# POLYGONAL BILLIARDS: SOME OPEN PROBLEMS

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A billiard ball, i.e. a point mass, moves inside a polygon $Q$ with unit speed along a straight line until it reaches the boundary $\partial Q$ of the polygon, then instantaneously changes direction according to the mirror law: "the angle of incidence is equal to the angle of reflection," and continues along the new line. The orbit of a point which reaches a vertex is not defined.

**Question 1. Is there at least one periodic billiard orbit in every polygon?**

This question is open even for non acute triangles. It is known that is $Q$ is a rational polygon, i.e. all the angles between sides are rational multiples of $\pi$, then there are periodic orbits [M1, GSV, B] and in fact they are dense in the phase space of $Q$ [BGKT]. There are constructive [GSV] and non constructive [GuTr, Tr2] methods of producing periodic orbits in certain irrational billiards.

**Question 2. For which polygons is the billiard flow topologically transitive? For which is it ergodic with respect to the invariant phase volume? Is it ever mixing? weak mixing?**

It is known that the typical polygon (in the topological sense) is both topologically transitive and ergodic [KZ, KMS]. There are explicit examples of ergodic polygons [Vo]. It is not known if polygonal billiards can be weak mixing or mixing. In a conversation J. Moser attributed the question of ergodicity to E. Artin.

We call an (oriented) orbit segment which starts and ends at a vertex of $Q$ a generalized diagonal. Let $N(t)$ be the the number of generalized diagonal whose geometric length is less than or equal to $t$.

**Question 3. What is the growth rate of $N(t)$?**

For rational polygons $N(t)$ has quadratic upper and lower bounds [M2, M3]. The limit $\lim_{t\to\infty} N(t)/t^2$ exists for a special class of polygons called Veech polygons, which included for example all regular polygons [V1, V2].

Label the sides of $Q$ by symbols from a finite alphabet $\mathcal{A}$ whose cardinality is equal to the number of sides of $Q$ and code the orbit by the sequence of sides it hits. Consider $L(n)$ be the set of all words of length $n$ which arise via this coding.

**Question 4 What is the growth rate of $\#L(n)$?**

The only general results known about the complexity function is that it grows slower than any exponential [K], at least quadratically [Tr1] and has cubic upper and lower bounds for convex rational polygons [CHTr]. For billiards in a square the complexity function has been explicitly calculated, albeit for a slightly different coding (the alphabet consists of two symbols, one for vertical sides one for horizontal sides) [Mi, BP].

If $Q$ is a rational polygon then the phase space is foliated by invariant surfaces. One can use the theory of quadratic differentials and Teichmüller theory as tools to prove results about the restriction of the billiard flow to an invariant surface.

Several good reviews exist [Gu1, Gu2, MT, T]. They include more open problems.

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