SIX PROBLEMS IN ALGEBRAIC DYNAMICS (UPDATED DECEMBER 2006)

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The notation and terminology used in these problems may be found in the lecture notes [22], and background for all of algebraic dynamics is in Schmidt's book [19].

Problem A: order of mixing. Consider the $\mathbb{Z}^2$-action defined by the polynomial

$$f(u_1, u_2) = 1 + u_1 u_2 + u_1^2 u_2 + u_1^3 u_2 + u_1^4 + u_2^2 + u_1^4 u_2^2,$$

(that is, the system corresponding via [14] to the module

$$\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]/\langle p, f\rangle$$

where p is a prime chosen to make f irreducible). The methods of [11] and [15] show that the exact order M of mixing satisfies

$$3 \leq M < 7.$$

What is it?

Expanding the definitions here goes as follows. The $\mathbb{Z}^2$-action is the natural shift action T on the compact group

$$X = \{x \in \mathbb{F}_p^{\mathbb{Z}^2} \mid x_{(n,m)} + x_{(n+1,m+1)} + x_{(n+2,m+1)} + x_{(n+3,m+1)}$$
$$+ x_{(n+4,m)} + x_{(n,m+2)} + x_{(n+4,m+2)} = 0 \bmod p \text{ for all } n, m\},$$

with a natural Haar measure μ. The order of mixing M is the largest k for which

$$\lim_{\mathbf{n}_i - \mathbf{n}_j \to \infty; i \neq j} \mu\left(T_{-\mathbf{n}_1}(A_1) \cap \cdots \cap T_{-\mathbf{n}_k}(A_k)\right) \to \prod_{i=1}^{k} \mu(A_i)$$

for all measurable sets $A_1, \ldots, A_k$.

Update: Considerable progress has been made on this circle of problems. Masser [17] proved a conjecture of Schmidt [20] by showing that the order of mixing for an algebraic $\mathbb{Z}^d$ by automorphisms of a zero-dimensional group as detected by studying mixing shapes coincides with the real order of mixing. It remains a considerable problem to actually compute either for non-trivial examples.

Problem B: mixing of all orders. For $\mathbb{Z}^d$-actions by automorphisms of a *connected* group, mixing actions are mixing of all orders. The proof in [21] uses Diophantine estimates (see also the discussion in [19]). Can this result be obtained using simpler ideas from dynamics?

Problem C: analogues of Pesin theory. Is there an analogue of Pesin theory for (suitably defined) 'smooth' maps of the objects that occur naturally in algebraic dynamical systems? That is, compact sets that locally look like a manifold cross a Cantor set or are totally disconnected. Can any of the systems from [19] with $d > 1$ be 'perturbed' in a meaningful way?

Problem D: typical group automorphisms. There is a setting in which it makes sense to talk about a 'typical' group automorphism with a given entropy (see [3], [25], [24]). The central question though is unresolved, and reduces to a purely number-theoretical question as follows. From the infinite set of primes $P = \{2, 3, 5, 7, 11, \dots\}$ select a subset Q by tossing a fair coin infinitely often and including the nth prime if and only if the nth toss is a Head. Then is it true that

$$\limsup_{n\to\infty} \frac{1}{n} \log(2^n - 1) \times \prod_{p\in Q} |2^n - 1|_p = \log 2 \tag{1}$$

almost surely? It is shown in [24] using an ergodic argument that this upper limit is $\geq \frac{1}{2}\log 2$ almost surely, and that the analogous problem in positive characteristic can be shown to have the right answer (indeed, more can be said there: see [23]). It is probably a red herring to point out that (1) is clear if there are infinitely many Mersenne primes – see [25].

Problem E: entropy values. Determining the set of values of entropies of algebraic $\mathbb{Z}^d$-actions is equivalent to solving Lehmer's problem by [16]. So it would be very nice to solve Lehmer's problem: given $\epsilon > 0$, is there a polynomial

$$f(x) = (x - \alpha_1) \dots (x - \alpha_d) \in \mathbb{Z}[x]$$

for which the logarithmic Mahler measure

$$m(f) = \sum_{i:|\alpha_i|>1} \log |\alpha_i|$$

satisfies

$$0 < m(f) < \epsilon?$$

For background on this problem, see also [2] and [12] or the 'Lehmer conjecture page' [18].

Problem F: entropy and Deligne periods. A very interesting problem has been raised by Deninger in the course of his work on Mahler measures. In [7] he showed – roughly speaking – that $m(f)$ is the Deligne period of a certain mixed motive associated in a canonical way to f. Using a p-adic analogue of Deligne cohomology gives an analogous p-adic *valued* Mahler measure, m_p, described in [1]. The question raised there is whether there is a p-adic *valued* notion of entropy that gives entropy $m_p(f)$ to the dynamical system associated to f. A specific form of this general question is the following. Define $\log_p : \mathbb{C}_p^* \to \mathbb{C}_p$ to be the branch of the p-adic logarithm with $\log_p(p) = 0$, and consider the map $T_\lambda : x \mapsto \lambda x$ on (say) $\mathbb{Q}_p$. Is there a meaningful entropy-like invariant h_p (invariant under topological conjugacy, for example) with $h_p(T_\lambda) = \log_p \lambda$? For more background on the theory behind this question, see [1, Sect. 1.8]; for background on these questions and mixed motives, see [6], [9], [8].

Update: There has been considerable progress on this problem, via a rather indirect route. Extending the entropy 'formula' from [16] to certain actions of amenable groups has enabled some progress on the p-adic problem. The Math Reviews entry for Deninger's paper [10] explains: Fix a discrete group Γ and some element $f \in \mathbb{Z}\Gamma$ (the integral group ring). By Pontryagin duality, this structure defines a left Γ-action α_f on the compact group $X_f = \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$ by continuous automorphisms. The group X_f may be described as a closed subgroup of $(\mathbb{S}^1)^\Gamma$, and in this description the action is by left shifts.

If Γ is amenable, then it makes sense to ask for the topological or Haar measure-theoretic entropy of the action. For $\Gamma = \mathbb{Z}$ the answer is given by the familiar formula of S. A. Yuzvinskiĭ [13]. For $\Gamma = \mathbb{Z}^d$, [16] showed that the answer is $m(f)$, the logarithmic Mahler measure of f.

For nonabelian Γ this paper represents the first progress of any sort. By using the von Neumann algebra $\mathcal{N}\Gamma$ of Γ, the author has shown how to associate a Fuglede-Kadison-Lck determinant $\det_{\mathcal{N}\Gamma} f$ to f, and in this paper he shows that the topological entropy of α_f is indeed this determinant under the assumptions that the group Γ has a log-strong Flner sequence, f is a convolution unit in $L^1(\Gamma)$ and f is positive in $\mathcal{N}\Gamma$. The first condition is a restrictive one (it is not very far from asking that Γ be virtually nilpotent); the second is a natural dynamical property related to expansiveness of α_f, and the third is somewhat technical. The proofs involve developing both the entropy side of the equation and ways to approximate the determinant.

Despite the strong hypotheses, this is an important result, and it has already stimulated much further research. C. Deninger and Schmidt [5] have used homoclinic points and specification arguments to replace the log-strong Flner sequence and positivity conditions with the much more modest requirement that Γ be residually finite and that f be a unit in $L^1(\Gamma)$; Deninger [4] has shown that a p-adic-valued determinant gives a meaningful notion of p-adic-valued entropy.

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