ENTROPY, PERIODIC POINTS, TRANSITIVITY AND MINIMALITY OF MAPS

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Let (X, ϱ) be a compact metric space and $I = [0, 1]$. All maps under consideration are supposed to be continuous. The set of all continuous maps $X \to X$ is denoted by $C(X)$.

A map $f \in C(X)$ is (topologically) transitive if for any two nonempty open sets U and V in X, there is a nonnegative integer k such that $f^k(U) \cap V \neq \emptyset$. If X has no isolated points then this definition is equivalent to the existence of a dense orbit. If every orbit of f is dense, the map f is called minimal. Denote by $\mathcal{T}(I^2)$ the set of transitive selfmaps of the square I^2.

A map $F : X \times I \longrightarrow X \times I$ is called triangular (or skew product) if it is of the form $F(x, y) = (f(x), g(x, y))$. Denote by $\mathcal{C}_\triangle(X \times I)$ the set of all such maps. In the above notation, the map f is called the basis map of F.

For a map φ, let $\mathcal{P}(\varphi)$ be the set of periodic points of φ and $h(\varphi)$ be the topological entropy of φ.

In [1] it is, among other results, proved the following

Theorem 1. *Let (X, ρ) be a compact metric space and let $f \in \mathcal{C}(X)$ be a transitive map which is not minimal. Then the map f can be extended to a map $F \in \mathcal{C}_\triangle(X \times I)$ (i.e., f is the basis map of F) in such a way that F is transitive and has the same entropy as f.*

Our first question is:

1. Does Theorem 1 hold true without the assumption that f is not minimal ?

The following questions are also motivated by results from the paper [1].

2. What is $\inf\{h(f) : f \in \mathcal{T}(I^2)$ **and** $\mathcal{P}(f)$ **is dense in** $I^2\}$ **?** (Without the restriction on $\mathcal{P}(f)$ there exists a minimum equal to 0.)

Answer: The infimum is again 0 and even in I^n, $n = 2, 3, \dots$ (see [2]).

3. What is $\inf\{h(F) : F \in \mathcal{T}(I^2) \cap \mathcal{C}_\triangle(I^2)$ **and** $\mathcal{P}(F)$ **is dense in** $I^2\}$ **?** (Without the restriction on $\mathcal{P}(F)$ there exists a minimum equal to $\frac{1}{2}\log 2$.)

Answer: The minimum is again $\frac{1}{2}\log 2$ and even in I^n, $n = 2, 3, \dots$ (see [2]). For $n = 2$ this was implicitly shown before, in [3], where one can find a map $F \in \mathcal{T}(I^2) \cap \mathcal{C}_\triangle(I^2)$ whose set of periodic points is dense, whose basis map is the tent map and whose entropy is $\log 2$ (one needs to replace the tent map by a transitive interval map with entropy $\frac{1}{2}\log 2$ and to modify the construction).

In [1] it is further proved the following

Theorem 2. *The map*

$$F : (x, y) \mapsto (1 - |2x - 1|, y^{\exp(x-\beta)})$$

where β is any irrational number from $(0, 2/3)$, is a transitive map from $\mathcal{C}_\triangle(I^2)$ with topological entropy $h(F) = \log 2$ (the same as the entropy of the basis map) *such that the set of periodic points is contained in* $I \times \{0, 1\}$ *(and hence is nowhere dense in* I^2*).*

The map F from Theorem 2 has another interesting property. First we note that since the basis map is the standard tent map, by the Ergodic Theorem, for almost all (in the sense of Lebesgue measure) points $x \in I$ we have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f^i(x) = 1/2$. Further, an easy calculation shows that $F^n(x, y) = (f^n(x), y^{\exp(\sum_{i=0}^{n-1} f^i(x)-n\beta)})$ for $n = 1, 2, \dots$. So we get the following result for the ω-limit sets $\omega_F(x, y)$ of points $(x, y) \in I^2$. If $\beta < 1/2$ (respectively $\beta > 1/2$), then for almost all (in the sense of Lebesgue measure) points $(x, y) \in I^2$, $\omega_F(x, y) \subset I \times \{0\}$ (respectively $\omega_F(x, y) \subset I \times \{1\}$). On the other hand it is well known that almost all (in the sense of Lebesgue measure) points $x \in I$ have dense orbits in I, i.e., $\omega_f(x) = I$. Therefore, if $\beta < 1/2$ (respectively $\beta > 1/2$), then for almost all points $(x, y) \in I^2$, $\omega_F(x, y) = I \times \{0\}$ (respectively $\omega_F(x, y) = I \times \{1\}$). More precisely, this is true for all points (x, y) where x belongs to a full Lebesgue measure subset of I and, respectively, $y \in [0, 1)$ or $y \in (0, 1]$.

As a consequence of the above mentioned fact we immediately get that the set $I \times \{0\}$ if $\beta < 1/2$ or the set $I \times \{1\}$ if $\beta > 1/2$ is the Milnor attractor in the sense of Lebesgue measure for the dynamical system (F, I^2) where F is the map from Theorem 1.

4. What is the Milnor attractor for the map F when $\beta = 1/2$?

5. Is the map F transitive when β is a rational number from $(0, 2/3)$?

Gerhard Keller's answer and remarks: For problem 4 I can write a sketch how to prove that the bottom and the top interval are contained in the Milnor attractor. I am not able to show that the Milnor attractor is the full square. Concerning problem 5 a similar argument would give that for a dense set of points the omega limit set contains the bottom and the top. However that would require some lengthy(?) calculation on the center of mass of certain invariant measures.

Let me try to indicate what I can say to Problem 4: Everything depends on the cocycle

$$A(n, x) := \sum_{i=0}^{n-1}(T^i x - \beta)$$

where T denotes the tent map. Let μ be an invariant measure for T with good exponential mixing properties (e.g. Lebesgue measure, but also many other equilibrium states would do) and suppose that $\beta = \int_0^1 x d\mu(x)$. From the "Law of the iterated logarithm" in [F. Hofbauer and G. Keller, Equilibrium states for piecewise monotonic transformations, Ergod. Th. & Dynam. Sys. 2 (1982), 23-43] it follows in particular that

$$\limsup_{n\to\infty} A(n, x) = +\infty$$

and

$$\liminf_{n\to\infty} A(n, x) = -\infty$$

for μ-a.e. x. Exactly the same reasoning applies to any first return map T_J of T to an interval J. It follows that the Milnor attractor of F contains the "bottom" and the "top". To prove that the Milnor attractor of F is the full square one would have to show that the cocycle $A(n, x)$ is μ-a.s. recurrent. I know two results that point into this direction, but do not furnish a final proof: 1) A very general statement is Theorem C.2 in [Ph. Thieullen, Journal d'Analyse mathematique 73 (1997), 19-64]. It says that $A(n, x)$ is recurrent in probability. 2) The local limit theorem from [J. Rousseau-Egele, Annals of Probability 11 (1983), 772-788] says that

$$\mu(A(n, x) \in J) \approx \mu(J)/\sqrt{n}$$

as $n \to \infty$. This is much stronger than the previous statement, and in view of the good mixing properties of T, it should come close to a.s. recurrence.

Finally I think that for each $\beta \in (0, 2/3)$ one finds a suitable measure μ with dense support. this would also answer Problem 5.

The next two questions are also motivated by Theorem 2.

For a periodic point x of period p of an interval map f, $\gamma_x = \frac{1}{p}\sum_{i=0}^{p-1} f^i(x)$ is said to be the *centre of gravity* of the orbit of x. A map f is said to be *centered* if the map $x \mapsto \gamma_x$ is constant on the set of periodic points, i.e., if the centre of gravity of each periodic orbit of f is the same and is said to be *anticentered* if no two different periodic orbits of f have the same centre of gravity.

We know from M. Misiurewicz that the standard tent map $\tau(x) = 1 - |2x - 1|$, $x \in \langle 0, 1\rangle$ is not anticentered since $22/127$ and $26/127$ belong to different periodic orbits of period 7 with the same centre of gravity $72/127$.

Let us also remark that, given $A > 0$, the map $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = Ax \exp(-x)$ is centered.

6. Let τ be the tent map. Does there exist a dense subset P_γ of $\mathcal{P}(\tau)$ such that the centre of gravity of the orbit of each point from P_γ is the same?

7. Let f be any continuous map $I \to I$. Can one always topologically conjugate f to an anticentered map $g : I \to I$?

References

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