Torus Doubling Cascade in Problems with Symmetries

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Switching the branch of the trivial states, at a Hopf bifurcation, to a branch of the standing and the travelling states is considered. A secondary bifurcation on period solutions, leading to chaotic behaviour, is dealt with the technique of canonical co-ordinates transformation. The techniques that are introduced in this paper are illustrated by a small system of ordinary differential equations.

1 Introduction

In this paper a multiple Hopf bifurcation is considered on the branch of trivial solutions in problems with $O(2)$ symmetry. The theory for this type of bifurcation is well understood and leads us to solutions in the form of standing and travelling waves (van Gils and Mallet-Paret [1], Golubitsky et al [2]). Golubitsky et al [2] distinguish the standing and the travelling (rotating) oscillations in the system of ordinary differential equations (ODEs) by their symmetries, however, these oscillations come from considering the ODE as an amplitude system for some partial differential equation (PDE) model. The triple zero bifurcation equations that we consider in this paper is a normal form for the amplitude equation for a PDE model Arneodo et al [3]. At the multiple Hopf bifurcation point, due to the four-dimensional null space resulting from the Jacobian of the linearized equations, swapping the branches is not straightforward. We show a possible way for swapping the branches, and this involves restricting the solutions in the two-dimensional fixed point subspaces associated with the isotropy subgroups of $O(2) \times S^1$. On the travelling waves we consider a further Hopf bifurcation leading to a tori branch. We deal with this bifurcation using canonical co-ordinates.

2 The travelling waves: multiple Hopf bifurcation

In a previous paper [4] we considered a system of the form

$$\dot{z} = g(z, \lambda), \tag{1}$$

where $z \in \mathbb{C}^3$ and $\lambda \in \mathbb{R}$. The system is symmetric with respect to the group $O(2)$. In problems with $O(2)$ symmetry there are typically many branches of symmetric steady state solutions, and the state space X can be decomposed into a symmetric subspace X^s (reflectional invariant subspace) and an anti-symmetric subspace X^a, i.e. $X = X^s \oplus X^a$, due to the underlying reflection symmetry [5]. On the symmetric solutions $z_s(\lambda)$, the Jacobian matrix can be also decomposed as $g_z(z_s, \lambda) = \mathrm{diag}(g_z^s(z_s, \lambda) : g_z^a(z_s, \lambda))$, where g_z^s and g_z^a are associated with symmetric and anti-symmetric subspaces, respectively [6]. In [4] a Hopf bifurcation from the circle of non-trivial steady states (or relative equilibria) is considered and it is shown that the waves emerging from a Hopf bifurcation change their direction of propagation in a periodic fashion. In that paper [4] we considered a couple of imaginary eigenvalues in the anti-symmetric block of the Jacobian matrix, and due to the symmetry the Hopf bifurcation was not standard. We used the canonical co-ordinates transformation to remove the degeneracy of the system.

The equation we consider in this paper is in the form of (1), which we assume symmetric with respect to the diagonal action of group $O(2)$ defined by

$$r_\alpha z = e^{i\alpha} z, \qquad sz = \bar{z},$$

where $z = (z_1, z_2, z_3, \ldots, z_n) \in \mathbb{C}^n =: X$. The problem has a branch of trivial solutions, with full $O(2)$ symmetry, inherited from the underlying symmetry of the system. We assume that on this branch, at $\lambda = \lambda_0$, the symmetric and the anti-symmetric blocks of the Jacobian matrix satisfy the following conditions

$$g_z^s(0, \lambda_0)\phi_s = \pm i\omega\phi_s, \qquad g_z^a(0, \lambda_0)\phi_a = \pm i\omega\phi_a, \tag{2}$$

where $\phi_s = \phi_{sr} + i\phi_{sj} \in X^s$ and $\phi_a = \phi_{ar} + i\phi_{aj} \in X^a$. Therefore, the Hopf bifurcation on the trivial solution is associated with the two-dimensional irreducible representation of the group $O(2)$. The irreducible representations of $O(2)$ are given by

(i) $r_\alpha = I$, $s = I$, one-dimensional,

(ii) $r_\alpha = I$, $s = -I$, one-dimensional,

(iii) $r_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, $s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, two-dimensional.

Bifurcations associated with the one-dimensional irreducible representations of $O(2)$ are not of interest. The analysis of this type of bifurcation is very similar to that for Hopf bifurcations of vector fields since an additional S^1 symmetry can be defined in this case [7]. Therefore the full symmetry of the system is $O(2) \times S^1$ and the Hopf bifurcation is associated with the two-dimensional irreducible representation of $O(2) \times S^1$. To analyse this bifurcation we linearize the equation (1) at the Hopf bifurcation giving rise to the solution of the form

$$\Phi_k(t) = e^{i\omega t}\phi_k, \qquad k = s, a. \tag{3}$$

We represent the corresponding real eigenspace as $E_i = \mathrm{sp}\{\phi_{sr}, \phi_{sj}, \phi_{ar}, \phi_{aj}\}$ which is a four-dimensional subspace. We can decompose the space X to its isotypic components as $X = X_0 \oplus X_1 \oplus X_2$, where $X_0 = X^s$, X_1 are associated with the one-dimensional representations, and X_2 is associated with the two-dimensional representation. These components are irreducible subspaces. Therefore ϕ_{sr}, ϕ_{sj}, ϕ_{ar}, $\phi_{aj} \in X_2$.

The group actions on the eigenspace can be easily obtained as:

$$\theta \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} \cos\omega\theta & 0 & -\sin\omega\theta & 0 \\ 0 & \cos\omega\theta & 0 & -\sin\omega\theta \\ \sin\omega\theta & 0 & \cos\omega\theta & 0 \\ 0 & \sin\omega\theta & 0 & \cos\omega\theta \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix}, \tag{4}$$

$$r_\alpha \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 & 0 \\ -\sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & \sin\omega t \\ 0 & 0 & -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix},$$

$$s \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix}.$$

Now, to use the results of Golubitsky et at [2] with regards to the isotropy subgroups of $O(2) \times S^1$ on $\mathbb{C}^2$, we identify the eigenspace E_i with $\mathbb{C}^2$ by

$$(x_1, y_1, x_2, y_2) \longleftrightarrow x_1\phi_{sr} + x_2\phi_{sj} + y_1\phi_{ar} + y_2\phi_{aj}, \tag{5}$$

where $x_j + iy_j =: \boldsymbol{z}_j \in \mathbb{C}^2$, $j = 1, 2$. From (4) and (5), we obtain the action of θ on $\mathbb{C}^2$:

$$\theta \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\omega\theta & 0 & \sin\omega\theta & 0 \\ 0 & \cos\omega\theta & 0 & \sin\omega\theta \\ -\sin\omega\theta & 0 & \cos\omega\theta & 0 \\ 0 & -\sin\omega\theta & 0 & \cos\omega\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}.$$

Now, we introduce new co-ordinates $(\tilde{\boldsymbol{z}}_1, \tilde{\boldsymbol{z}}_2) = (\bar{\boldsymbol{z}}_1 - i\bar{\boldsymbol{z}}_2, \boldsymbol{z}_1 - i\boldsymbol{z}_2)$ [2], so that in these new co-ordinates θ acts diagonally on $\mathbb{C}^2$. It can be shown that the representation T of $O(2) \times S^1$ in these new coordinates is given by

$$T(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T(r_\alpha) = \begin{bmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \qquad T(\theta) = \begin{bmatrix} e^{i\omega\theta} & 0 \\ 0 & e^{i\omega\theta} \end{bmatrix}.$$

These results correspond to the results given in [2]. The isotropy subgroups of $O(2) \times S^1$ acting on $\mathbb{C}^2$ are given by Golubitsky, et al [2]. They have shown that there are only two isotropy subgroups, which give two-dimensional fixed point subspaces and these are used here, to reduce the four-dimensional eigenspace E_i to a two-dimensional one. These subgroups and their corresponding fixed point subspaces are given bellow:

	Isotropy subgroup	**Fixed point space**	**dim**
(a)	$Z_2(s) = \{(I, 0), (s, 0)\}$	$\{(\tilde{\boldsymbol{z}}_1, \tilde{\boldsymbol{z}}_1)\}$	2
(b)	$\tilde{SO}(2) = \{(I, 0), (r_{\omega\theta}, \omega\theta)\}$	$\{(\tilde{\boldsymbol{z}}_1, 0)\}$	2

In case (a), from $\tilde{\boldsymbol{z}}_1 = \tilde{\boldsymbol{z}}_2$ and reverting to the original co-ordinates we obtain $y_1 = y_2 = 0$. Thus the identification (5) implies that $(x_1, 0, x_2, 0) \longleftrightarrow x_1\phi_{sr} + x_2\phi_{sj}$. Therefore in this case, subgroup $\sigma_1 = Z_2(s)$ with $\mathrm{Fix}(\sigma_1) \cap E_i = \mathrm{sp}\{\phi_{sr}, \phi_{sj}\}$, reduces the four-dimensional eigenspace E_i to a two-dimensional one, and the Equivariant Hopf Theorem implies that there exists a branch of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$ having σ_1 as its group of symmetries. This branch corresponds to the standing waves. From $\tilde{\boldsymbol{z}}_2 = 0$ in case (b), we obtain $x_1 = -y_2$ and $y_1 = x_2$. Thus the identification (5) implies that $(x_1, y_1, y_1, -x_1) \longleftrightarrow x_1(\phi_{sr} - \phi_{aj}) + y_1(\phi_{sj} + \phi_{ar})$. Therefore the subgroup $\sigma_2 = \tilde{SO}(2)$, reduces the four-dimensional eigenspace to a two-dimensional one with $\mathrm{Fix}(\sigma_2) \cap E_i = \mathrm{sp}\{\phi_{sj} + \phi_{ar}, \phi_{sr} - \phi_{aj}\}$. Hence the Equivariant Hopf Theorem implies that there exists a branch of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$ having σ_2 as its group of symmetries. This branch corresponds to the branch of travelling waves. Thus generically there are two branches of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$.

Once a multiple Hopf bifurcation has been detected at $(0, \lambda_0)$, a starting solution for the two branches of periodic solutions can be obtained in usual way for the variable $z(t)$, using the solution (3). Near the symmetry breaking Hopf bifurcation, the solution of (1) to first order is given by

$$\begin{aligned} z(t) &= \alpha\Phi_j(t) + O(\alpha^2), \qquad j = s, a, \\ \lambda &= \lambda_0 + O(\alpha^2), \end{aligned} \tag{6}$$

where $\Phi_j(t)$ is defined by (3). Restricting the solutions to the fixed point subspace with symmetry group σ_1 implies that $\Phi_s(t) = e^{i\omega t}(\phi_{sr} + i\phi_{sj})$ and this solution gives rise to the branch of standing wave solutions. Restricting to the fixed point subspace with the subgroup of symmetry σ_2 leads to a solution of the form $\Phi_a(t) = e^{i\omega t}((\phi_{sj} + \phi_{ar}) + i(\phi_{sr} - \phi_{aj}))$ which gives rise to the branch of travelling wave solutions. Therefore, there exist two branches of solutions bifurcating from trivial solutions at $\lambda = \lambda_0$. Note that ϕ_{kr}, ϕ_{kr}, $k = s, a$ can be obtained by (2).

3 The torus bifurcation: canonical co-ordinates transformation

Assume that a Hopf bifurcation occurs on the branch of travelling wave solutions. The branch switching at this bifurcation is not standard, due to the fact that solutions emanating at this Hopf bifurcation are in the form of torus. This can be achieved by transforming the equations to the canonical co-ordinates [4, 9], so that in this co-ordinates system the drift velocity of the travelling waves will be decoupled, and therefore we observe travelling wave solutions in travelling (rotating) frame as steady state solutions. The torus bifurcation in this co-ordinates system becomes a standard Hopf bifurcation point and the package AUTO [10] deals with it automatically.

In general we make a change of co-ordinates

$$w = W(z) = (w_1(z), w_2(z), \ldots, w_n(z)), \qquad z = (z_1, z_2, \ldots, z_n).$$

Then $(w_1, w_2, \ldots, w_n)$ defines canonical co-ordinates for the one parameter Lie group of transformations $z^* = X(z, \alpha)$, $\alpha \in \mathbb{R}$, if in terms of such co-ordinates the group action is

$$w_i^* = w_i, \quad i = 1, 2, \ldots n-1, \qquad w_n^* = w_n + \alpha.$$

The advantage of these co-ordinates is that we can decouple the velocity of travelling waves. In this paper the canonical co-ordinates is defined as follows:

$$w_1 = \frac{z_2}{z_1}, \quad w_2 = \frac{z_3}{z_1}, \quad \ldots, \quad w_{n-1} = \frac{z_n}{z_1}, \quad r = |z_1|, \quad \phi = \arg(z_1),$$

where $w_j = u_j + iv_j \in \mathbb{C}$, $j = 1, 2, \ldots, n-1$ and $r \in \mathbb{R}$ are all invariant under the rotation and $\phi \to \phi + \alpha$.

4 An example in $\mathbb{C}^3$

We consider the following system

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = z_3, \qquad \dot{z}_3 = -\lambda z_1 - 2.0 z_2 - 1.5 z_3 + |z_1|^2 z_1, \tag{7}$$

and $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, $\lambda \in \mathbb{R}$ [3, 4, 8]. The system is symmetric with respect to the diagonal action of $O(2)$ defined by

$$r_\alpha(z_1, z_2, z_3) = (e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\alpha} z_3), \qquad s(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3).$$

The trivial branch of solutions, $z = 0$, $\forall \lambda$, with full $O(2)$ symmetry, is unstable for $\lambda < 0$ and gains stability at $\lambda = 0$ in a subcritical bifurcation to an unstable non-trivial branch of solutions defined by $\lambda = |z_1|^2$. These solutions lie in the symmetric subspace X^s, characterized by $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = \mathrm{Im}(z_3) = 0$. The Hopf bifurcation from non-trivial states is considered in [4, 8]; here we consider a Hopf bifurcation on trivial states which occurs at $\lambda = 3.0$. The full symmetry of the system (7) is defined as $O(2) \times S^1$ and the Hopf bifurcation is associated with the two-dimensional irreducible representation of $O(2) \times S^1$.

To follow the branches of the standing and the travelling waves, respectively, we start with the initial solutions of the form

$$\begin{aligned}
&z_1 = \alpha(c_1(\cos\omega t\phi_{sr} - \sin\omega t\phi_{sj}) + c_2(\cos\omega t\phi_{sj} + \sin\omega t\phi_{sr})), \ \lambda = 3.0\\
&z_2 = \alpha(c_1(\cos\omega t(\phi_{sj} + \phi_{ar}) - \sin\omega t(\phi_{sr} - \phi_{aj})) + c_2(\cos\omega t(\phi_{sr} - \phi_{aj})\\
&\qquad + \sin\omega t(\phi_{sj} + \phi_{ar}))),\\
&\lambda = 3.0,
\end{aligned}$$

where

$$\phi_{sr} = [\,0,\ 0,\ 0,\ -0.38,\ 0,\ 0.76]^T, \qquad \phi_{sj} = [0,\ 0,\ 0,\ 0,\ 0.54,\ 0]^T$$

and

$$\phi_{ar} = [-0.38,\ 0,\ 0.76,\ 0,\ 0,\ 0]^T, \qquad \phi_{aj} = [0,\ 0.54,\ 0,\ 0,\ 0,\ 0]^T,$$

are the eigenvectors associated with the multiple pure-imaginary eigenvalues, α is a suitable small real value and c_1, c_2 are the arbitrary real constants.

We followed the unstable standing wave solutions and found a symmetry breaking bifurcation at $\lambda = 6.613$ giving rise to a branch of unstable asymmetric standing wave solutions. Note that the standing wave solutions lie in $\mathrm{Fix}(\sigma_1) = X^s$. The branch of asymmetric standing waves terminate at $\lambda = 8.01$ when an asymmetric standing wave collides with an unstable steady state solution in a homoclinic connection. The standing wave solutions terminate at $\lambda = 12.45$ in a heteroclinic connection with the travelling wave and the non-trivial steady states.

Following the travelling waves a torus bifurcation is obtained at $\lambda = 6.176$ giving rise to a branch of stable tori (here after T-branch). Switching the branch of travelling waves to the T-branch is obtained by using the canonical co-ordinate transformations discussed in the previous section. Therefore the system in canonical co-ordinates can be written as:

$$\begin{aligned}
\dot{R} &= 2Ru_1,\\
\dot{u}_1 &= u_2 - u_1^2 + v_1^2,\\
\dot{u}_2 &= -\lambda - 2.0u_1 - 1.5u_2 + R - (u_1u_2 - v_1v_2),\\
\dot{v}_1 &= v_2 - 2.0u_1v_1,\\
\dot{v}_2 &= -2.0v_1 - 1.5v_2 - (u_1v_2 + v_1u_2),\\
\dot{\phi} &= v_1,
\end{aligned} \tag{8}$$

where $R = r^2$.

A bifurcation diagram is given in Fig. 1 (see also Fig. 2 for better understanding of the bifurcations involved). The bifurcation diagram is obtained using both the original (7) and the transformed system (8).

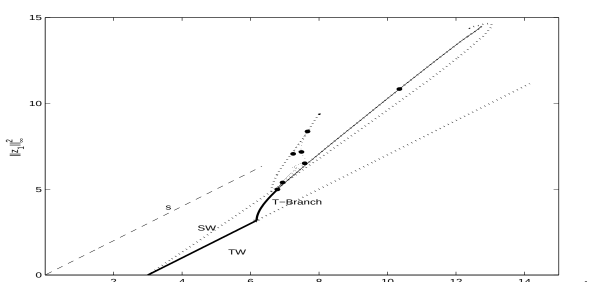

Figure 1. A bifurcation diagram is obtained by solving system (7). The two branches of travelling waves and the standing waves are obtained at $\lambda = 3.0$. Following the travelling waves a torus bifurcation is obtained at $\lambda = 6.176$. Solid circles indicate period doubling bifurcations and unstable branches are shown using dashed lines. A plus sign indicates the torus bifurcation on the travelling wave solutions.

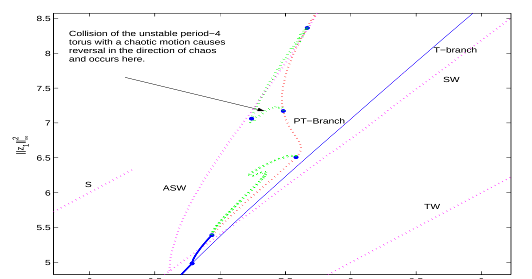

Figure 2. To illustrate a better picture of the bifurcation points and the branches, Fig. 1 is enlarged. The letter T represent a torus; TW (travelling wave); SW (standing Wave); ASW (asymmetric standing wave); PT (period-doubled torus) and s (steady state).

Note that in this new system (8) the rotation ϕ is decoupled from the remaining equations. The drift velocity of the pattern, $\dot{\phi}$, must vanish in the reflection invariant subspace X^s, hence periodic motions in this space are standing waves. The travelling wave solutions are the steady states of the above equations (ignoring the $\dot{\phi}$ equation) and the torus bifurcation appears as a standard Hopf bifurcation, resulting a standard branch swapping. The torus bifurcation is far from the multiple Hopf bifurcation point which occurred on the trivial states, therefore we do not expect any interaction between these two bifurcations. For better understanding of the motion of the torus we investigate the behaviour of its velocity defined by the variable $\dot{\phi}$ in the transformed equations (8). The torus can be obtained by the numerical integration of the original system (7). A torus and its velocity is shown in Fig. 3.

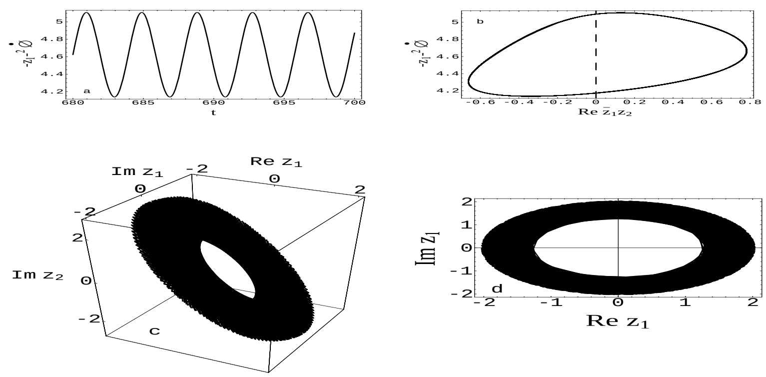

Figure 3. (a): The velocity of the torus is obtained by the numerical integration of the system in the canonical co-ordinates system for $\lambda = 6.39$, it is periodic in time. (b) The phase plot of the velocity in $(\mathrm{Re}\,(\bar{z}_1 z_2), \mathrm{Im}\,\bar{z}_1 z_2))$-plane for $\lambda = 6.39$, it is easy to show that $|z_1|^2\dot{\phi} = \mathrm{Im}\,\bar{z}_1 z_2$. (c): A torus is obtained by the numerical integration of the original equations (7) for $\lambda = 6.39$. (d): A projection of the torus in $(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1)$-plane.

Now, we consider a numerical investigation of chaotic behaviour in the above system (7). In many experiments, changes in the system behavior are studied as some system parameter is varied. We study the qualitative changes in dynamic behavior, which produces the oscillatory waves that reverse their direction of propagation in a chaotic fashion. Therefore we vary the bifurcation parameter λ and observe that the branch of tori loses stability at $\lambda = 6.786$ to a branch of period-doubled tori (PT-branch) and hereafter undergoes torus-doubling cascade bifurcations. PT-branch loses stability at $\lambda = 6.94$ to a branch of period-4 tori and this branch joins with PT-branch at $\lambda = 7.58$. A period-4 tori also bifurcates at $\lambda = 7.48$ from PT-branch. Later we will see that this branch is in the basin of attraction of the chaotic attractors which give rises to the occurrence of reversal in the direction of chaos. This branch collides with PT-branch at $\lambda = 7.60$. As λ increases close to 7.38, the torus-doubled cascade forms a modulated asymmetric strange attractor; the velocity of these modulated waves remain positive for $\lambda < 7.38$ As we vary λ further, a bifurcation occurs at $\lambda = 7.38$. At this point the chaotic attractor gains symmetry, and the velocity of the attractor changes sign irregularly and no longer remains positive (see Fig. 4) indicating chaotic reversal in the direction of propagation. The qualitative changes in chaotic behavior occurs when an asymmetric chaotic attractor describing the chaotic motion collides with an unstable period-4 torus. This increases the size of the attractor, produces a symmetric attractor and hence results in reversal in the direction of propagation (we call this an interior crisis). A crisis is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly disappears or suddenly changes in size as some control parameter is adjusted. In other words, if a parameter is changed in the opposite direction, the chaotic attractor can suddenly appear or the size of the attractor can suddenly be reduced. The interior and the boundary crisis is investigated by Grebogi et al [11] in the context of a one-dimensional quadratic map, when at a tangent bifurcation the stable and unstable orbits are created. The stable fixed point undergoes a period-doubling cascade, forming chaotic behavior and hence colliding with an unstable periodic orbit. The result is the disappearance of the chaotic motion. The mechanism of qualitative changes in this paper involves the symmetry-increasing bifurcation of the strange attractor in which the asymmetric trajectories collide with an unstable period-4 torus Grebogi et al [12]. In [12] sudden qualitative changes in chaotic dynamical behaviour occur when an attractor collides with an unstable periodic orbit.

In Fig. 4 we show time evolution of the velocity of the travelling waves. The velocity of travelling waves eventually perform complex behaviour. This complex chaotic behaviour reverses its direction of propagation, indicating an irregular sign changes of the wave's velocity.

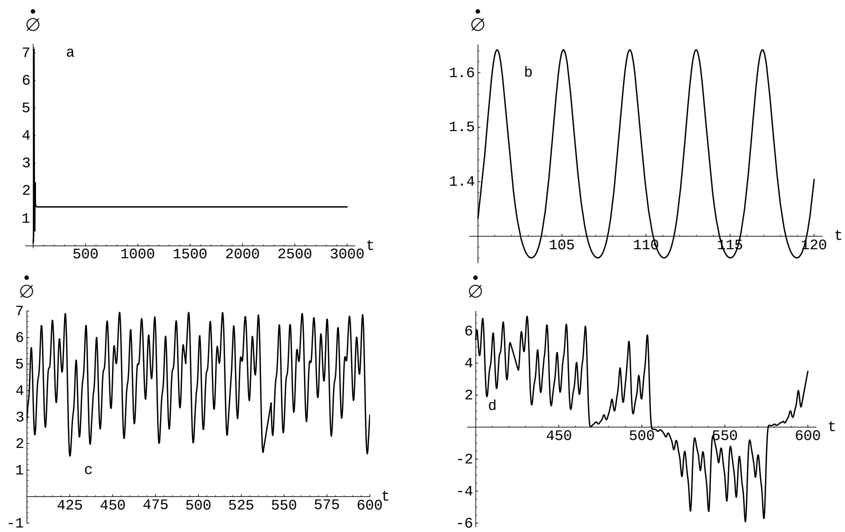

Figure 4.

A detailed study of this problem is submitted to CHAOS, American Institute of Physics [13].

[1] van Gils S.A. and Mallet-Paret J., Hopf bifurcation and symmetry: travelling and standing waves on the circle, *Proc. Roy. Soc. Edinburgh Sect. A*, 1986, V.104, 27–307.

[2] Golubitsky M., Stewart I. and Schaeffer D.G., Singularities and Groups in Bifurcation Theory, Vol. II, *Appl. Math. Sci.*, Vol. 69, New York, Springer, 1988.

[3] Arneodo A., Coullet P.H. and Spiegel E.A., The dynamics of triple convection, *Geophys. Astrophys. Fluid Dynamics*, 1985, V.31, N 1–2, 1–48.

[4] Amdjadi F. and Gomatam J., Oscillating waves arising from $O(2)$ symmetry, *J. Phys. Letts. A*, 1997, V.231, N 5–6, 379–388.

[5] Aston P.J., Spence A. and Wu W., An investigation of the bifurcation from travelling waves to modulated travelling waves, in Bifurcation and Symmetry, ISNM 104, Editors E. Allgower, K. Bohmer and M. Golubitsky, Birkhauser, 1992, 35–47.

[6] Werner B. and Spence A., The computation of symmetry-breaking bifurcation points, *SIAM J. Num. Anal.*, 1984, V.21, 388–399.

[7] Chossat P. and Golubitsky M., Iterates of maps with symmetry, *SIAM. J. Math. Anal.*, 1988, V.19, 1259–1270.

[8] Amdjadi F., The calculation of the Hopf/Hopf mode interaction point in problems with $\mathbb{Z}_2$ symmetry, *Inter. J. of Bifurcation and Chaos*, 2002, V.12, N 8, 1925–1935.

[9] Bluman G.W. and Kumei S., Symmetries and differential equations, New York, Springer-Verlag, 1989.

[10] AUTO: A program for the automatic bifurcation analysis of autonomous systems, *Congressus Numerantium*, 1981, V.30, 265–284.

[11] Grebogi C., Ott E. and Yorke J.A., Chaotic attractors in crisis, *Phys. Rev. Letts.*, 1982, V.48, 1507–1510.

[12] Grebogi C., Ott E., Romeiras F. and Yorke J.A., Critical exponents for crisis induced inermittency, *Phys. Rev. A*, 1987, V.36, 5365–5380.

[13] Amdajdi F. and Gomatam J., The travelling wave and torus bifurcation in problems with $O(2)$ symmetry, to appear.