{12}$$

where $\widetilde{x}_{\tau,\beta} = 0$ for $|\beta| \geq 1$ and $\widetilde{x}_{\tau,\varnothing} = \widetilde{x}_\tau$,

$$\varphi_{k;\tau,\beta}(t) = \sum_{\substack{\gamma_1\cup\cdots\cup\gamma_\ell=\tau\\ |\gamma_i|\geq 1,\ \ell\geq 1}} \ \sum_{\substack{\sigma_0\cup\cdots\cup\sigma_\ell=\beta\\ |\sigma_0|\geq 2-\ell,\ |\sigma_i|\geq 0}} t^{|\sigma_0|}\delta_k^{\sigma_0} F^{(\ell+|\sigma_0|)}(\xi_k^0)\,\mathbb{D}^{\sigma_1}\xi_{k,\gamma_1}\cdots\mathbb{D}^{\sigma_\ell}\xi_{k,\gamma_\ell}$$

and vectors $c_{\tau,\beta} \in \mathbb{P}$ *fulfill hierarchy*

$$\exists\, K_c \quad \delta_k^{\sigma_0}[c_{k;\,\tau,\beta}]^{|\tau|} a_k^{-\frac{\mathbf{k}+1}{2} m_1} \le K_c [c_{k;\,\gamma_1,\sigma_1}]^{|\gamma_1|} \cdots [c_{k;\,\gamma_\ell,\sigma_\ell}]^{|\gamma_\ell|}, \qquad k \in \mathbb{Z}^d. \tag{13}$$

Above $\delta_j^\beta = \prod_{i\in\beta} \delta_j^i$ *is a product of Kronnecker symbols and the subdivisions of sets* $\tau = \gamma_1 \cup \cdots \cup \gamma_\ell$, $\beta = \sigma_0 \cup \cdots \cup \sigma_\ell$ *are such that* $1 \le \ell \le |\tau|$ *and for* $\ell = 1$, $|\sigma_0| \ge 1$*; for* $\ell \ge 2$ $|\sigma_0| \ge 0$. *Moreover, at* $t = 0$ *there is asymptotic* $\forall\, R > 0\ \exists\, K_R \in \bigcap_{p\ge1} L^p(\Omega, \mathbf{P})$ *such that*

$$\forall\, |\beta| \ge 1 \quad \|\mathbb{D}^\beta \xi_\tau(t, x^0)\|_{\ell_{m_\tau}(c_{\tau,\beta})} \le t^{|\beta|+1} K_R(\omega), \qquad t \in [0, T]$$

uniformly on $\max(\|x^0\|_{\ell_2(a)}, \|\widetilde{x}_{\gamma,\varnothing}\|_{\ell_{m_\gamma}(c_{\gamma,\varnothing})}, \gamma \subset \tau) \le R$.

The proof of this Theorem is quite complicated and will be given in forthcoming paper [13]. Remark that the equations (12) are obtained by direct action of $\mathbb{D}^\beta$ on equations (5), because (8) and chain rule (7) for F give $\mathbb{D}^\beta F^{(\ell)}(\xi_j^0(t, x^0)) = \delta_j^\beta t^{|\beta|} F^{(\ell+|\beta|)}(\xi_j^0(t, x^0))$.

The property $\mathbb{D}^k \xi_t^0 = t e_k$ implies not only the simple integration-by-parts formula, but also a simplified structure of coefficients in (12). In particular, one sees the nonlinear symmetries in this equation, i.e. that the terms $F'(\xi^0)\mathbb{D}^\beta \xi_\tau$, $t^{|\beta|} F^{(1+|\beta|)}(\xi^0)\xi_\tau$, $t^{|\beta|} F^{(|\tau|+|\beta|)}(\xi^0)\xi_{j_1} \cdots \xi_{j_n}$, $\tau = \{j_1, \ldots, j_n\}$ appear in the r.h.s. of (12) simultaneously. Taking into account this reason we introduce a nonlinear expression

$$\rho_{\tau,\beta}(t) = \sum_{\gamma\subset\tau,\ \sigma\subset\beta,\ \gamma\neq\varnothing} \mathbf{E}\, p_{\gamma,\sigma}(\|\xi^0(t, x^0)\|^2_{\ell_2(a)}) \|\frac{\mathbb{D}^\sigma \xi_\gamma}{t^{|\sigma|}}\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}, \qquad m_\gamma = m_1/|\gamma|.$$

The following Theorem gives a quasi-contractive estimate on $\rho_{\tau,\beta}$, which will be the principal tool for investigation of the raise of smoothness properties of semigroup.

Theorem 3. *Let* F *fulfill* (3), $x^0 \in \ell_2(a)$ *and vectors* $c_{\gamma,\sigma} \in \mathbb{P}$ *satisfy hierarchy* (13). *Suppose that monotone functions* $p_{\gamma,\sigma} \in C^2(\mathbb{R}^1_+)$ *are such that* $\exists\, \varepsilon > 0\ \exists\, K_p > 0\ \forall\, z \in \mathbb{R}_+$

$$\begin{aligned} &p_{\gamma,\sigma}(z) \ge \varepsilon \qquad \text{and} \qquad (1+z)(|p'_{\gamma,\sigma}(z)| + |p''_{\gamma,\sigma}(z)|) \le K_p p_{\gamma,\sigma}(z), \\ &(p_{\tau,\beta})^{|\tau|}(1+z)^{\frac{\mathbf{k}+1}{2}m_1} \le K_p (p_{\gamma_1,\sigma_1})^{|\gamma_1|} \cdots (p_{\gamma_\ell,\sigma_\ell}) z^{|\gamma_\ell|} \end{aligned} \tag{14}$$

for any subdivision $\tau = \gamma_1 \cup \cdots \cup \gamma_\ell$, $\beta = \sigma_0 \cup \cdots \cup \sigma_\ell$ *such that* $2 - \ell \le |\sigma_0|$.

Then there is a constant $M = M_{\tau,\beta} \in \mathbb{R}^1$ *such that the* **quasi-contractive nonlinear estimate** *holds*

$$\rho_{\tau,\beta}(t) \le e^{Mt} \rho_{\tau,\beta}(0). \tag{15}$$

The r.h.s. limit $t = 0$ *is substituted by Theorem 2.*

Proof. If we introduce function $h_i(t) = \sum_{\ell=0}^{i} \sum_{\substack{\sigma\subset\beta \\ |\sigma|=\ell}} \sum_{\substack{\gamma\subset\tau \\ \gamma\neq\varnothing}} \mathbf{E}\, p_{\gamma,\sigma}(\|\xi^0(t, x^0)\|^2_{\ell_2(a)}) \left\|\frac{\mathbb{D}^\sigma \xi_\gamma}{t^{|\sigma|}}\right\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$ then it is clear that for estimate (15) it is sufficient to prove that for any $i \in \{0, \ldots, m\}$ $h_i(t) \le e^{M_i t} h_i(0)$ or an estimate

$$g_\sigma(t) \le e^{K_1 t} g_\sigma(0) + K_2 \int_0^t e^{K_1(t-s)} h_{i-1}(s) ds$$

for function $g_\sigma(t) = \sum_{\gamma\subset\tau,\, \gamma\neq\varnothing} \mathbf{E}\, p_{\gamma,\sigma}(\|\xi^0(t, x^0)\|^2_{\ell_2(a)}) \left\|\frac{\mathbb{D}^\sigma \xi_\gamma}{t^{|\sigma|}}\right\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$.

The proof is based on Ito formula, estimate $H_\mu p_{\gamma,\sigma}(z) \ge K_{\gamma,\sigma} p_{\gamma,\sigma}(z)$, $z \in \mathbb{R}_+$ (see [5, Hint 9]) and Young inequality. ■

The following Proposition is an immediate consequence of Theorem 3.

Proposition 1. *Let F fulfill* (3), $\psi \in \mathbb{P}$ *and function* $Q \in C^2(\mathbb{R}^1)$ *satisfy* $(14)_1$. *Then* $\forall n \in \mathbb{N}$ $\exists M = M_n(\psi, Q)$ *such that for any* $t \in [0,T]$ *the estimate holds*

$$\mathbf{E}\, Q(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\, |\mathbb{D}^\beta \xi_{k,\tau}|^{m_\tau} \le \frac{t^{|\beta| m_\tau} e^{Mt} |\tau| \psi_0 Q(\|x^0\|^2_{\ell_2(a)})(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|+|\beta|-1)}}{a_k^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|-1)} \prod_{i\in\beta} a_i^{\frac{\mathbf{k}+1}{2} m_\tau} \prod_{j\in\tau} \psi_{k-j}^{m_1/|\tau|}}$$

for all $1 \le m_1 \le n$, $|\tau| \le m_1$, $|\beta| \le n$ *and* $m_\tau = m_1/|\tau|$.

3 Smoothing properties of semigroups

Proposition 1 shows that for $\mathbb{D}^\beta \xi_\gamma$ there is a *certain ordering of behaviour* with respect to $\beta, \gamma \subset \mathbb{Z}^d$, generated by weights $\left((1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}, \{a_k^{-\frac{\mathbf{k}+1}{2}}\}_{k\in\mathbb{Z}^d}\right)$. Due to the representation (11) it influences corresponding relations between different order derivatives of semigroup P_t and therefore requires a reduction of weights in seminorms on partial derivatives $\partial_\tau P_t f$ in C_Θ, which we are going to introduce.

Let $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$ denote any array of pairs $(p, \mathcal{G}) \in \Theta_i$ with i-tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$, constructed by vectors $G^1, \ldots, G^i \in \mathbb{P}$, and monotone functions $p \in C^2(\mathbb{R}^1_+)$ with property $(14)_1$. Array Θ_0 should consist of pairs $(p, \varnothing)$ with empty tensor $\varnothing$ such that $\varnothing \otimes G = G \otimes \varnothing = G$, $G \in \mathbb{P}$. The array $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$, $n \in \mathbb{N}$, is *quasi-contractive with parameter* $\mathbf{k}$ iff for all $(p, \mathcal{G}) \in \Theta_i$, $i = 2, \ldots, n$, and all pair of indexes $k, j \in \{1, \ldots, i\}$, $k \ne j$, there is $(\widetilde{p}, \widetilde{\mathcal{G}}) \in \Theta_{i-1}$ such that $\exists K > 0\ \forall z \in \mathbb{R}_+$

$$(1+z)^{\frac{\mathbf{k}+1}{2}} \widetilde{p}(z) \le K p(z) \qquad \textit{and} \qquad (\widehat{\mathcal{G}}^{\{k,j\}})^\ell \le K \widetilde{\mathcal{G}}^\ell, \qquad \ell = 1, \ldots, i-1. \tag{16}$$

Above for tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$, we used notation $\mathcal{G}^i = G^i$ and $(i-1)$-tensor $\widehat{\mathcal{G}}^{\{k,j\}}$ is constructed from i-tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$ by the rule $\widehat{\mathcal{G}}^{\{k,j\}} = G^1 \otimes \cdots \otimes G^{k-1} \otimes G^{k+1} \otimes \cdots \otimes G^{j-1} \otimes A^{-(\mathbf{k}+1)} G^k G^j \otimes G^{j+1} \otimes \cdots \otimes G^i$ with $A^{-(\mathbf{k}+1)} = \{a_k^{-(\mathbf{k}+1)}\}_{k\in\mathbb{Z}^d}$.

Definition 3. We say that $f \in C_\Theta(\ell_2(a))$, $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$ iff $f \in C(\ell_2(a))$ and $\forall \tau = \{k_1, \ldots, k_i\}$, $|\tau| \le n$ there are partial derivatives $\partial_\tau f = \partial_{k_1} \cdots \partial_{k_i} f \in C(\ell_2(a))$ such that the norm $\|f\|_\Theta = \max_{i=0,\ldots,n} |||\partial^{(i)} f|||_{\Theta_i}$ is finite, where

$$|||\partial^{(i)} f|||_{\Theta_i} = \sup_{x\in\ell_2(a)} \max_{(p,\mathcal{G})\in\Theta_i} \frac{|\partial^{(i)} f(x)|_{\mathcal{G}}}{p(\|x\|^2_{\ell_2(a)})}, \qquad |\partial^{(i)} f(x)|^2_{\mathcal{G}} = \sum_{\tau=\{k_1,\ldots,k_i\}\in\mathbb{Z}^d} G^1_{k_1} \ldots G^i_{k_i} |\partial_\tau f(x)|^2.$$

Above partial derivatives $\partial_\tau f$ are understood in the sense that $\forall x^0 \in \ell_2(a)\ \forall h \in \mathbf{X}_\infty([a,b]) = \bigcap_{p\ge1,\, c\in\mathbb{P}} AC_\infty([a,b], \ell_p(c))$ representations hold

$$\forall |\tau| = 0, \ldots, n-1 \qquad \partial_\tau f(x^0 + h(\bullet)) \Big|_a^b = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} f(x^0 + h(s)) h'_k(s) ds,$$

where we used notation $AC_\infty([a,b], X) = \{h \in C([a,b], X);\ \exists h' \in L^\infty([a,b], X)\}$ for Banach space X.

In [14] it was shown that the semigroup P_t preserves spaces $C_\Theta(\ell_2(a))$ and fulfills estimate $\exists M_\Theta, K_\Theta\colon \forall f \in C_\Theta(\ell_2(a))$

$$\|P_t f\|_{C_\Theta} \le K_\Theta e^{M_\Theta t} \|f\|_{C_\Theta} \tag{17}$$

if the array Θ is quasi-contractive with parameter $\mathbf{k}$.

Introduce $T_{\mathbf{k}}\Theta = \{(\,(1+z^2)^{\frac{\mathbf{k}+1}{2}}p(z),\, \mathrm{sym}(\mathcal{G}\otimes^{\mathbf{k}+2})\,);\ (p,\mathcal{G})\in\Theta\}$ and denote $(\Theta)^m = \bigcup_{i=0}^{m} T^i_{\mathbf{k}}\Theta$, $(\Theta)^0=\Theta$. Remark that for quasi-contractive with parameter $\mathbf{k}$ array Θ the array $(\Theta)^m$ is also quasi-contractive. This follows from $(\Theta)^i = (\Theta)^{i-1}\cup T_{\mathbf{k}}(\Theta)^{i-1}$ and ordering (16).

Next Theorem gives the raise of smoothness in scale C_Θ under the action of semigroup P_t. Denote by $\mathcal{D}_\Theta$ the closure in C_Θ of $f\in\mathcal{P}^\infty_{cyl}(\ell_2(a))$ such that $\|f\|_{C_\Theta}<\infty$.

Theorem 4. *Let Θ be quasi-contractive array with parameter $\mathbf{k}$ (3). Then $\forall\, m\ge 1\ \exists\, K_{\Theta,m}$, $M_{\Theta,m}$ such that $\forall\, f\in\mathcal{D}_\Theta$ we have $P_tf\in C_{(\Theta)^m},\ t>0$ and*

$$\|P_tf\|_{C_{(\Theta)^m}} \le \frac{1}{t^{m/2}}K_{\Theta,m}e^{M_{\Theta,m}t}\|f\|_{C_\Theta}, \qquad t>0. \tag{18}$$

Proof. Let $\Theta=\Theta_0\cup\cdots\cup\Theta_n$ be a quasi-contractive array. Consider $f\in\mathcal{P}^\infty_{\mathrm{cyl}}(\ell_2(a))$ such that $\|f\|_{C_\Theta}<\infty$. It is not difficult to see that for (18) it is sufficient to prove: $\forall\, i\in\{0,\dots,n\}\ \forall\, t>0$

$$|\!|\!|\partial^{(i+1)}P_tf|\!|\!|_{T_{\mathbf{k}}\Theta_i} \le \frac{1}{\sqrt{t}}Ke^{Mt}\|f\|_{C_\Theta} \tag{19}$$

or using $P_tf=P_{t/2}P_{t/2}f$ it can be shown inductively that (19) follows from: $\forall\, i=0,\dots,n$ and $\Theta_i\in\Theta$:

$$|\!|\!|\partial^{(i+1)}P_tf|\!|\!|_{T_{\mathbf{k}}\Theta_i} \le Ke^{Mt}\left\{\frac{1}{\sqrt{t}}|\!|\!|\partial^{(i)}f|\!|\!|_{\Theta_i} + \max_{\ell=1,\dots,i}(|\!|\!|\partial^{(\ell)}f|\!|\!|_{\Theta_\ell}, |\!|\!|\partial^{(\ell)}f|\!|\!|_{T_{\mathbf{k}}\Theta_{\ell-1}})\right\}.$$

Due to integration-by-parts formula (10) we have

$$\partial_{k_{i+1}}\cdots\partial_{k_1}P_tf = \sum_{\ell=1}^{i}\ \sum_{\gamma_1\cup\cdots\cup\gamma_\ell=\{k_1,\dots,k_{i+1}\}} \mathbf{E}\,\langle\partial^{(\ell)}f(\xi^0),\xi_{\gamma_1}\otimes\cdots\otimes\xi_{\gamma_\ell}\rangle \tag{20}$$

$$+\frac{1}{t}\sum_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\mathbf{E}\ \partial_{j_i}\cdots\partial_{j_1}f(\xi^0)\xi_{j_1,k_1}\cdots\xi_{j_{i+1},k_{i+1}}\int_0^t\langle\Gamma_se_{j_{i+1}},dW(s)\rangle_{\ell_2(1)} \tag{21}$$

$$-\frac{1}{t}\sum_{\ell=1}^{i+1}\sum_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\mathbf{E}\ \partial_{j_i}\cdots\partial_{j_1}f(\xi^0)\xi_{j_1,k_1}\cdots\mathbb{D}^{j_{i+1}}\xi_{j_\ell,k_\ell}\cdots\xi_{j_{i+1},k_{i+1}}. \tag{22}$$

The following estimates finish the proof:

$$\begin{aligned}
&|\!|\!|(20)|\!|\!|_{T_{\mathbf{k}}\Theta_i} \le Ke^{Mt}\max_{\ell=1,\dots,i}(|\!|\!|\partial^{(\ell)}f|\!|\!|_{\Theta_\ell}, |\!|\!|\partial^{(\ell)}f|\!|\!|_{T_{\mathbf{k}}\Theta_{\ell-1}})\\
&|\!|\!|(21)|\!|\!|_{T_{\mathbf{k}}(p,\mathcal{G})} \le \frac{1}{\sqrt{t}}Ke^{Mt}|\!|\!|\partial^{(i)}f|\!|\!|_{\Theta_i},\\
&|\!|\!|(22)|\!|\!|_{T_{\mathbf{k}}(p,\mathcal{G})} \le Ke^{Mt}|\!|\!|\partial^{(i)}f|\!|\!|_{\Theta_i}.
\end{aligned} \tag{23}$$

Estimate (23_1) follows from (17). To obtain (23_2) we apply the Hölder inequality

$$|\!|\!|(21)|\!|\!|_{T_{\mathbf{k}}(p,\mathcal{G})} \le \frac{\left|\displaystyle\sum_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\left(\mathbf{E}\,\frac{|\partial_{j_i}\cdots\partial_{j_1}f(\xi^0)|^2}{p^2(z_t)}\right)^{1/2}A_{j_1,\dots,j_{i+1}}\right|_{\mathcal{G}\otimes A^{\mathbf{k}+2}}}{t\,(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}p(\|x^0\|^2_{\ell_2(a)})} \tag{24}$$

with $z_t=\|\xi^0(t,x^0)\|^2_{\ell_2(a)}$ and

$$A^{k_1,\dots,k_{i+1}}_{j_1,\dots,j_{i+1}} = \left(\mathbf{E}\,p^2(z_t)\,|\xi_{j_1,k_1}\cdots\xi_{j_{i+1},k_{i+1}}\int_0^t\langle\Gamma_se_{j_{i+1}},dW(s)\rangle_{\ell_2(1)}|^2\right)^{1/2}.$$

A simple consequence of Ito formula gives

$$\left(\mathbf{E}\,(\int_0^t \langle\Gamma_s e_{j_{i+1}}, dW(s)\rangle_{\ell_2(1)})^{2(i+2)}\right)^{1/2(i+2)} \le K e^{Mt}\sqrt{t} a_{j_{i+1}}^{-\frac{\mathbf{k}+1}{2}} (1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}, \tag{25}$$

the Hölder inequality with $q_\ell = 1/(i+2)$, (25) and Proposition 1 with $|\beta| = 0$, $\tau = \{j\}$ imply

$$A^{k_1,\ldots,k_{i+1}}_{j_1,\ldots,j_{i+1}} \le \prod_{\ell=1}^{i+1}\left(\mathbf{E}\, p^{2\frac{i+2}{i+1}}|\xi_{j_\ell,k_\ell}|^{2(i+2)}\right)^{1/2(i+2)} \left\{\mathbf{E}\left(\int_0^t \langle\Gamma_s e_{j_{i+1}}, dW\rangle\right)^{2(i+2)}\right\}^{1/2(i+2)}$$

$$\le K e^{Mt}\sqrt{t}(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}} p(\|x^0\|^2_{\ell_2(a)}) \frac{a_{j_{i+1}}^{-\frac{\mathbf{k}+1}{2}}}{\prod\limits_{\ell=1}^{i+1}\psi_{j_\ell-k_\ell}}. \tag{26}$$

Substituting (26) in (24) we obtain

$$|||(21)|||_{T_{\mathbf{k}}(p,\mathcal{G})} \le \frac{Ke^{Mt}}{\sqrt{t}}(\mathrm{tr}\, a)^{1/2}$$

$$\times \sum_{j\in\mathbb{Z}^d} \frac{\delta_a^{\frac{\mathbf{k}+1}{2}|j|}}{\psi_j}\left(\sum_{k_1,\ldots,k_i} G^1_{k_1}\cdots G^i_{k_i}\,|\sum_{j_1,\ldots,j_i}\left(\mathbf{E}\,\frac{|\partial_{j_i}\cdots\partial_{j_1} f(\xi^0)|^2}{p^2(z_t)}\right)^{1/2}\prod_{\ell=1}^{i}\frac{1}{\psi_{j_\ell-k_\ell}}|^2\right)^{1/2}$$

$$\le \frac{K' e^{Mt}}{\sqrt{t}} K_{G,\psi}|||\partial^{(i)} f|||_{(p,\mathcal{G})} \le \frac{K' e^{Mt}}{\sqrt{t}} K_{G,\psi}|||\partial^{(i)} f|||_{\Theta_i}.$$

Estimate (23_3) can be done in a similar way. ■

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