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Group classification of nonlinear wave equations

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We perform complete group classification of the general class of quasilinear wave equations in two variables. This class may be seen as a generalization of the nonlinear d'Alembert, Liouville, sin/sinh-Gordon and Tzitzeica equations. We derive a number of new genuinely nonlinear invariant models with high symmetry properties. In particular, we obtain four classes of nonlinear wave equations that admit five-dimensional invariance groups. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1884886]

INTRODUCTION

More than a century ago Lie introduced the concept of continuous transformation group into mathematical physics and mechanics. His initial motivation was to develop a theory of integration of ordinary differential equations enabling to answer the basic questions, like, why some equations are integrable and others are not. His fundamental results obtained on this way, can be seen as a far reaching generalization of the Galois's and Abel's theory of solubility of algebraic equations by radicals. Since that time the Lie's theory of continuous transformation groups has become applicable to an astonishingly wide range of mathematical and physical problems.

It was Lie who was the first to utilize group properties of differential equations for constructing of their exact solutions. In particular, he computed the maximal invariance group of the one-dimensional heat conductivity equation and applied this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Since late 1970s symmetry reduction becomes one of the most popular tools for solving nonlinear partial differential equations (PDEs).

By now symmetry properties of the majority of fundamental equations of mathematical and theoretical physics are well known. It turns out that for the most part these equations admit wide symmetry groups. Especially this is the case for linear PDEs and it is this rich symmetry that enables developing a variety of efficient methods for mathematical analysis of linear differential equations. However, linear equations give mathematical description of physical, chemical or biological processes in a first approximation only. To provide a more detailed and precise description a mathematical model must incorporate nonlinear terms. Note that some important differential equations are intrinsically nonlinear and have no linear counterpart.

Hyperbolic type second-order nonlinear PDEs in two independent variables play a fundamental role in modern mathematical physics. Equations of this type are utilized to describe various types of wave propagation. They are used in differential geometry, in various fields of hydrodynamics and gas dynamics, chemical technology, superconductivity, crystal dislocation to mention only a few applications areas. Surprisingly the list of equations utilized is rather narrow. In fact, it is comprised by the Liouville, sine/sinh-Gordon, Goursat, d'Alembert, and Tzitzeica equations and a couple of others. Popularity of these very models has a natural group-theoretical interpretation, namely, all of them have nontrivial Lie or Lie–Bäcklund symmetry. By this very reason some of them are integrable by the inverse problem methods (see, e.g., Refs. 1–3) or linearizable[4–6] and completely integrable.[7,8]

Knowing symmetry group of the equation under study provides us with the powerful equation exploration tool. So it is natural to attempt classifying a reasonably extensive class of nonlinear hyperbolic type PDEs into subclasses of equations enjoying the best symmetry properties. Saying reasonably extensive we mean that (i) this class should contain the above enumerated equations as particular cases, and (ii) it should contain a variety of new invariant models of potential interest for applications. The list of the so obtained invariant equations will contain candidates for realistic nonlinear mathematical models of the physical and chemical processes mentioned above.

The history of group classification methods goes back to Lie itself. Probably, the very first paper on this subject is Ref. 9, where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite-parameter symmetry group, which is due to linearity).

The modern formulation of the problem of group classification of PDEs was suggested by Ovsyannikov in Ref. 10. He developed a regular method (we will refer to it as the Lie–Ovsyannikov method) for classifying differential equations with nontrivial symmetry and performed complete group classification of the nonlinear heat conductivity equation. In a number of subsequent publications more general types of nonlinear heat equations were classified (review of these results can be found in Ref. 11).

However, even a very quick analysis of the papers on group classification of PDEs reveals that an overwhelming majority of them deals with equations whose arbitrary elements (functions) depend on one variable only. The reason for this is that Lie–Ovsyannikov method becomes inefficient for PDEs containing arbitrary functions of several variables. To achieve a complete classification one either needs to specify the transformation group realization or restrict somehow an arbitrariness of the functions contained in the equation under study. We have recently, developed an efficient approach enabling to overcome this difficulty for low dimensional PDEs.[12,13] Utilizing it we have derived the complete group classification of the general quasilinear heat conductivity equation in two independent variables. In this paper we apply the approach in question to perform group classification of the most general quasilinear hyperbolic PDE in two independent variables.

I. GROUP CLASSIFICATION ALGORITHM

While classifying a given class of differential equations into subclasses, one can use different classifying features, like linearity, order, the number of independent or dependent variables, etc. In group analysis of differential equations the principal classifying features are symmetry properties of equations under study. This means that classification objects are equations considered together with their symmetry groups. This point of view is based on the well-known fact that any PDE admits a (possibly trivial) Lie transformation group. And what is more, any transformation group corresponds to a class of PDEs, which are invariant under this group. So the problem of group classification of a class of PDEs reduces to describing all possible (inequivalent) pairs (PDE, maximal invariance group), where PDE should belong to the class of equations under consideration.

We perform group classification of the following class of quasilinear wave equations:

$$u_{tt} = u_{xx} + F(t,x,u,u_x). \tag{1.1}$$

Here F is an arbitrary smooth function, $u=u(t,x)$. Hereafter we adopt notations $u_t=\partial u/\partial t$, $u_x=\partial_u/\partial_x$, $u_{tt}=\partial^2 u/\partial t^2$, etc.

Our aim is describing *all* equations of the form (1.1) that admit nontrivial symmetry groups. The challenge of this task is in the word *all*. If, for example, we somehow constrain the form of invariance group to be found, then the classification problem simplifies enormously. A slightly more cumbersome (but still tractable with the standard Lie–Ovsyannikov approach) is the problem of group classification of equation with arbitrary functions of, at most, one variable.

As equations invariant under similar Lie groups are identical within the group-theoretic framework, it makes sense to consider nonsimilar transformation groups[14,15] only. The important example of similar Lie groups is provided by Lie transformation groups obtained one from another by a suitable change of variables. Consequently, equations obtained one from another by a change of variables have similar symmetry groups and cannot be distinguished within the group-theoretical viewpoint. That is why, we perform group classification of (1.1) within a change of variables preserving the class of PDEs (1.1).

The problem of group classification of linear hyperbolic type equation

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0 \tag{1.2}$$

with $u = u(t,x)$, was solved by Lie[9] (see, also, Ref. 16). In view of this fact, we consider only those equations of the form (1.1) which are not (locally) equivalent to the linear equation (1.2).

As we have already mentioned in the Introduction, the Lie–Ovsyannikov method of group classification of differential equations has been suggested in Ref. 10. Utilizing this method enabled solving the group classification problem for a number of important one-dimensional nonlinear wave equations:

$$u_{tt} = u_{xx} + F(u) \quad \text{(Refs. 17–19)},$$

$$u_{tt} = [f(u)u_x]_x \quad \text{(Refs. 20–22)},$$

$$u_{tt} = f(u_x)u_{xx} \quad \text{(Refs. 22,23)},$$

$$u_{tt} = F(u_x)u_{xx} + H(u_x) \quad \text{(Ref. 24)},$$

$$u_{tt} = F(u_{xx}) \quad \text{(Ref. 22)},$$

$$u_{tt} = u_x^m u_{xx} + f(u) \quad \text{(Ref. 25)},$$

$$u_{tt} + f(u)u_t = (g(u)u_x)_x + h(u)u_x \quad \text{(Ref. 26)},$$

$$u_{tt} = (f(x,u)u_x)_x \quad \text{(Ref. 27)}.$$

Analysis of the above list shows that most of all arbitrary elements (=arbitrary functions) depend on one variable. This is not coincidental. As we already mentioned, the Lie–Ovsyannikov approach works smoothly for the case when the arbitrary elements are functions of one variable. The reason for this is that the obtained system of determining equations is still over-determined. So it can be effectively solved within the same techniques used to compute maximal symmetry group of PDEs containing no arbitrary elements.

The situation becomes much more complicated for the case when arbitrary elements are functions of two (or more) arguments. By this very reason the group classification of nonlinear wave equations,

$$u_{tt} + \lambda u_{xx} = g(u,u_x) \quad \text{(Refs. 28,29)},$$

$$u_{tt} = [f(u)u_x + g(x,u)]_x \quad \text{(Ref. 30)},$$

$$u_{tt} = f(x,u_x)u_{xx} + g(x,u_x) \quad \text{(Ref. 31)},$$

is not complete.

We suggest an efficient approach to the problem of group classification of low dimensional PDEs in Refs. 12 and 13. This approach is based on the Lie–Ovsyannikov infinitesimal method and classification results for abstract finite-dimensional Lie algebras. It enables us to obtain the complete solution of the group classification problem for the general heat equation with a nonlinear source

$$u_t = u_{xx} + F(t,x,u,u_x).$$

Later on, we perform complete group classification of the most general quasilinear evolution equation,[32–34]

$$u_t = f(t,x,u,u_x)u_{xx} + g(t,x,u,u_x).$$

We utilize the above approach to obtain complete solution of the group classification problem for the class of Eqs. (1.1).

Our algorithm of group classification of the class of PDEs (1.1) is implemented in the following three steps (further details can be found in Ref. 34):

(I) Using the infinitesimal Lie method we derive the system of determining equations for coefficients of the first-order operator that generates symmetry group of equation (1.1). (Note that the determining equations which explicitly depend on the function F and its derivatives are called classifying equations.) Integrating equations that do not depend on F we obtain the form of the most general infinitesimal operator admitted by Eq. (1.1) under arbitrary F. Another task of this step is calculating the equivalence group $\mathcal{E}$ of the class of PDEs (1.1).

(II) We construct all realizations of Lie algebras A_n of the dimension $n \leqslant 3$ in the class of operators obtained at the first step within the equivalence relation defined by transformations from the equivalence group $\mathcal{E}$. Inserting the so obtained operators into classifying equations we select those realizations that can be symmetry algebras of a differential equation of the form (1.1).

(III) We compute all possible extensions of realizations constructed at the previous step to realizations of higher dimensional $(n>3)$ Lie algebras. Since extending symmetry algebras results in reducing arbitrariness of the function F, at some point this function will contain either arbitrary functions of at most one variable or arbitrary constants. At this point, we apply the standard classification method (which is due to Lie and Ovsyannikov) to derive the maximal symmetry group of the equation under study. This completes group classification of (1.1).

Performing the above enumerated steps yields the complete list of inequivalent equations of the form (1.1) together with their maximal (in Lie's sense) symmetry algebras.

We say that the group classification problem is completely solved when it is proved that

(1) the constructed symmetry algebras are maximal invariance algebras of the equations under consideration;

(2) the list of invariant equations contains only inequivalent ones, namely, no equation can be transformed into another one from the list by a transformation from the equivalence group $\mathcal{E}$.

II. PRELIMINARY GROUP CLASSIFICATION OF Eq. (1.1)

We look for the infinitesimal operator of symmetry group of equation (1.1) in the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{2.1}$$

where τ, ξ, η are smooth functions defined on an open domain Ω of the space $V=\mathbb{R}^2\times\mathbb{R}^1$ of independent $\mathbb{R}^2=\langle t,x\rangle$ and dependent $\mathbb{R}^1=\langle u\rangle=u(t,x)$ variables.

Operator (2.1) generates one-parameter invariance group of (1.1) iff its coefficients τ, ξ, η, ϵ satisfy the equation (Lie's invariance criterion)

$$\varphi^{tt} - \varphi^{xx} - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{u_x}|_{(1.1)} = 0, \tag{2.2}$$

where

$$\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),$$

$$\varphi^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),$$

$$\varphi^{tt} = D_t(\varphi^t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi),$$

$$\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),$$

and D_t, D_x are operators of total differentiation with respect to the variables t, x. As customary, by writing (1.1) we mean that one needs to replace u_{tt} and its differential consequences with the expression $u_{xx}+F$ and its differential consequences in (2.2).

After a simple transformations algebra we reduce (2.2) to the form

$$\begin{aligned} &(1) \quad \xi_u = \tau_u = \eta_{uu} = 0,\\ &(2) \quad \tau_t - \xi_x = 0, \quad \xi_t - \tau_x = 0,\\ &(3) \quad 2\eta_{tu} + \tau_x F_{u_x} = 0,\\ &(4) \quad \eta_{tt} - \eta_{xx} - 2u_x \eta_{xu} + [\eta_u - 2\tau_t]\\ &\qquad \times F - \tau F_t - \xi F_x - \eta F_u - [\eta_x + u_x(\eta_u - \xi_x)]F_{u_x} = 0. \end{aligned} \tag{2.3}$$

The first two groups of PDEs from (2.3) are to be used to derive the form of the most general infinitesimal operator admitted by (1.1). The remaining PDEs are classifying equations.

We prove in Ref. 35 that the following assertion holds.

Theorem 1: *Provided $F_{u_x u_x} \neq 0$, the maximal invariance group of equation (1.1) is generated by the following infinitesimal operator:*

$$Q = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + [h(x)u + r(t,x)]\partial_u, \tag{2.4}$$

where λ, λ_1, λ_2 are real constants and $h=h(x)$, $r=r(t,x)$, $F=F(t,x,u,u_x)$ are functions obeying the constraint

$$r_{tt} - r_{xx} - \frac{d^2h}{dx^2}u - 2\frac{dh}{dx}u_x + (h-2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x - (hu + r)F_u - \left(r_x + \frac{dh}{dx}u + (h-\lambda)u_x\right)F_{u_x} = 0. \tag{2.5}$$

If $F=g(t,x,u)u_x + f(t,x,u)$, $g_u \neq 0$, then the maximal invariance group of equation (1.1) is generated by infinitesimal operator (2.4), where λ, λ_1, λ_2 are real constants h, r, g, f are functions satisfying system of two equations

$$\begin{aligned} -2h' - \lambda g &= (\lambda t + \lambda_1)g_t + (\lambda x + \lambda_2)g_x + (hu + r)g_u,\\ -h''u + r_{tt} - r_{xx} + (h-2\lambda)f &= (\lambda t + \lambda_1)f_t + (\lambda x + \lambda_2)f_x + (hu + r)f_u + g(h'u + r_x). \end{aligned} \tag{2.6}$$

Next, if $F=g(t,x)u_x + f(t,x,u)$, $q \not\equiv 0$, $f_{uu} \neq 0$, then the infinitesimal operator of the invariance group of equation (1.1) reads as

$$Q=\tau(t,x)\partial_t+\xi(t,x)\partial_x+(h(t,x)u+r(t,x))\partial_u,$$

where τ, ξ, h, r, g, f *are functions satisfying system of PDEs*

$$\tau_t-\xi_x=0,\quad \xi_t-\tau_x=0,$$

$$2h_t=-\tau_x g,\quad 2h_x=-\tau_t g-\tau g_t-\xi g_x,$$

$$(h_{tt}-h_{xx})u+r_{tt}-r_{xx}+f(h-2\tau_t)-\tau f_t-\xi f_x-(hu+r)f_u-(h_x u+r_x)g=0.$$

Finally, if $F=f(t,x,u)$, $f_{uu}\neq 0$, *then the maximal invariance group of equation (1.1) is generated by infinitesimal operator*

$$Q=[\varphi(\theta)+\psi(\sigma)]\partial_t-[\varphi(\theta)-\psi(\sigma)]\partial_x+[ku+r(t,x)]\partial_u,$$

where $k\in\mathbb{R}$, $\theta=t-x$, $\sigma=t+x$ *and functions* φ, ψ, r, f *and constant k satisfy the following equation:*

$$r_{tt}-r_{xx}+[k-2\varphi'-2\psi']f-(\varphi+\psi)f_t+(\varphi-\psi)f_x-(ku+r)f_u=0,\quad \varphi'=\frac{\mathrm{d}\varphi}{\mathrm{d}\theta},\quad \psi'=\frac{\mathrm{d}\psi}{\mathrm{d}\theta}.$$

By virtue of the above theorem the problem of group classification of equation (1.1) reduces to the one of classifying equations of more specific forms,

$$u_{tt}=u_{xx}+F(t,x,u,u_x),\quad F_{u_xu_x}\neq 0,\tag{2.7}$$

$$u_{tt}=u_{xx}+g(t,x,u)u_x+f(t,x,u),\quad g_u\neq 0,\tag{2.8}$$

$$u_{tx}=g(t,x)u_x+f(t,x,u),\quad g_x\neq 0,\quad f_{uu}\neq 0,\tag{2.9}$$

$$u_{tx}=f(t,x,u),\quad f_{uu}\neq 0.\tag{2.10}$$

Note that condition $g_x\neq 0$ is essential, since otherwise (2.9) is locally equivalent (2.10).

Summing up, we conclude that the problem of group classification of (1.1) reduces to classifying more specific classes of PDEs (2.7)–(2.10).

First, we consider equations (2.8)–(2.10).

III. GROUP CLASSIFICATION OF EQ. (2.8)

According to Theorem 1 invariance group of equation (2.8) is generated by infinitesimal operator (2.4). And what is more, the real constants λ, λ_1, λ_2 and functions h, r, g, f satisfy equations (2.6). System (2.6) is to be used to specify both the form of functions f, g from (2.8) and functions h, r and constants λ, λ_1, λ_2 in (2.4). It is called the determining (sometimes classifying) equations.

Efficiency of the Lie method for calculation of maximal invariance group of PDE is essentially based on the fact that routinely system of determining equations is over-determined. This is clearly not the case, since we have only one equation for four (!) arbitrary functions and three of the latter depend on two variables. By this very reason direct application of Lie approach in the Ovsyannikov's spirit is no longer efficient when we attempt classifying PDEs with arbitrary functions of several variables.

Compute the equivalence group E of equation (2.8). This group is generated by invertible transformations of the space V preserving the differential structure of equation (2.8) (see, e.g., Ref. 14). Saying it another way, group transformation from $\mathcal{E}$

$$\bar{t} = \alpha(t,x,u), \quad \bar{x} = \beta(t,x,u), \quad v = U(t,x,u), \quad \frac{D(\bar{t},\bar{x},v)}{D(t,x,u)} \neq 0,$$

should reduce (2.8) to equation of the same form

$$v_{\bar{t}\bar{t}} = v_{\bar{x}\bar{x}} + \tilde{g}(\bar{t},\bar{x},v)v_{\bar{x}} + \tilde{f}(\bar{t},\bar{x},v), \quad \tilde{g}_v \neq 0$$

with possibly different $\tilde{f}$, $\tilde{g}$.

As proved by Ovsyannikov,[14] it is possible to modify the Lie's infinitesimal approach to calculate equivalence group in essentially same way as invariance group. We omit the simple intermediate calculations and present the final result.

Assertion 1: The maximal equivalence group $\mathcal{E}$ of Eq. (2.8) reads as

$$\bar{t} = kt + k_1, \quad \bar{x} = \epsilon kx + k_2, \quad v = X(x)u + Y(t,x), \tag{3.1}$$

where $k \neq 0$, $X \neq 0$, $\epsilon = \pm 1, k, k_1, k_2 \in \mathbb{R}$, and X, Y are arbitrary smooth functions.

This completes the first step of the algorithm.

A. Preliminary group classification of Eq. (2.8)

First, we derive inequivalent classes of equations of the form (2.8) admitting one-parameter invariance groups.

Lemma 1: There exist transformations (3.1) that reduce operator (2.4) to one of the six possible forms,

$$Q = m(t\partial_t + x\partial_x), \quad m \neq 0, \quad Q = \partial_t + \beta\partial_x, \quad \beta \geqslant 0,$$

$$Q = \partial_t + \sigma(x)u\partial_u, \quad \sigma \neq 0, \quad Q = \partial_x, \tag{3.2}$$

$$Q = \sigma(x)u\partial_u, \quad \sigma \neq 0, Q = \theta(t,x)\partial_u, \quad \theta \neq 0.$$

Proof: Change of variables (3.1) reduces operator (2.4) to become

$$\tilde{Q} = k(\lambda t + \lambda_1)\partial_{\bar{t}} + \epsilon k(\lambda x + \lambda_2)\partial_{\bar{x}} + [Y_t(\lambda t + \lambda_1) + (\lambda x + \lambda_2)(X'u + Y_x) + X(hu + r)]\partial_v. \tag{3.3}$$

If $\lambda \neq 0$ in (2.4), then setting $k_1 = \lambda^{-1}\lambda_1 k$, $k_2 = \epsilon\lambda^{-1}\lambda_2 k$, and taking as X, Y $(X \neq 0)$ integrals of system of PDEs,

$$X'(\lambda x + \lambda_2) + Xh = 0,$$

$$Y_t(\lambda t + \lambda_1) + Y_x(\lambda x + \lambda_2) + Xr = 0,$$

we reduce (3.3) to the form

$$\tilde{Q} = \lambda(\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}).$$

Provided $\lambda = 0$ and $\lambda_1 \neq 0$, we similarly obtain

$$\tilde{Q} = \partial_{\bar{t}} + \beta\partial_{\bar{x}}, \quad \beta \geqslant 0, \quad Q = \partial_{\bar{t}} + \sigma(\bar{x})v\,\partial_v, \quad \sigma \neq 0.$$

Next, if $\lambda = \lambda_1 = 0$, $\lambda_2 \neq 0$ in (2.4), then setting $k = \epsilon\lambda_2^{-1}$, and taking as X, Y $(X \neq 0)$ integrals of equations

$$\lambda_2 X' + hX = 0, \quad Y_x + rX = 0,$$

we reduce operator (3.3) to become $\tilde{Q} = \partial_{\bar{x}}$.

Finally, the case $\lambda=\lambda_1=\lambda_2=0$, gives rise to operators $\tilde{Q}=\sigma(\overline{x})v\partial_v$, $\tilde{Q}=\theta(\overline{t},\overline{x})\partial_v$.

Rewriting the above operators in the initial variables t, x completes the proof.

Theorem 2: *There are exactly five inequivalent equations of the form (2.8) that admit one-parameter transformation groups. Below we list these equations together with one-dimensional Lie algebras generating their invariance groups (note that we do not present the full form of invariant PDEs we just give the functions f and g),*

$$A_1^1=\langle t\partial_t+x\partial_x\rangle,\quad g=x^{-1}\tilde{g}(\psi,u),$$

$$f=x^{-2}\tilde{f}(\psi,u),\quad \psi=tx^{-1},\quad \tilde{g}_u\neq 0,$$

$$A_1^2=\langle \partial_t+\beta\partial_x\rangle,\quad g=\tilde{g}(\eta,u),\quad f=\tilde{f}(\eta,u),$$

$$\eta=x-\beta t,\quad \beta\geqslant 0,\quad \tilde{g}_u\neq 0,$$

$$A_1^3=\langle \partial_t+\sigma(x)u\partial_u\rangle,\quad g=-2\sigma'\sigma^{-1}\ln|u|+\tilde{g}(\rho,x),$$

$$f=(\sigma'\sigma^{-1})^2u\ln^2|u|-\sigma'\sigma^{-1}\tilde{g}(\rho,x)u\ln|u|-\sigma^{-1}\sigma''u\ln|u|+u\tilde{f}(\rho,x),$$

$$\rho=u\exp(-t\sigma),\quad \sigma\neq 0,$$

$$A_1^4=\langle\partial_x\rangle,\quad g=\tilde{g}(t,u),\quad f=\tilde{f}(t,u),\quad \tilde{g}_u\neq 0,$$

$$A_1^5=\langle\sigma(x)u\partial_u\rangle,\quad g=-2\sigma'\sigma^{-1}\ln|u|+\tilde{g}(t,x),\quad f=(\sigma'\sigma^{-1})^2u\ln^2|u|$$
$$-(\sigma^{-1}\sigma''+\sigma^{-1}\sigma'\tilde{g}(t,x))u\ln|u|+u\tilde{f}(t,x),\quad \sigma'\neq 0.$$

Proof: If Eq. (2.8) admits a one-parameter invariance group, then it is generated by operator of the form (2.4). According to Lemma 1, the latter is equivalent to one of the six operators (3.2). That is why, all we need to do is integrate six systems of determining equations corresponding to operators (2.6). For the first five operators solutions of determining equations are easily shown to have the form given in the statement of the theorem.

We consider in more detail the operator $Q=\theta(t,u)\partial_u$. Determining Eqs. (2.6) for this operator reduce to the form

$$\theta_{tt}-\theta_{xx}=\theta f_u+\theta_x g,\quad \theta g_u=0,$$

whence we get $g_u=0$. Consequently, the system of determining equations is incompatible and the corresponding invariant equation fails to exist.

Nonequivalence of the invariant equations follows from nonequivalence of the corresponding symmetry operators.

The theorem is proved.

Note that in the sequel we give the formulations of theorems omitting routine proofs. The detailed proofs of the most of the statements presented in this paper can be found in Ref. 35.

It is a common knowledge that there exist two inequivalent two-dimensional solvable Lie algebras[36–40]

$$A_{2.1}=\langle e_1,e_2\rangle,\quad [e_1,e_2]=0,$$

$$A_{2.2} = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_2.$$

To construct all possible realizations of the above algebras we take as the first basis element one of the realizations of one-dimensional invariance algebras listed in Lemma 1. The second operator is looked for in the generic form (2.4).

Algebra $A_{2.1}$: Let operator e_1 be of the form $\partial_t + x\partial_x$ and operator e_2 read as (2.4). Then it follows from the relation $[e_1, e_2]=0$ that $\lambda_1 = \lambda_2 = xh' = 0$, $tr_t + xr_x = 0$. Consequently, we can choose basis elements of the algebra in question in the form $\langle t\partial_t + x\partial_x, (mu + r(\psi))\partial_u \rangle$, where $m \in \mathbb{R}$, $\psi = tx^{-1}$. Provided $m=0$, the operator e_2 becomes $r(\psi)\partial_u$. It is straightforward to verify that this realization does not satisfy the determining equations. Hence, $m \neq 0$. Making the change of variables

$$\overline{t} = t, \quad \overline{x} = x, \quad v = u + m^{-1} r(\psi)$$

reduces the basis operators in question to the form $\overline{t}\partial_{\overline{t}} + \overline{x}\partial_{\overline{x}}$, $mv\partial_v$. That is why we can restrict our considerations to the realization $\langle t\partial_t + x\partial_x, u\partial_u \rangle$.

The second determining equation from (2.6) takes the form $ug_u = 0$. Hence it follows that the realization under consideration does not satisfy the determining equations. Consequently, the realization A_1^1 cannot be extended to a realization of the two-dimensional algebra $A_{2.1}$.

Algebra $A_{2.2}$: If operator e_1 is of the form $t\partial_t + x\partial_x$, then it follows from $[e_1, e_2] = e_2$ that $\lambda = \lambda_1 = \lambda_2 = 0$, $xh' = h$, $tr_t + xr_x = r$.

Next, if e_2 reads as $t\partial_t + x\partial_x$, then we get from $[e_1, e_2] = e_2$ the erroneous equality $1=0$.

That is why, the only possible case is when $e_2 = (mxu + xr(\psi))\partial_u$, $m \neq 0$, $\psi = tx^{-1}$, which gives rise to the following realization of the algebra $A_{2.2}$: $\langle t\partial_t + x\partial_x, xu\partial_u \rangle$. This is indeed invariance algebra of an equation from the class (2.8) and the functions f and g read as $g = -2x^{-1}\ln|u| + x^{-1}\tilde{g}(\psi)$, $f = x^{-2}u\ln^2|u| - x^{-2}\tilde{g}(\psi)u\ln|u| + x^{-2}u\tilde{f}(\psi)$, $\psi = tx^{-1}$.

Analysis of the remaining realizations of one-dimensional Lie algebras yields 10 inequivalent $A_{2.1}$- and $A_{2.2}$-invariant equations (see the assertions below). What is more, the obtained (two-dimensional) algebras are maximal symmetry algebras of the corresponding equations.

Theorem 3: *There are, at most, four inequivalent $A_{2.1}$ -invariant nonlinear equations (2.8). Below we list the realizations of $A_{2.1}$ and the corresponding expressions for f and g.*

$$(1) \quad \langle \partial_t, \sigma(x)u\partial_u, \rangle, \quad g = -2\sigma'\sigma^{-1}\ln|u|,$$

$$f = (\sigma'\sigma^{-1})^2 u \ln^2|u| - \sigma^{-1}\sigma' u \ln|u| + u\tilde{f}(x), \quad \sigma' \neq 0,$$

$$(2) \quad \langle \partial_t, \partial_x \rangle, \quad g = \tilde{g}(u), \quad f = \tilde{f}(u), \quad \tilde{g}_u \neq 0,$$

$$(3) \quad \langle \partial_x, \partial_t + u\partial_u \rangle, \quad g = \tilde{g}(\omega), \quad f = \exp(t)\tilde{f}(\omega), \quad \omega = \exp(-t), \quad \tilde{g}_\omega \neq 0,$$

$$(4) \quad \left\langle \sigma(x)u\partial_u, \partial_t - \tfrac{1}{2}k\sigma(x)\psi(x)u\partial_u \right\rangle, \quad g = -2\sigma'\sigma^{-1}\ln|u| + kt + \tilde{g}(x),$$

$$f = (\sigma'\sigma^{-1})^2 u \ln^2|u| - \sigma^{-1}\sigma'' u \ln|u| - \sigma^{-1}\sigma'(kt + \tilde{g}(x))u\ln|u| + u\left[\tfrac{1}{2}k\sigma'\sigma^{-1}t + \tfrac{1}{4}k^2t^2 + \tfrac{1}{2}k\tilde{g}(x) + \tilde{f}(x)\right],$$

$$k \neq 0, \quad \sigma' \neq 0, \quad \psi = \int \sigma^{-1}\, dx.$$

Theorem 4: *There exist, at most, six inequivalent $A_{2.2}$-invariant nonlinear equations (2.8). Below we list the realizations of $A_{2.1}$ and the corresponding expressions for f and g.*

$$(1) \quad \langle t\partial_t + x\partial_x, k^{-1}|x|^k u\partial_u\rangle, \quad g = x^{-1}(-2k\ln|u| + \tilde{g}(\psi)),$$

$$f = x^{-2}u(-k^2\ln^2|u| + k\tilde{g}(\psi)\ln|u| + k(k-1)\ln|u| + \tilde{f}(\psi)),$$

$$k \neq 0, \quad \psi = tx^{-1},$$

$$(2) \quad \langle \partial_t + \beta\partial_x, \exp(\beta^{-1}x)u\partial_u\rangle, \quad g = -2\beta^{-1}\ln|u| + \tilde{g}(\eta),$$

$$f = \beta^{-2}u\ln^2|u| - (\beta^{-2} + \beta^{-1}\tilde{g}(\eta))u\ln|u| + u\tilde{f}(\eta),$$

$$\beta > 0, \quad \eta = x - \beta t,$$

$$(3) \quad \langle -t\partial_t - x\partial_x, \partial_t + \beta\partial_x\rangle, \quad g = \eta^{-1}\tilde{g}(u), \quad f = \eta^{-2}\tilde{f}(u), \quad \beta \geqslant 0,$$

$$\eta = x - \beta t, \quad \tilde{g}_u \neq 0,$$

$$(4) \quad \langle -t\partial_t - x\partial_x, \partial_t + mx^{-1}u\partial_u\rangle, \quad g = x^{-1}(2m\psi + \tilde{g}(\omega)),$$

$$f = x^{-1}[-2m\psi u - 2m\psi - 2 - \tilde{g}(\omega) + \exp(m\psi)\tilde{g}(\omega)],$$

$$m > 0, \quad \omega = u\exp(-m\psi), \quad \psi = tx^{-1}, \quad \tilde{g}_\omega \neq 0,$$

$$(5) \quad \langle \partial_x, e^x u\partial_u\rangle, \quad g = -2\ln|u| + \tilde{g}(t), \quad f = u\ln^2|u| - u\ln|u|(1 + \tilde{g}(t)) + u\tilde{f}(t),$$

$$(6) \quad \langle -t\partial_t - x\partial_x, \partial_x\rangle, \quad g = t^{-1}\tilde{g}(u), \quad f = t^{-2}\tilde{f}(u), \quad \tilde{g}_u \neq 0.$$

B. Completing group classification of (2.8)

As the invariant equations obtained in the previous subsection contain arbitrary functions of, at most, one variable, we can now apply the standard Lie–Ovsyannikov routine to complete the group classification of (2.8). We give the computation details for the case of the first $A_{2.1}$-invariant equation, the remaining cases are handled in a similar way.

Setting $g=-2\sigma'\sigma^{-1}\ln|u|$, $f=(\sigma'\sigma^{-1})u\ln^2|u|-\sigma^{-1}\sigma''u\ln|u|+u\tilde{f}(x)$, $\sigma=\sigma(x)$, $\sigma'\neq 0$ we rewrite the first determining equation to become

$$-2h' + 2\lambda\sigma'\sigma^{-1}\ln|u| = -2(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'_x\ln|u| - 2h\sigma'\sigma^{-1} - 2r\sigma'\sigma^{-1}u^{-1}.$$

As $h=f(x)$, $\sigma=\sigma(x)$, $r=r(t,x)$, λ, $\lambda_2\in\mathbb{R}$, the above relation is equivalent to the following ones:

$$h' = \sigma'\sigma^{-1}h, \quad r = 0, \quad \lambda\sigma'\sigma^{-1} = -(\lambda x + \lambda_2)(\sigma'\sigma^{-1})'.$$

If σ is an arbitrary function, then $\lambda=\lambda_2=r=0$, $h=C\sigma$, $C\in\mathbb{R}$ and we get $\langle\partial_t, \sigma(x)u\partial_u\rangle$ as the maximal symmetry algebra. Hence, extension of the symmetry algebra is only possible when the function $\psi=\sigma'\sigma^{-1}$ is a (nonvanishing identically) solution of the equation

$$(\alpha x + \beta)\psi' + \alpha\psi = 0, \quad \alpha,\beta \in R, \quad |\alpha| + |\beta| \neq 0.$$

If $\alpha\neq 0$, then at the expense of displacements by x we can get $\beta=0$, so that $\psi=mx^{-1}$, $m\neq 0$. Integrating the remaining determining equations yields

$$g=-2mx^{-1}\ln|u|, \quad f=mx^{-2}[mu\ln^2|u|-(m-1)u\ln|u|+nu], \quad m\neq 0, \quad m,n\in\mathbb{R}.$$

The maximal invariance algebra of the obtained equation is the three-dimensional Lie algebra $\langle\partial_t,|x|^m u\partial_u, t\partial_t+x\partial_x\rangle$ isomorphic to $A_{3.7}$.

Next, if $\alpha=0$, then $\beta\neq 0$ and $\psi=m$, $m\neq 0$. If this is the case, we have

$$g=\ln|u|, \quad f=\tfrac{1}{4}u\ln^2|u|-\tfrac{1}{4}u\ln|u|+nu, \quad n\in\mathbb{R}.$$

The maximal invariance algebra of the above equation reads as

$$\left\langle\partial_t,\partial_x,\exp\left(-\tfrac{1}{2}x\right)u\partial_u\right\rangle.$$

It is isomorphic to $A_{3.2}$.

Similarly, we prove that the list of inequivalent equations of the form (2.8) admitting three-dimensional symmetry algebras is exhausted by the equations given below. Note that the presented algebras are maximal. This means, in particular, that maximal symmetry algebra of Eq. (2.8) is, at most, three dimensional.

$A_{3.2}$-invariant equations,

$$(1) \quad u_{tt}=u_{xx}+u_x\ln|u|+\tfrac{1}{4}u\ln^2|u|-\tfrac{1}{4}u\ln|u|+nu(n\in\mathbb{R}),\left\langle\partial_t,\partial_x,\exp\left(-\tfrac{1}{2}x\right)u\partial_u\right\rangle,$$

$$(2) \quad u_{tt}=u_{xx}+m[\ln|u|-t]u_x+(m^2/4)u[(\ln|u|-t)(\ln|u|-t-1)]+nu(m>0,n\in\mathbb{R}),$$
$$\left\langle\partial_x,\partial_t+u\partial_u,\exp\left(-\tfrac{1}{2}mx\right)u\partial_u\right\rangle.$$

$A_{3.4}$-invariant equations,

$$(1) \quad u_{tt}=u_{xx}+x^{-1}[2\ln|u|+mx^{-1}t+n]u_x+x^{-2}u\ln|u|+(mx^{-1}t+n-2)x^{-2}u\ln|u|+\tfrac{1}{4}m^2x^{-4}t^2u$$
$$+\tfrac{1}{2}m(n-3)x^{-3}tu+px^{-2}u(m\neq 0,n,p\in\mathbb{R}), \quad \langle t\partial_t+x\partial_x,x^{-1}u\partial_u,\partial_t-(m/2)x^{-1}\ln|x|u\partial_u\rangle.$$

$A_{3.5}$-invariant equations,

$$(1) \quad u_{tt}=u_{xx}+|u|^m u_x+n|u|^{1+2m} \quad (m\neq 0,n\in\mathbb{R}), \quad \langle\partial_t,\partial_x,t\partial_t+x\partial_x-m^{-1}u\partial_u\rangle,$$

$$(2) \quad u_{tt}=u_{xx}+e^u u_x+ne^{2u} \quad (n\in\mathbb{R}), \quad \langle\partial_t,\partial_x,t\partial_t+x\partial_x-\partial_u\rangle,$$

$$(3) \quad u_{tt}=u_{xx}-x^{-1}[2\ln|u|-mx^{-1}t-n]u_x+x^{-2}u\ln^2|u|-x^{-2}(mx^{-1}t+n)u\ln|u|+ux^{-2}[(m/4)x^{-2}t^2$$
$$+(m/2)(n-1)x^{-1}t+p](m,n,p\in\mathbb{R}), \quad \langle t\partial_t+x\partial_x,xu\partial_u,\partial_t+(m/4)x^{-1}u\partial_u\rangle.$$

$A_{3.7}$-invariant equations,

$$(1) \quad u_{tt}=u_{xx}-2mx^{-1}u_x\ln|u|+mx^{-2}[mu\ln^2|u|-(m-1)u\ln|u|+nu]$$
$$\times(m\neq 0,1;n\in\mathbb{R}), \quad \langle\partial_t,|x|^m u\partial_u,t\partial_t+x\partial_x\rangle,$$

$$(2) \quad u_{tt}=u_{xx}-x^{-1}[2k+\ln|u|-mx^{-1}t-n]u_x+k^2x^{-2}u\ln^2|u|-kx^{-2}[mtx^{-1}+k+n-1]u\ln|u|+\tfrac{1}{2}m(k$$
$$-2+n)tx^{-3}u+\tfrac{1}{4}m^2t^2x^{-4}u+px^{-2}u \quad (|k|\neq 0,1;m\neq 0,n,p\in\mathbb{R}),$$
$$\langle t\partial_t+x\partial_x,|x|^k u\partial_u,\partial_t+[m/2(1+k)]x^{-1}u\partial_u\rangle.$$

This completes the group classification of nonlinear equations (2.8).

IV. GROUP CLASSIFICATION OF Eq. (2.9)

Omitting calculation details we present below the determining equations for symmetry operators admitted by Eq. (2.9).

Assertion 2: The maximal invariance group of PDE (2.9) is generated by the infinitesimal operator,

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + [h(t)u + r(t,x)]\partial_u, \tag{4.1}$$

where τ, ξ, h, r, f, g are smooth functions satisfying the conditions

$$\begin{gathered} r_{tx} + f[h - \tau_t - \xi_x] = g r_x + \tau f_t + \xi f_x + [hu + r] f_u, \\ h_t = \tau_t g + \tau g_t + \xi g_x. \end{gathered} \tag{4.2}$$

Assertion 3: The equivalence group $\mathcal{E}$ of (2.9) is formed by the following transformations of the space V:

$$(1) \quad \bar{t} = T(t), \quad \bar{x} = X(x), \quad v = U(t)u + Y(t,x), \quad t'X'U \neq 0,$$

$$(2) \quad \bar{t} = T(x), \quad \bar{x} = X(t), \quad v = \Psi(x)\Phi(t,x)u + Y(t,x), \quad t'X'\Psi \neq 0, \tag{4.3}$$

$$\Phi(t,x) = \exp\left(-\int g(t,x)\mathrm{d}t\right), g_x \neq 0.$$

As the direct verification shows, given arbitrary functions g and f, it follows from (4.2) that $\tau = h = \xi = r = 0$. So that in the generic case the maximal invariance group of (2.9) is the trivial group of identical transformations.

We begin classification of (2.9) by constructing equations that admit one-dimensional symmetry algebras. The following assertions hold.

Lemma 2: There exist transformations (4.3) reducing operator (4.1) to one of the seven canonical forms given below

$$\begin{gathered} Q = t\partial_t + x\partial_x, \quad Q = \partial_t, \quad Q = \partial_x + tu\partial_u, \\ Q = \partial_x + \epsilon u\partial_u, \quad \epsilon = 0,1, \quad Q = tu\partial_u, \\ Q = u\partial_u, \quad Q = r(t,x)\partial_u, \quad r \neq 0. \end{gathered} \tag{4.4}$$

Theorem 5: *There exist, at most, three inequivalent nonlinear equations (2.9) that admit one-dimensional invariance algebras. The form of functions f, g and the corresponding symmetry algebras are given below,*

$$A_1^1 = \langle t\partial_t + x\partial_x\rangle, \quad g = t^{-1}\tilde{g}(\omega), \quad f = t^{-2}f(u,\omega), \quad \omega = tx^{-1}, \quad \tilde{g}_\omega \neq 0, \quad f_{uu} \neq 0,$$

$$A_1^2 = \langle \partial_t\rangle, \quad g = \tilde{g}(x), \quad f = \tilde{f}(x,u), \quad \tilde{g}' \neq 0, \quad \tilde{f}_{uu} \neq 0,$$

$$A_1^3 = \langle \partial_x + tu\partial_u\rangle, \quad g = x + \tilde{g}(t), \quad f = e^{tx}\tilde{f}(t,\omega), \quad \omega = e^{-tx}u, \quad \tilde{f}_{\omega\omega} \neq 0.$$

We proceed now to analyzing Eqs. (2.9) admitting two-dimensional symmetry algebras.

Theorem 6: *There exist, at most, three inequivalent nonlinear equations (2.9) that admit two-dimensional symmetry algebras, all of them being $A_{2.2}$-invariant equations. The forms of functions f and g and the corresponding realizations of the Lie algebra $A_{2.2}$ read as*

$$A_{2.2}^1 = \langle t\partial_t + x\partial_x, t^2\partial_t + x^2\partial_x + mut\partial_u\rangle \quad (m \in \mathbb{R}),$$

$$g = [mt + (k-m)x]t^{-1}(t-x)^{-1}, \quad k \neq 0,$$

$$f = |t - x|^{m-2}|x|^{-m}\tilde{f}(\omega),$$

$$\omega = u|t - x|^{-m}|x|^{m}, \quad \tilde{f}_{\omega\omega} \neq 0,$$

$$A^2_{2.2} = \langle t\partial_t + x\partial_x, t^2\partial_t + mtu\partial_u\rangle \quad (m \in \mathbb{R}),$$

$$g = t^{-2}[kx + mt], \quad k \neq 0, \quad f = |t|^{m-2}|x|^{-m}\tilde{f}(\omega),$$

$$\omega = |t|^{-m}|x|^{m}u, \quad \tilde{f}_{\omega\omega} \neq 0,$$

$$A^3_{2.2} = \langle t\partial_t + x\partial_x, x^2\partial_x + tu\partial_u\rangle,$$

$$g = (tx)^{-1}(mx - t)(m \in \mathbb{R}), \quad f = x^{-2}\exp(-tx^{-1})\tilde{f}(\omega),$$

$$\omega = u\exp(tx^{-1}), \quad \tilde{f}_{\omega\omega} \neq 0.$$

Note that if the function $\tilde{f}$ is arbitrary, then the invariance algebras given in the statement of Theorem 6 are maximal.

It turns out that the above theorems provide complete group classification of the class of PDEs (2.9). Namely, the following assertion holds true.

Theorem 7: *A nonlinear equation (2.9) having nontrivial symmetry properties is equivalent to one of the equations listed in Theorems 5 and 6.*

V. GROUP CLASSIFICATION OF Eq. (2.10)

As earlier, we present the results of the first step of our group classification algorithm skipping derivation details.

Assertion 4: Invariance group of equation (2.10) is generated by infinitesimal operator

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + (ku + r(t,x))\partial_u, \tag{5.1}$$

where k is a constant τ, ξ, r, f are functions satisfying the relation

$$r_{tx} + [k - \tau' - \xi']f = \tau f_t + \xi f_x + [ku + r]f_u. \tag{5.2}$$

Assertion 5: Equivalence group $\mathcal{E}$ of the class of equations (2.10) is formed by the following transformations:

$$\begin{aligned} &(1) \quad \bar{t} = T(t), \quad \bar{x} = X(x), \quad v = mu + Y(t,x), \\ &(2) \quad \bar{t} = T(x), \quad \bar{x} = X(t), \quad v = mu + Y(t,x), \quad T'X'm \neq 0. \end{aligned} \tag{5.3}$$

Note that given an arbitrary function f, it follows from (5.2) that $\tau = \xi = k = r = 0$, i.e., the group admitted is trivial. To obtain equations with nontrivial symmetry we need to specify properly the function f. To this end we perform classification of equations under study admitting one-dimensional invariance algebras. The following assertions give exhaustive classification of those.

Lemma 3: There exist transformations from the group $\mathcal{E}$ (5.3) that reduce (5.1) to one of the four canonical forms,

$$Q = \partial_t + \partial_x + \epsilon u\partial_u \quad (\epsilon = 0,1),$$

TABLE I. Invariant equations (2.10).

Number	Function f	Symmetry operators	Invariance algebra type
1	$e^t\tilde{f}(\omega)$, $\omega = ue^{-t}$, $\tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u$, ∂_x	$A_{2.1}$
2	$e^{t+x}\tilde{f}(\omega)$, $\omega = ue^{-t-x}$, $\tilde{f}_{\omega\omega} \neq 0$	$\partial_t + u\partial_u$, $\partial_x + u\partial_u$	$A_{2.1}$
3	$(t-x)^{-3}\tilde{f}(\omega)$, $\omega = (t-x)u$, $\tilde{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x + u\partial_u$, $\partial_t + \partial_x$	$A_{2.2}$
4	$x^{-1}\tilde{f}(\omega)$, $\omega = x^{-1}u$, $\tilde{f}_{\omega\omega} \neq 0$	$-t\partial_t - x\partial_x - u\partial_u$, ∂_t	$A_{2.2}$
5	$(t-x)^{-2}\tilde{f}(u)$, $\tilde{f}_{uu} \neq 0$	$\partial_t + \partial_x$, $t\partial_t + x\partial_x$, $t^2\partial_t + x^2\partial_x$	$\mathrm{sl}(2,R)$
6	$\exp(x^{-1}u)$	$-t\partial_t + x\partial_u$, ∂_t, $x\partial_x + u\partial_u$	$A_{2.2} \oplus A_1$
7	$\lambda\|x\|^{-m-2}\|u\|^{m+1}$, $\lambda \neq 0$, $m \neq 0, -, 1-2$	∂_t, $t\partial_t - 1/m u\partial_u$, $x\partial_x + m+1/m u\partial_u$	$A_{2.2} \oplus A_1$
8	$\tilde{f}(u)$, $\tilde{f}_{uu} \neq 0$	∂_t, ∂_x, $-t\partial_t - x\partial_x$	$A_{3.6}$
9	$\lambda\|u\|^{n+1}$, $\lambda \neq 0$, $n \neq 0, -1$	$t\partial_t - 1/n u\partial_u$ $x\partial_x - 1/n u\partial_u$ ∂_t, ∂_x	$A_{2.2} \oplus A_{2.2}$

$$Q = \partial_t + \epsilon u\partial_u \quad (\epsilon = 0,1),$$

$$Q = u\partial_u, \quad Q = g(t,x)\partial_u \quad (g \neq 0).$$

Theorem 8: *There exist exactly two nonlinear equations of the form (2.10) admitting one-dimensional invariance algebras. The corresponding expressions for function f and invariance algebras are given below,*

$$A_1^1 = \langle \partial_t + \partial_x + \epsilon u\partial_u \rangle \quad (\epsilon = 0,1), \quad f = e^{\epsilon t}\tilde{f}(\theta,\omega), \quad \theta = t - x, \quad \omega = e^{-\epsilon t}u, \quad \tilde{f}_{\omega\omega} \neq 0,$$

$$A_1^2 = \langle \partial_t + \epsilon u\partial_u \rangle \quad (\epsilon = 0,1), \quad f = e^{\epsilon t}\tilde{f}(x,\omega), \quad \omega = e^{-\epsilon t}u, \quad \tilde{f}_{\omega\omega} \neq 0.$$

Our analysis of Eqs. (2.10) admitting higher dimensional invariance algebras yields the following assertion.

Theorem 9: *The Liouville equation $u_{tx} = \lambda e^u$, $\lambda \neq 0$, has the highest symmetry among equations (2.10). Its maximal invariance algebra is infinite-dimensional and is spanned by the following infinite set of basis operators:*

$$Q = h(t)\partial_t + g(x)\partial_x - (h'(t) + g'(x))\partial_u,$$

where h and g are arbitrary smooth functions. Next, there exist exactly nine inequivalent equations of the form (2.10), whose maximal invariance algebras have dimension higher that one. We give these equations and their invariance algebras in Table I.

Details of the proof can be found in Ref. 35.

VI. GROUP CLASSIFICATION OF Eq. (2.7)

The first step of the algorithm of group classification of (2.7),

$$u_{tt} = u_{xx} + F(t,x,u,u_x), \quad F_{u_x u_x} \neq 0,$$

has been partially performed in Sec. II. It follows from Theorem 1 that the invariance group of Eq. (2.7) is generated by infinitesimal operator (2.4). What is more, the real constants λ, λ_1, λ_2 and real-valued functions $h=h(x)$, $r=r(t,x)$, $F=F(t,x,u,u_x)$ obey the relation (2.5). The equivalence group of the class of equations (2.7) is formed by transformations (3.1).

With these facts in hand we can utilize results of group classification of Eq. (2.8) in order to classify Eq. (2.7). In particular, using Lemma 1 and Lemma 2 from Ref. 35 it is straightforward to verify that the following assertions hold true.

Theorem 10: *There are, at most, seven inequivalent classes of nonlinear equations (2.7) invariant under the one-dimensional Lie algebras.*

Below we give the full list of the invariant equations and the corresponding invariance algebras,

$$A_1^1 = \langle t\partial_t + x\partial_x \rangle, \quad F = t^{-2}G(\xi,u,\omega), \quad \xi = tx^{-1}, \quad \omega = xu_x,$$

$$A_1^2 = \langle \partial_t + k\partial_x \rangle \quad (k > 0), \quad F = G(\eta,u,u_x), \quad \eta = x - kt,$$

$$A_1^3 = \langle \partial_x \rangle, \quad F = G(t,u,u_x),$$

$$A_1^4 = \langle \partial_t \rangle, \quad F = G(x,u,u_x),$$

$$A_1^5 = \langle \partial_t + f(x)u\partial_u \rangle \quad (f \neq 0),$$

$$F = -tf''u + t^2(f')^2u - 2tf'u_x + e^{tf}G(x,v,\omega),$$

$$v = e^{-tf}u, \quad \omega = u^{-1}u_x - f'f^{-1}\ln|u|,$$

$$A_1^6 = \langle f(x)u\partial_u \rangle \quad (f \neq 0), \quad F = -f^{-1}f''u\ln|u| - 2f^{-1}f'u_x\ln|u| + f^{-2}(f')^2u\ln^2|u| + uG(t,x,\omega),$$

$$\omega = u^{-1}u_x - f'f^{-1}\ln|u|,$$

$$A_1^7 = \langle f(t,x)\partial_u \rangle \quad (f \neq 0), \quad F = f^{-1}(f_{tt} - f_{xx})u + G(t,x,\omega),$$

$$\omega = u_x - f^{-1}f_xu.$$

Note that if the functions F and G are arbitrary, then the presented algebras are maximal (in Lie's sense) symmetry algebras of the respective equations.

Theorem 11: *An equation of the form (2.7) cannot admit Lie algebra which has a subalgebra having nontrivial Levi factor.*

With account of the above facts we conclude that nonlinear equations (2.7) admit a symmetry algebra of the dimension higher than one only if the latter is a solvable real Lie algebra. That is why, we turn to classifying equations (2.7) whose invariance algebras are two-dimensional solvable Lie algebras.

Below we present the list of invariant equations and the corresponding realizations of the two-dimensional invariance algebras.

(I) $A_{2.1}$-invariant equations,

$$A_{2.1}^1 = \langle t\partial_t + x\partial_x, u\partial_u \rangle, \quad F = x^{-2}uG(\xi,\omega),$$

$$\xi = tx^{-1}, \quad \omega = u^{-1}xu_x,$$

$$A_{2.1}^2 = \langle t\partial_t + x\partial_x, \sigma(\xi)\partial_u \rangle \quad (\sigma \neq 0, \xi = tx^{-1}),$$

$$F = x^{-2}[\sigma^{-1}((1-\xi^2)\sigma'' - 2\xi\sigma')u + G(\xi,\omega)],$$

$$\omega = \xi\sigma' u + \sigma x u_x,$$

$$A_{2.1}^3 = \langle \partial_t + k\partial_x, u\partial_u \rangle \quad (k > 0), \quad F = uG(\eta,\omega),$$

$$\eta = x - kt, \quad \omega = u^{-1}u_x,$$

$$A_{2.1}^4 = \langle \partial_t + k\partial_x, \varphi(\eta)\partial_u \rangle \quad (k > 0, \eta = x - kt, \varphi \neq 0),$$

$$F = (k^2-1)\varphi''\varphi^{-1}u + G(\eta,\omega), \quad \omega = \varphi u_x - \varphi' u,$$

$$A_{2.1}^5 = \langle \partial_t + k\partial_x, \partial_x + u\partial_u \rangle \quad (k > 0),$$

$$F = e^{\eta}G(\omega,v), \quad \eta = x - kt, \quad \omega = ue^{-\eta}, \quad v = u^{-1}u_x,$$

$$A_{2.1}^6 = \langle \partial_t, \partial_x \rangle, \quad F = G(u,u_x),$$

$$A_{2.1}^7 = \langle \partial_x, u\partial_u \rangle, \quad F = uG(t,\omega), \quad \omega = u^{-1}u_x,$$

$$A_{2.1}^8 = \langle \partial_x, \varphi(t)\partial_u \rangle \quad (\varphi \neq 0),$$

$$F = \varphi^{-1}\varphi'' u + G(t,u_x),$$

$$A_{2.1}^9 = \langle \partial_t, \partial_u \rangle, \quad F = G(x,u_x),$$

$$A_{2.1}^{10} = \langle \partial_t, f(x)u\partial_u \rangle \quad (f \neq 0),$$

$$F = -u^{-1}u_x^2 + uG(x,\omega),$$

$$\omega = u^{-1}u_x - f'f^{-1}\ln|u|,$$

$$A_{2.1}^{11} = \langle \partial_t + f(x)u\partial_u, g(x)u\partial_u \rangle \quad (\delta = f^{-1}f' - g^{-1}g' \neq 0),$$

$$F = -g^{-1}g''u\ln|u| - 2g^{-1}g'u_x\ln|u| + g^{-2}(g')^2u\ln^2|u| - 2f\delta tu_x + 2f\delta g'g^{-1}tu\ln|u| + f^2\delta^2t^2u + f(g^{-1}g'' - f^{-1}f'')tu + uG(x,\omega),$$

$$\omega = u^{-1}u_x - g'g^{-1}\ln|u| - tf\delta,$$

$$A_{2.1}^{12} = \langle \partial_t + f(x)u\partial_u, e^{tf}\partial_u \rangle \quad (f \neq 0),$$

$$F=[f^2-tf''+t^2(f')^2]u-2tf'u_x+e^{tf}G(x,\omega),$$

$$\omega=e^{-tf}(u_x-tf'u),$$

$$A_{2.1}^{13}=\langle f(x)u\partial_u,g(x)u\partial_u\rangle \quad (\delta=f'g-g'f\neq 0),$$

$$F=-u^{-1}u_x^2-\delta^{-1}\delta'u_x+\delta^{-1}[f''g'-g''f']u\ln|u|+uG(t,x),$$

$$A_{2.1}^{14}=\langle \varphi(t)\partial_u,\psi(t)\partial_u\rangle \quad (\varphi'\psi-\varphi\psi'\neq 0),$$

$$F=\varphi^{-1}\varphi''u+G(t,x,u_x), \quad \varphi''\psi-\varphi\psi''=0.$$

(II) $A_{2.2}$-invariant equations,

$$A_{2.2}^1=\langle t\partial_t+x\partial_x,xu\partial_u\rangle, \quad F=x^{-2}u\ln^2|u|-2x^{-1}u_x\ln|u|+t^{-2}uG(\xi,\omega), \quad \xi=tx^{-1},$$

$$\omega=xu^{-1}u_x-\ln|u|,$$

$$A_{2.2}^2=\langle t\partial_t+x\partial_x,t\varphi(\xi)\partial_u\rangle \quad (\varphi\neq 0,\xi=tx^{-1}),$$

$$F=t^{-2}(1-\xi^2)\varphi^{-1}\xi(2\varphi'+\xi\varphi'')u+t^{-2}G(\xi,\omega),$$

$$\omega=x\varphi u_x+\xi\varphi'u,$$

$$A_{2.2}^3=\langle \partial_t+k\partial_x,\exp(k^{-1}x)u\partial_u\rangle(k>0),$$

$$F=k^{-2}u\ln^2|u|-2k^{-1}u_x\ln|u|-k^{-2}u\ln|u|+uG(\eta,\omega), \quad \eta=x-kt, \quad \omega=u^{-1}u_x-k^{-1}\ln|u|,$$

$$A_{2.2}^4=\langle \partial_t+k\partial_x,e^t\varphi(\eta)\partial_u\rangle \quad (\eta=x-kt,k>0,\varphi\neq 0),$$

$$F=((k^2-1)\varphi''\varphi^{-1}-2k\varphi'\varphi^{-1}+1)u+G(\eta,\omega),$$

$$\omega=\varphi u_x-\varphi'u, \quad \varphi'=\frac{\mathrm{d}\varphi}{\mathrm{d}\eta},$$

$$A_{2.2}^5=\langle -t\partial_t-x\partial_x,\partial_t+k\partial_x\rangle \quad (k>0),$$

$$F=\eta^{-2}G(u,\omega), \quad \eta=x-kt, \quad \omega=u_x\eta,$$

$$A_{2.2}^6=\langle -t\partial_t-x\partial_x+mu\partial_u,\partial_t+k\partial_x\rangle \quad (k>0,m\neq 0),$$

$$F=|\eta|^{-2-m}G(v,\omega), \quad \eta=x-kt,$$

$$\omega=u|\eta|^m, \quad v=u_x|\eta|^{m+1},$$

$$A_{2.2}^7=\langle \partial_x,e^xu\partial_u\rangle, \quad F=u\ln^2|u|-u\ln|u|-2u_x\ln|u|+uG(t,\omega), \quad \omega=u^{-1}u_x-\ln|u|,$$

$$A^8_{2.2} = \langle \partial_x, e^x \varphi(t)\partial_u \rangle \quad (\varphi \neq 0),$$

$$F = (\varphi^{-1}\varphi'' - 1)u + G(t,\omega), \quad \omega = u_x - u,$$

$$A^9_{2.2} = \langle -t\partial_t - x\partial_x, \partial_x \rangle, \quad F = t^{-2}G(u, tu_x),$$

$$A^{10}_{2.2} = \langle -t\partial_t - x\partial_x + ku\partial_u, \partial_x \rangle \quad (k \neq 0),$$

$$F = |t|^{-2-k}G(v,\omega), \quad v = |t|^k u, \quad \omega = |t|^{k+1}u_x,$$

$$A^{11}_{2.2} = \langle \partial_t, e^t \partial_u \rangle, \quad F = u + G(x, u_x),$$

$$A^{12}_{2.2} = \langle -t\partial_t - x\partial_x, \partial_t \rangle, \quad F = x^{-2}G(u,\omega), \quad \omega = xu_x,$$

$$A^{13}_{2.2} = \langle \partial_t + f(x)u\partial_u, e^{(1+f)t}\partial_u \rangle \quad (f \neq 0),$$

$$F = -(tf'' - t^2(f')^2 - (1+f^2))u - 2tf'u_x + e^{tf}G(x,\omega), \quad \omega = e^{-tf}(u_x - f'(t+f^{-1})u),$$

$$A^{14}_{2.2} = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u \rangle \quad (k > 0),$$

$$F = -2ktx^{-3}u + k^2t^2x^{-4}u + 2ktx^{-2}u_x + x^{-2}\exp(ktx^{-1})G(v,\omega), \quad v = \exp(-kx^{-1}t)u,$$

$$\omega = xu^{-1}u_x + \ln|u|,$$

$$A^{15}_{2.2} = \langle k(t\partial_t + x\partial_x), |x|^{k^{-1}}u\partial_u \rangle \quad (k \neq 0,1),$$

$$F = -k^{-2}(1-k)x^{-2}u\ln|u| - 2k^{-1}x^{-1}u_x\ln|u| + k^{-2}x^{-2}u\ln^2|u| + x^{-2}uG(v,\omega),$$

$$v = tx^{-1}, \quad \omega = xu^{-1}u_x - k^{-1}\ln|u|,$$

$$A^{16}_{2.2} = \langle k(t\partial_t + x\partial_x), |t|^{k^{-1}}\varphi(\xi)\partial_u \rangle \quad (k \neq 0,1, \varphi \neq 0, \xi = tx^{-1}), \quad F = [k^{-1}(k^{-1}-1) + 2\xi(k^{-1} - \xi^2)\varphi^{-1}\varphi' + \xi^2(1-\xi)^2\varphi^{-1}\varphi'']t^{-2}u + t^{-2}G(\xi,\omega),$$

$$\omega = x\varphi u_x + \xi\varphi' u.$$

In the above formulas G stands for an arbitrary smooth function. As customary, the prime denotes the derivative of a function of one variable.

A. Group classification of the equation $u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u\ln|u| + uD(t,x)$

Before analyzing Eqs. (2.7) admitting algebras of the dimension higher than two we perform group classification of the equation

$$u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u\ln|u| + uD(t,x). \tag{6.1}$$

Here $A(x), B(x), D(t,x)$ are arbitrary smooth functions. Note that the above class of PDEs contains $A^{13}_{2.1}$-invariant equation. Importantly, class (6.1) contains a major part of equations of the form (2.7), whose maximal symmetry algebras have dimension three or four. This fact is used to simplify group classification of Eqs. (2.7).

The complete account of symmetry properties of PDE (6.1) is given in the following assertions.

Lemma 4: If A, B, and D are arbitrary, then the maximal invariance algebra of PDE (6.1) is the two-dimensional Lie algebra equivalent to $A_{2.1}^{13}$ and (6.1) reduces to $A_{2.1}^{13}$-invariant equation. Next, if the maximal symmetry algebra of an equation of the form (6.1) is three-dimensional (we denote it as A_3), then this equation is equivalent to one of the following ones:

$$\text{(I)}\ A_3 \sim A_{3.1}, \quad A_3 = \langle \partial_t, f(x)u\partial_u, \varphi(x)u\partial_u \rangle,$$

$$A = -\sigma^{-1}\sigma', \quad B = \sigma^{-1}\rho, \quad D = 0, \quad \sigma = f'\varphi - f\varphi' \neq 0,$$

$$\rho = \varphi' f'' - \varphi'' f',$$

$$\text{(II)}\ A_3 \sim A_{3.1}, \quad A_3 = \langle f(x)u\partial_u, \varphi(x)u\partial_u, \partial_t + \psi(x)u\partial_u \rangle,$$

$$A = -\sigma^{-1}\sigma', \quad B = \sigma^{-1}\rho,$$

$$D = t\sigma^{-1}[\sigma'\psi' - \psi\rho - \sigma\psi''],$$

$$\sigma = f'\varphi - \varphi' f \neq 0, \quad \rho = f''\varphi' - \varphi'' f',$$

$$f'\psi - f\psi' \neq 0, \quad \varphi'\psi - \varphi\psi' \neq 0,$$

$$\text{(III)}\ D = x^{-2}G(\xi), \quad \xi = tx^{-1}, G \neq 0,$$

$$(1)\ A_3 \sim A_{3.2},\ A_3 = \langle t\partial_t + x\partial_x, u\partial_u, |x|^{1-n}u\partial_u \rangle,$$

$$A = nx^{-1} \quad (n \neq 1), \quad B = 0,$$

$$(2)\ A_3 \sim A_{3.3}, \quad A_3 = \langle t\partial_t + x\partial_x, u\partial_u, u \ln|x|\partial_u \rangle,$$
$$A = x^{-1}, \quad B = 0,$$

$$(3)\ A_3 \sim A_{3.4}, \quad A_3 = \langle t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, \sqrt{|x|}\ \ln|x|u\partial_u \rangle,$$

$$A = 0, \quad B = \tfrac{1}{4}x^{-2},$$

$$(4)\ A_3 \sim A_{3.9}, \quad A_3 = \left\langle t\partial_t + x\partial_x, \sqrt{|x|}\ \cos\left(\tfrac{1}{2}\beta \ln|x|\right)u\partial_u, \sqrt{|x|}\ \sin\left(\tfrac{1}{2}\beta \ln|x|\right)u\partial_u \right\rangle,$$
$$A = 0, \quad B = mx^{-2},$$

$$m > \tfrac{1}{4}, \quad \beta = \sqrt{4m-1},$$

$$(5)\ A_3 \sim A_{3.7}, \quad A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1+\beta}u\partial_u, (\sqrt{|x|})^{1-\beta}u\partial_u \rangle,$$

$$A = 0, \quad B = mx^{-2}, \quad m < \tfrac{1}{4}, \quad m \neq 0, \quad \beta = \sqrt{1-4m},$$

$$(6)\ A_3 \sim A_{3.8}, \quad A_3 = \langle t\partial_t + x\partial_x, \cos(\sqrt{m}\ln|x|)u\partial_u, \sin(\sqrt{m}\ln|x|)u\partial_u\rangle,$$
$$A = x^{-1}, \quad B = mx^{-2}, \quad m > 0,$$

$$(7)\ A_3 \sim A_{3.6}, \quad A_3 = \langle t\partial_t + x\partial_x, |x|^{\sqrt{|m|}}u\partial_u, |x|^{-\sqrt{|m|}}u\partial_u\rangle,$$

$$A = x^{-1}, \quad B = mx^{-2}, \quad m < 0,$$

$$(8)\ A_3 \sim A_{3.4}, \quad A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-n}u\partial_u, (\sqrt{|x|})^{1-n} \times \ln|x|u\partial_u\rangle,$$
$$A = nx^{-1} \quad (n \neq 0,1), \quad B = \tfrac{1}{4}(n-1)^2x^{-2},$$

$$(9)\ A_3 \sim A_{3.9}, \quad A_3 = \left\langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-n}\cos\left(\tfrac{1}{2}\beta\ln|x|\right)u\partial_u, (\sqrt{|x|})^{1-n}\sin\left(\tfrac{1}{2}\beta\ln|x|\right)u\partial_u\right\rangle,$$
$$A = nx^{-1} \quad (n \neq 0,1),$$

$$B = mx^{-2} \quad \left(m > \tfrac{1}{4}(n-1)^2\right), \quad \beta = \sqrt{4m-(n-1)^2},$$

$$(10)\ A_3 \sim A_{3.7}, \quad A_3 = \langle t\partial_t + x\partial_x, (\sqrt{|x|})^{1-\beta-n}u\partial_u, (\sqrt{|x|})^{1-n+\beta} \times u\partial_u\rangle,$$
$$A = nx^{-1} \quad (n \neq 0,1), \quad B = mx^{-2}$$

$$\left(m < \tfrac{1}{4}(n-1)^2, m \neq 0\right), \quad \beta = \sqrt{(n-1)^2 - 4m}.$$

$$(\mathrm{IV})\ D = G(t),$$

$$(1)\ A_3 \sim A_{3.3}, \quad A_3 = \langle \partial_x, u\partial_u, xu\partial_u\rangle,$$

$$A = B = 0,$$

$$(2)\ A_3 = A_{3.2}, \quad A_3 = \langle \partial_x, u\partial_u, e^x u\partial_u\rangle,$$

$$A = -1, \quad B = 0,$$

$$(3)\ A_3 \sim A_{3.8}, \quad A_3 = \langle \partial_x, \cos(x)u\partial_u, \sin(x)u\partial_u\rangle,$$

$$A = 0, \quad B = 1,$$

$$(4)\ A_3 \sim A_{3.6}, \quad A_3 = \langle \partial_x, e^x u\partial_u, e^{-x}u\partial_u\rangle,$$

$$A = 0, \quad B = -1,$$

$$(5)\ A_3 \sim A_{3.4}, \quad A_3 = \left\langle \partial_x, \exp\left(\tfrac{1}{2}x\right)u\partial_u, \exp\left(\tfrac{1}{2}x\right)xu\partial_u\right\rangle,$$

$$A = -1, \quad B = \tfrac{1}{4},$$

$$(6)\ A_3 \sim A_{3.7}, \quad A_3 = \left\langle \partial_x, \exp\left(\tfrac{1}{2}(1+\beta)x\right)u\partial_u, \exp\left(\tfrac{1}{2}(1-\beta)x\right)u\partial_u\right\rangle,$$

$$A=-1,\quad B=m\quad \left(m<\tfrac{1}{4}\right),\quad m\neq 0,\quad \beta=\sqrt{1-4m},$$

$$(7)\ A_3\sim A_{3.9}, A_3=\left\langle \partial_x, \exp\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}\beta x\right)u\partial_u, \exp\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}\beta x\right)u\partial_u\right\rangle,$$

$$A=-1,\quad B=m\quad \left(m>\tfrac{1}{4}\right),\quad \beta=\sqrt{4m-1},$$

$$(\mathrm{V})\ D=G(\eta),\quad \eta=x-kt,\quad k>0,$$

$$(1)\ A_3\sim A_{3.3},\quad A_3=\langle \partial_t+k\partial_x, u\partial_u, xu\partial_u\rangle,$$

$$A=B=0,$$

$$(2)\ A_3=A_{3.2},\quad A_3=\langle \partial_t+k\partial_x, u\partial_u, e^x u\partial_u\rangle,$$

$$A=-1,\quad B=0,$$

$$(3)\ A_3\sim A_{3.8},\quad A_3=\langle \partial_t+k\partial_x, \cos(x)u\partial_u, \sin(x)u\partial_u\rangle,$$

$$A=0,\quad B=1,$$

$$(4)\ A_3\sim A_{3.6},\quad A_3=\langle \partial_t+k\partial_x, e^x u\partial_u, e^{-x}u\partial_u\rangle,$$

$$A=n,\quad B=-1,$$

$$(5)\ A_3\sim A_{3.4},\quad A_3=\left\langle \partial_t+k\partial_x, \exp\left(\tfrac{1}{2}x\right)u\partial_u, \exp\left(\tfrac{1}{2}x\right)xu\partial_u\right\rangle,$$

$$A=-1,\quad B=\tfrac{1}{4},$$

$$(6)\ A_3\sim A_{3.7},\quad A_3=\left\langle \partial_t+k\partial_x, \exp\left(\tfrac{1}{2}(1+\beta)x\right)u\partial_u, \exp\left(\tfrac{1}{2}(1-\beta)x\right)u\partial_u\right\rangle,$$

$$A=-1,\quad B=m\quad \left(m<\tfrac{1}{4}\right),\quad m\neq 0,\quad \beta=\sqrt{1-4m},$$

$$(7)\ A_3\sim A_{3.9},\quad A_3=\left\langle \partial_t+k\partial_x, \exp\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}\beta x\right)u\partial_u, \exp\left(\tfrac{1}{2}x\right)\sin\left(\tfrac{1}{2}\beta x\right)u\partial_u\right\rangle,$$

$$A=-1,\quad B=m\quad \left(m>\tfrac{1}{4}\right),\quad \beta=\sqrt{4m-1}.$$

Theorem 12: *Equation $u_{tt}=u_{xx}-u^{-1}u_x^2$ has the widest symmetry group amongst equations of the form (6.1). Its maximal invariance algebra is the five-dimensional Lie algebra,*

$$A_5^1=\langle \partial_t, \partial_x, t\partial_t+x\partial_x, xu\partial_u, u\partial_u\rangle.$$

There are no equations of the form (6.1) which are inequivalent to the above equation and admit invariance algebra of the dimension higher than four. Inequivalent equations (6.1) admitting four-dimensional algebras are listed below together with their symmetry algebras.

$$(\mathrm{I})\ D=0,$$

$$(1)\ A_4 \sim A_{3.6} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, u\ \mathrm{ch}(\beta x)\partial_u, u\ \sinh(\beta x)\partial_u \rangle,$$

$$A = 0, \quad B = -\beta^2, \quad \beta \neq 0,$$

$$(2)\ A_4 \sim A_{3.8} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, u \cos(\beta x)\partial_u, u \sin(\beta x)\partial_u \rangle,$$

$$A = 0, \quad B = \beta^2, \quad \beta \neq 0,$$

$$(3)\ A_4 \sim A_{2.1} \oplus A_{2.2}, \quad A_4 = \langle \partial_t, \partial_x, u\partial_u, e^{-x}u\partial_u \rangle, \quad A = 1, \quad B = 0,$$

$$(4)\ A_4 \sim A_{3.4} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, e^{-x}u\partial_u, xe^{-x}u\partial_u \rangle, \quad A = 2, \quad B = 1,$$

$$(5)\ A_4 \sim A_{3.9} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, ue^{-x} \cos(\beta x)\partial_u, ue^{-x} \sin(\beta x)\partial_u \rangle,$$

$$A = 2, \quad B = m, \quad m > 1, \quad \beta = \sqrt{m-1},$$

$$(6)\ A_4 \sim A_{3.7} \oplus A_1, \quad A_4 = \langle \partial_t, \partial_x, ue^{-x}\ \mathrm{ch}(\beta x)\partial_u, ue^{-x} \sinh(\beta x)\partial_u \rangle,$$

$$A = 2, \quad B = m, \quad m > 1, \quad m \neq 0, \quad \beta = \sqrt{1-m},$$

$$(7)\ A_4 \sim A_{4.2}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, \sqrt{|x|}u\partial_u, u\sqrt{|x|}\ \ln|x|\partial_u \rangle,$$

$$A = 0, \quad B = \tfrac{1}{4}x^{-2},$$

$$(8)\ A_4 \sim A_{4.5}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}+\beta}u\partial_u, |x|^{\frac{1}{2}-\beta}u\partial_u \rangle, \quad A = 0,$$

$$B = mx^{-2}, \quad m < \tfrac{1}{4}, \quad m \neq 0, \quad \beta = \sqrt{\tfrac{1}{4} - m},$$

$$(9)\ A_4 \sim A_{4.6}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, \sqrt{|x|} \cos(\beta \ln|x|)u\partial_u, \sqrt{|x|} \sin(\beta \ln|x|)u\partial_u \rangle,$$

$$A = 0, \quad B = mx^{-2}, \quad m > \tfrac{1}{4}, \quad \beta = \sqrt{m - \tfrac{1}{4}},$$

$$(10)\ A_4 \sim A_{4.3}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, u \ln|x|\partial_u, u\partial_u \rangle, \quad A = x^{-1}, \quad B = 0,$$

$$(11)\ A_4 \sim A_{3.7} \oplus A_1, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{1-n}u\partial_u, u\partial_u \rangle,$$

$$A = nx^{-1}, \quad B = 0, \quad n \neq 0,1,$$

$$(12)\ A_4 \sim A_{4.5}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n)}u\partial_u, |x|^{\frac{1}{2}(1-n)}u \ln|x|\partial_u \rangle,$$

$$A = nx^{-1}, \quad B = \tfrac{1}{4}(n-1)^2x^{-2}, \quad n \neq 0,1,$$

$$(13)\ A_4 \sim A_{4.5}, \quad A_4 = \langle \partial_t, t\partial_t + x\partial_x, |x|^{\frac{1}{2}(1-n+\beta)}u\partial_u, |x|^{\frac{1}{2}(1-n-\beta)}u\partial_u \rangle,$$

$$A=nx^{-1},\quad B=mx^{-2},\quad m<\tfrac{1}{4}(n-1)^2,\quad m\neq 0,\quad n\neq 0,$$

$$\beta=\sqrt{(n-1)^2-4m},$$

(14) $A_4\sim A_{4.6},\quad A_4=\langle\partial_t, t\partial_t+x\partial_x, |x|^{\frac{1}{2}(1-n)}\cos(\beta\ln|x|)u\partial_u, |x|^{\frac{1}{2}(1-n)}\sin(\beta\ln|x|)u\partial_u\rangle,$
$A=nx^{-1},\quad B=mx^{-2},$

$$m\neq 0,\quad n\neq 0,\quad m>\tfrac{1}{4}(n-1)^2,\quad \beta=\sqrt{m-\tfrac{1}{4}(n-1)^2},$$

(II) $D=ktx^{-3},\quad k>0,$

(1) $A_4\sim A_{4.1},\quad A_4=\left\langle\partial_t-\tfrac{1}{2}kx^{-1}u\partial_u, t\partial_t+x\partial_x, xu\partial_u, u\partial_u\right\rangle,\quad A=B=0,$

(2) $A_4\sim A_{4.2},\quad A_4=\left\langle\partial_t-\tfrac{4}{9}kx^{-1}u\partial_u, t\partial_t+x\partial_x, \sqrt{|x|}u\partial_u, \sqrt{|x|}\ln|x|u\partial_u\right\rangle,$

$$A=0,\quad B=\tfrac{1}{4}x^{-2},$$

(3) $A_4\sim A_{4.5},\quad A_4=\langle\partial_t-[k/(m+2)]x^{-1}u\partial_u, t\partial_t+x\partial_x, |x|^{\frac{1}{2}+\beta}u\partial_u, |x|^{\frac{1}{2}-\beta}u\partial_u\rangle,$

$$A=0,\quad B=mx^{-2},\quad m\neq 0,-2,\quad m<\tfrac{1}{4},\quad \beta=\sqrt{\tfrac{1}{4}-m},$$

(4) $A_4\sim A_{4.2},\quad A_4=\left\langle\partial_t+\tfrac{1}{9}kx^{-1}(1+3\ln|x|u)\partial_u, t\partial_t+x\partial_x, x^2u\partial_u, x^{-1}u\partial_u\right\rangle,\quad A=0,\quad B=-2x^{-2},$

(5) $A_4\sim A_{4.6},\quad A_4=\langle\partial_t-[k/(m+2)]x^{-1}u\partial_u, t\partial_t+x\partial_x, \sqrt{|x|}u\cos(\beta\ln|x|)\partial_u, \sqrt{|x|}u\sin(\beta\ln|x|)\partial_u\rangle,$
$A=0,\quad B=mx^{-2},\quad m>\tfrac{1}{4},\quad \beta=\sqrt{m-\tfrac{1}{4}},$

(6) $A_4\sim A_{4.3},\quad A_4=\langle\partial_t-kx^{-1}u\partial_u, t\partial_t+x\partial_x, u\partial_u, u\ln|x|\partial_u\rangle,$

$$A=x^{-1},\quad B=0,$$

(7) $A_4\sim A_{3.4}\oplus A_1,\quad A_4=\langle\partial_t+kx^{-1}(1+\ln|x|)u\partial_u, t\partial_t+x\partial_x, u\partial_u, x^{-1}u\partial_u\rangle,$
$A=2x^{-1},\quad B=0,$

(8) $A_4\sim A_{3.7}\oplus A_1,\quad A_4=\langle\partial_t+[k/(n-2)]x^{-1}u\partial_u, t\partial_t+x\partial_x, u\partial_u, |x|^{1-n}u\partial_u\rangle,$

$$A=nx^{-1},\quad B=0,\quad n\neq 0,1,2,$$

(9) $A_4=A_{4.4},\quad A_4=\left\langle\partial_t-\tfrac{1}{2}kx^{-1}\ln^2|x|u\partial_u, t\partial_t+x\partial_x, x^{-1}u\partial_u, x^{-1}\ln|x|u\partial_u\right\rangle,$
$A=3x^{-1},\quad B=x^{-2},$

(10) $A_4\sim A_{4.2},\quad A_4=\langle\partial_t-[4k/(n-3)^2]x^{-1}u\partial_u, t\partial_t+x\partial_x, |x|^{\frac{1}{2}(1-n)}u\partial_u, |x|^{\frac{1}{2}(1-n)}\ln|x|u\partial_u\rangle,$
$A=nx^{-1},\quad B=\tfrac{1}{4}(n-1)^2x^{-2},\quad n\neq 0,3,$

(11) $A_4 \sim A_{4.5}$, $A_4 = \langle t\partial_t + x\partial_x, \partial_t - [k(2-n+m)]x^{-1}u\partial_u, |x|^{\frac{1}{2}(1-n+\beta)}u\partial_u, |x|^{\frac{1}{2}(1-n-\beta)}u\partial_u \rangle$,
$A = nx^{-1}$, $B = mx^{-2}$,

$$n \neq 0,2, \quad m \neq n-2, \quad m < \tfrac{1}{4}(n-1)^2, \quad \beta = \sqrt{(n-1)^2 - 4m},$$

(12) $A_4 \sim A_{4.2}$, $A_4 = \langle t\partial_t + x\partial_x, \partial_t + [k/(3-n)]x^{-1}\ln|x|u\partial_u, x^{-1}u\partial_u, |x|^{2-n}u\partial_u \rangle$,

$$A = nx^{-1}, \quad B = (n-2)x^{-2}, \quad n \neq 0,2,3,$$

(13) $A_4 \sim A_{4.6}$, $A_4 = \langle t\partial_t + x\partial_x, \partial_t - [k/(2-n+m)]x^{-1}u\partial_u$,
$|x|^{\frac{1}{2}(1-n)}u\cos(\beta\ln|x|)\partial_u, |x|^{\frac{1}{2}(1-n)}u\sin(\beta\ln|x|)\partial_u \rangle$,

$$A = nx^{-1}, \quad B = mx^{-2}, \quad n \neq 0, \quad m \neq 0, \quad m > \tfrac{1}{4}(n-1)^2,$$

$$\beta = \sqrt{m - \tfrac{1}{4}(n-1)^2},$$

(III) $D = kt$, $k > 0$,

(1) $A_4 \sim A_{4.1}$, $A_4 = \left\langle \partial_x, \partial_t - \frac{1}{2}kx^2u\partial_u, xu\partial_u, u\partial_u \right\rangle$, $A = B = 0$,

(2) $A_4 \sim A_{4.3}$, $A_4 = \langle \partial_x, \partial_t - kxu\partial_u, e^{-x}u\partial_u, u\partial_u \rangle$, $A = 1$, $B = 0$,

(3) $A_4 \sim A_{3.8} \oplus A_1$, $A_4 = \langle \partial_x, \partial_t - k\beta^{-2}u\partial_u, u\cos(\beta x)\partial_u, u\sin(\beta x)\partial_u \rangle$,

$$A = 0, \quad B = \beta^2, \quad \beta \neq 0,$$

(4) $A_4 \sim A_{3.6} \oplus A_1$, $A_4 = \langle \partial_x, \partial_t + k\beta^{-2}u\partial_u, u\,\mathrm{ch}(\beta x)\partial_u, u\sinh(\beta x)\partial_u \rangle$,

$$A = 0, \quad B = -\beta^2, \quad \beta \neq 0,$$

(5) $A_4 \sim A_{3.4} \oplus A_1$, $A_4 = \left\langle \partial_x, \partial_t - 4ku\partial_u, \exp\left(-\frac{1}{2}x\right)u\partial_u, x\exp\left(-\frac{1}{2}x\right)u\partial_u \right\rangle$,

$$A = 1, \quad B = \tfrac{1}{4},$$

(6) $A_4 \sim A_{3.7} \oplus A_1$, $A_4 = \left\langle \partial_x, \partial_t - km^{-1}u\partial_u, \exp\left(-\frac{1}{2}(1-\beta)x\right)u\partial_u, \exp\left(-\frac{1}{2}(1+\beta)x\right)u\partial_u \right\rangle$,

$$A = 1, \quad B = m, \quad m < \tfrac{1}{4}, \quad m \neq 0, \quad \beta = \sqrt{1-4m},$$

(7) $A_4 \sim A_{3.9} \oplus A_1$, $A_4 = \left\langle \partial_x, \partial_t - km^{-1}u\partial_u, \exp\left(-\frac{1}{2}x\right)\cos(\beta x)u\partial_u, \exp\left(-\frac{1}{2}x\right)\sin(\beta x)u\partial_u \right\rangle$,

$$A = 1, \quad B = m, \quad m > \tfrac{1}{4}, \quad \beta = \sqrt{m - \tfrac{1}{4}},$$

(IV) $D = kt^{-2}$, $k \neq 0$,

$$A_4 \sim A_{4.8} \quad (q=-1), \quad A_4 = \langle \partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle, \quad A=B=0,$$

$$\text{(V) } D = m(x-kt)^{-2}, \quad k>0, \quad m \neq 0,$$

$$A_4 \sim A_{4.8} \quad (q=-1), \quad A_4 = \langle \partial_t + k\partial_x, t\partial_t + x\partial_x, xu\partial_u, u\partial_u \rangle, \quad A=B=0.$$

Proof can be found in Ref. 35.

B. Nonlinear equations (2.7) invariant under three-dimensional Lie algebras

Equations of the form (2.7) cannot be invariant under the algebra which is isomorphic to a Lie algebra with a nontrivial Levi ideal.[35] That is why, to complete the second step of our classification algorithm it suffices to consider three-dimensional solvable real Lie algebras. We begin by considering two decomposable three-dimensional solvable Lie algebras.

Note that while classifying invariant equations (2.7) we skip those belonging to the class (6.1), since the latter has already been analyzed.

1. Invariance under decomposable Lie algebras

As $A_{3.1}=3A_1=A_{2.1}\oplus A_1$, $A_{3.2}=A_{2.2}\oplus A_1$, to construct all realizations of $A_{3.1}$ it suffices to compute all possible extensions of the (already known) realizations of the algebras $A_{2.1}=\langle e_1,e_2\rangle$ and $A_{2.2}=\langle e_1,e_2\rangle$. To this end we need to supplement the latter by a basis operator e_3 of the form (2.4) in order to satisfy the commutation relations

$$[e_1,e_3]=[e_2,e_3]=0. \tag{6.2}$$

What is more, to simplify the form of e_3 we may use those transformations from $\mathcal{E}$ that do not alter the remaining basis operators of the corresponding two-dimensional Lie algebras.

We skip the full calculation details and give a couple of examples illustrating the main calculation steps needed to extend $A_{2.1}$ to a realization of $A_{3.1}$.

Consider the realization $A^1_{2.1}$. Upon checking commutation relations (6.2), where e_3 is of form (2.4), we get

$$\lambda_1 = \lambda_2 = r(t,x) = 0, \quad h = k = \text{const}.$$

Consequently, e_3 is the linear combination of e_1, e_2, namely, $e_3=\lambda e_1 + k e_2$, which is impossible by the assumption that the algebra under study is three dimensional. Hence we conclude that the above realization of $A^1_{2.1}$ cannot be extended to a realization of the algebra $A_{3.1}$.

Turn now to the realization $A^2_{2.1}$. Checking commutation relations (6.2), where e_3 is of form (2.4) yields the following realization of $A_{3.1}$:

$$\langle t\partial_t + x\partial_x, \sigma(\xi)\partial_u, \gamma(\xi)\partial_u \rangle, \quad \xi = tx^{-1},$$

where $\gamma'\sigma - \gamma\sigma' \neq 0$ However, the corresponding invariant equation (2.7) is linear.

Finally, consider the realization $A^3_{2.1}$. Inserting its basis operators and the operator e_3 of the form (2.4) into (6.2) and solving the obtained equations gives the following realization of $A_{3.1}$:

$$\langle \partial_t, \partial_x, u\partial_u \rangle.$$

Inserting the obtained coefficients for e_3 into the classifying equation (2.5) we get invariant equation

$$u_{tt} = u_{xx} + uG(\omega), \quad \omega = u^{-1}u_x,$$

where (to ensure nonlinearity) we need to have $G_{\omega\omega} \neq 0$.

Similar analysis of the realizations $A^i_{2.1}$ ($i=4,5,\ldots,12,14$) yields three new invariant equations. For two of thus obtained $A_{3.1}$-invariant equations the corresponding three-dimensional algebras are maximal. The other two may admit four-dimensional invariance algebras provided arbitrary elements are properly specified.

Handling in a similar way the extensions of $A_{2.2}$ up to realizations of $A_{3.2}$ gives 10 inequivalent nonlinear equations whose maximal invariance algebras are realizations of the three-dimensional algebra $A_{3.2}$ and four inequivalent equations (2.7) admitting four-dimensional symmetry algebras.

We perform analysis of equations admitting four-dimensional algebras in the next section. Here we present the complete list of nonlinear equations (2.7) whose maximal symmetry algebras are realizations of three-dimensional Lie algebras $A_{3.1}$ and $A_{3.2}$.

$A_{3.1}$-invariant equations,

$$A^1_{3.1} = \langle \partial_t, \partial_x, u\partial_u \rangle,$$

$$F = uG(\omega), \quad \omega = u^{-1}u_x,$$

$$A^2_{3.1} = \langle \partial_x, \varphi(t)\partial_u, \psi(t)\partial_u \rangle,$$

$$\sigma = \psi'\varphi - \psi\varphi' \neq 0, \quad \sigma' = 0,$$

$$F = \varphi^{-1}\varphi'' u + G(t,u_x).$$

$A_{3.2}$-invariant equations,

$$A^1_{3.2} = \langle \partial_t, \partial_x, e^x u\partial_u \rangle,$$

$$F = -u^{-1}u_x^2 - u\ln|u| + uG(\omega),$$

$$\omega = u^{-1}u_x - \ln|u|,$$

$$A^2_{3.2} = \langle -t\partial_t - x\partial_x, \partial_t + k\partial_x, u\partial_u \rangle \quad (k \geqslant 0),$$

$$F = u\eta^{-2}G(\omega), \quad \eta = x - kt,$$

$$\omega = \eta u^{-1}u_x,$$

$$A^3_{3.2} = \langle -t\partial_t - x\partial_x + mu\partial_u, \partial_t + k\partial_x, |\eta|^{-m}\partial_u \rangle$$

$$(\eta = x - kt,\ k = m = 0 \text{ or } k > 0,\ m \in \mathbb{R}),$$

$$F = m(k^2-1)(m+1)\eta^{-2}u + |\eta|^{-2-m}G(\omega),$$

$$\omega = |\eta|^m(mu + \eta u_x),$$

$$A^4_{3.2} = \langle \partial_x, e^x u\partial_u, \partial_t + mu\partial_u \rangle \quad (m > 0),$$

$$F = -u^{-1}u_x^2 - u_x + uG(\omega),$$

$$\omega = u^{-1}u_x - \ln|u| + mt,$$

$$A^5_{3.2} = \langle -t\partial_t - x\partial_x, \partial_x, u\partial_u \rangle,$$

$$F = ut^{-2}G(\omega), \quad \omega = tu^{-1}u_x,$$

$$A^6_{3.2} = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, u\partial_u \rangle \quad (k > 0),$$

$$F = 2ktx^{-2}u_x - 2ktx^{-3}u + k^2t^2x^{-4}u + x^{-2}uG(\omega),$$

$$\omega = xu^{-1}u_x + ktx^{-1},$$

$$A^7_{3.2} = \langle -t\partial_t - x\partial_x, \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u \rangle \quad (k > 0),$$

$$F = 2ktx^{-2}u_x + (k^2t^2x^{-4} - 2ktx^{-3} + k^2x^{-2})u + x^{-2}\exp(ktx^{-1})G(\omega), \quad \omega = \exp(-ktx^{-1})(xu_x + ktx^{-1}u),$$

$$A^8_{3.2} = \left\langle \frac{1}{2k}(\partial_t + k\partial_x), e^{x+kt}\partial_u, e^{\eta}\partial_u \right\rangle \quad (k > 0, \ \eta = x - kt),$$

$$F = (k^2 - 1)u + G(\eta, \omega), \quad \omega = u_x - u,$$

$$A^9_{3.2} = \langle \partial_t + f(x)u\partial_u, e^{(1+f(x))t}\partial_u, f(x)e^{f(x)t}\partial_u \rangle,$$

$$F = -(tf'' - t^2(f')^2 - (1+f)^2)u - 2tf'u_x + e^{tf}G(x, \omega),$$

$$\omega = e^{-tf}(u_x - f'(t + f^{-1})u), \quad f'' + 2f^2 + f = 0, \quad f \neq 0,$$

$$A^{10}_{3.2} = \langle k(t\partial_t + x\partial_x), |t|^{k^{-1}}|\xi|^{(k-1)/2k}\partial_u, |\xi|^{(k-1)/2k}\partial_u \rangle \quad (k \neq 0;1),$$

$$F = \left[\frac{1-k}{k}\xi^2 + \frac{1-k^2}{4k^2}(1-\xi^2)\right]t^{-2}u + t^{-2}G(\xi, \omega),$$

$$\omega = |\xi|^{(k-1)/2k}\left[xu_x + \frac{k-1}{2k}u\right], \quad \xi = tx^{-1}.$$

2. Invariance under nondecomposable three-dimensional solvable Lie algebras

There exist seven nondecomposable three-dimensional solvable Lie algebras over the field of real numbers. All those algebras contain a subalgebra which is the two-dimensional Abelian ideal. Consequently, we can use the results of classification of $A_{2.1}$-invariant equations in order to describe equations admitting nondecomposable three-dimensional solvable real Lie algebras. We remind that equations of the form (6.1) has already been analyzed and therefore are not considered in the sequel.

Note that there are nonlinear PDEs of the considered form that admits four-dimensional invariance algebras. As four-dimensional algebras will be considered separately in the next section, we give below only those nonlinear invariant equations whose maximal symmetry algebras are three-dimensional nondecomposable solvable real Lie algebras.

$A_{3.3}$-invariant equations,

$$A_{3.3}^1 = \langle u\partial_u, \partial_t + k\partial_x, m\partial_t + k^{-1}xu\partial_u\rangle \quad (k > 0, m \neq 0),$$

$$F = -u^{-1}u_x^2 + uG(\omega), \quad \omega = x - kt + mk^2u^{-1}u_x,$$

$$A_{3.3}^2 = \langle u\partial_u, \partial_x, m\partial_t + xu\partial_u\rangle \quad (m > 0),$$

$$F = -u^{-1}u_x^2 + uG(\omega), \quad \omega = t - mu^{-1}u_x,$$

$$A_{3.3}^3 = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u\right\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + u_x^3G(\xi,\omega), \quad \xi = tx^{-1}, \quad \omega = xu_x^2,$$

$$A_{3.3}^4 = \langle \partial_u, -t\partial_u, \partial_t + k\partial_x\rangle \quad (k \geqslant 0),$$

$$F = G(\eta, u_x), \quad \eta = x - kt.$$

$A_{3.4}$-invariant equations,

$$A_{3.4}^1 = \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + (mu + t|\eta|^{m-1})\partial_u\rangle$$

$$(\eta = x - kt,\ k > 0,\ m \neq 1),$$

$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u - 2k(m-1)\eta^{m-2}\ln|\eta| + |\eta|^{m-2}G(\omega),$$
$$\omega = [\eta u_x - (m-1)u]|\eta|^{-m},$$

$$A_{3.4}^2 = \langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u\rangle \quad (k \geqslant 0),$$

$$F = e^tG(\eta,\omega), \quad \eta = x - kt, \quad \omega = e^{-t}u_x,$$

$$A_{3.4}^3 = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{3}{2}u\partial_u\right\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + u_x^{-1}G(\xi,\omega), \quad \xi = tx^{-1}, \quad \omega = x^{-1}u_x^2,$$

$$A_{3.4}^4 = \langle kx^{-1}u\partial_u, \partial_t - kx^{-1}\ln|x|u\partial_u, t\partial_t + x\partial_x\rangle \quad (k > 0),$$

$$F = -3ktx^{-3}u - 2x^{-2}u\ln|u| - u^{-1}u_x^2 + x^{-2}uG(\omega),$$

$$\omega = xu^{-1}u_x + \ln|u| + ktx^{-1},$$

$$A_{3.4}^5 = \langle \exp(ktx^{-1})\partial_u, \partial_t + kx^{-1}u\partial_u, t\partial_t + x\partial_x + (u + t\exp(ktx^{-1}))\partial_u\rangle \quad (k > 0),$$

$$F = k^2x^{-4}u(t^2 + x^2) + 2x^{-1}(ktx^{-1} + 1)u_x + 2k\exp(ktx^{-1})x^{-1}\ln|x| + x^{-1}\exp(ktx^{-1})G(\omega),$$

$$\omega = \exp(-ktx^{-1})(u_x + ktx^{-2}u).$$

$A_{3.5}$-invariant equations,

$$A^1_{3.5} = \langle |\eta|^{m-1}\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + mu\partial_u\rangle \quad (k > 0,\ m \neq 1),$$

$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$

$$\omega = |\eta|^{-m}[\eta u_x - (m-1)u], \quad \eta = x - kt,$$

$$A^2_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x\rangle,$$

$$F = u_x^2 G(u),$$

$$A^3_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + mu\partial_u\rangle \quad (m \neq 0),$$

$$F = |u|^{1-(2/m)}G(\omega), \quad \omega = |u_x|^m |u|^{1-m},$$

$$A^4_{3.5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + \partial_u\rangle,$$

$$F = e^{-2u}G(\omega), \quad \omega = e^u u_x,$$

$$A^5_{3.5} = \langle \partial_t, x^{-1}u\partial_u, t\partial_t + x\partial_x\rangle,$$

$$F = -u^{-1}u_x^2 - 2x^{-2}u \ln|u| + x^{-2}uG(\omega),$$

$$\omega = xu^{-1}u_x + \ln|u|,$$

$$A^6_{3.5} = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + u\partial_u\rangle \quad (k > 0),$$

$$F = kx^{-4}u[kt^2 - 2tx + kx^2] + 2ktx^{-2}u_x + x^{-1}\exp(ktx^{-1})G(\omega),$$

$$\omega = \exp(-ktx^{-1})(u_x + ktx^{-2}u),$$

$$A^7_{3.5} = \langle \varphi(t)\partial_u, \psi(t)\partial_u, \partial_x + u\partial_u\rangle \quad (\varphi'\psi - \varphi\psi' \neq 0),$$

$$F = \varphi^{-1}\varphi'' u + u_x G(t,\omega),$$

$$\omega = e^{-x}u_x, \quad \varphi''\psi - \varphi\psi'' = 0.$$

$A_{3.6}$-invariant equations,

$$A^1_{3.6} = \langle \partial_t + k\partial_x, |\eta|^{m+1}\partial_u, t\partial_t + x\partial_x + mu\partial_u\rangle \quad (k > 0,\ m \neq -1),$$

$$F = m(k^2-1)(m+1)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$

$$\omega = |\eta|^{1-m}[u_x - \eta^{-1}(m+1)u], \quad \eta = x - kt,$$

$$A^2_{3.6} = \langle \partial_t + mx^{-1}u\partial_u, xu\partial_u, t\partial_t + x\partial_x\rangle \quad (m \geqslant 0),$$

$$F = -u^{-1}u_x^2 - 2mtx^{-3}u + x^{-2}uG(\omega),$$

$$\omega = xu^{-1}u_x - \ln|u| + 2mtx^{-1},$$

$$A_{3.6}^3 = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x - u\partial_u \rangle \quad (k > 0),$$

$$F = x^{-4}[k^2x^2 - 2ktx + k^2t^2]u + 2ktx^{-2}u_x + x^{-3}\exp(ktx^{-1})G(\omega),$$

$$\omega = \exp(-ktx^{-1})(x^2u_x + ktu),$$

$$A_{3.6}^4 = \langle e^{-t}\partial_u, e^t\partial_u, \partial_t + k\partial_x \rangle \quad (k \geqslant 0),$$

$$F = u + G(\eta, u_x), \quad \eta = x - kt,$$

$$A_{3.6}^5 = \left\langle |t|^{-\frac{1}{2}}\partial_u, |t|^{\frac{3}{2}}\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u \right\rangle,$$

$$F = \tfrac{3}{4}t^{-2}u + |t|^{-3/2}G(\xi,\omega), \quad \xi = tx^{-1}, \quad \omega = x^{-1}u_x^2.$$

$A_{3.7}$-invariant equations,

$$A_{3.7}^1 = \langle \partial_t + k\partial_x, |\eta|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle$$

$$(k > 0,\ m \neq q,\ 0 < |q| < 1),$$

$$F = (k^2-1)(m-q)(m-q-1)\eta^{-2}u + |\eta|^{m-2}G(\omega),$$

$$\omega = |\eta|^{1-m}[u_x - (m-q)\eta^{-1}u], \quad \eta = x - kt,$$

$$A_{3.7}^2 = \langle \partial_t + kx^{-1}u\partial_u, \exp(ktx^{-1})\partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle$$

$$(k > 0, 0 < |q| < 1),$$

$$F = [k^2x^{-2} + k^2x^{-4}t^2 - 2ktx^{-3}]u + 2ktx^{-2}u_x + |x|^{q-2}\exp(ktx^{-1})G(\omega),$$

$$\omega = |x|^{1-q}\exp(-ktx^{-1})(u_x + ktx^{-2}u),$$

$$A_{3.7}^3 = \left\langle |t|^{\frac{1}{2}q}\partial_u, |t|^{1-\frac{1}{2}q}\partial_u, t\partial_t + x\partial_x + \left(1 + \tfrac{1}{2}q\right)u\partial_u \right\rangle \quad (q \neq 0, \pm 1),$$

$$F = \tfrac{1}{4}q(q-2)t^{-2}u + |t|^{\frac{1}{2}(q-2)}G(\xi,\omega),$$

$$\xi = tx^{-1}, \quad \omega = |t|^{-\frac{1}{2}q}u_x,$$

$$A_{3.7}^4 = \left\langle \exp\left(\tfrac{1}{2}(q-1)t\right)\partial_u, \exp\left(\tfrac{1}{2}(1-q)t\right)\partial_u, \partial_t + k\partial_x + \tfrac{1}{2}(1+q)u\partial_u \right\rangle$$

$$(q \neq 0, \pm 1; k \geqslant 0),$$

$$F = \tfrac{1}{4}(q-1)^2 u + \exp\left(\tfrac{1}{2}(1+q)t\right)G(\eta,\omega),$$

$$\eta = x - kt, \quad \omega = \exp\left(-\tfrac{1}{2}(1+q)t\right)u_x,$$

$$A^5_{3.7} = \langle \partial_t + kx^{-1}u\partial_u, |x|^{-q}u\partial_u, t\partial_t + x\partial_x\rangle \quad (k \geqslant 0,\ q \neq 0, \pm 1),$$

$$F = -u^{-1}u_x^2 - q(q+1)x^{-2}u\ln|u| + k(q-1)(q+2)tx^{-3}u + ux^{-2}G(\omega),$$
$$\omega = xu^{-1}u_x + q\ln|u| + k(1-q)tx^{-1}.$$

$A_{3.8}$-invariant equations

$$A^1_{3.8} = \langle \cos t\partial_u, -\sin t\partial_u, \partial_t + k\partial_x\rangle \quad (k \geqslant 0),$$

$$F = -u + G(\eta, u_x), \quad \eta = x - kt,$$

$$A^2_{3.8} = \left\langle |t|^{\frac{1}{2}}\cos(\ln|t|)\partial_u, -|t|^{\frac{1}{2}}\sin(\ln|t|)\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u\right\rangle,$$

$$F = -\tfrac{5}{4}t^{-2}u + |t|^{-3/2}G(\xi,\omega),$$

$$\xi = tx^{-1}, \quad \omega = |t|^{1/2}u_x.$$

$A_{3.9}$-invariant equations

$$A^1_{3.9} = \langle \sin t\partial_u, \cos t\partial_u, \partial_t + k\partial_x + qu\partial_u\rangle \quad (k \geqslant 0,\ q > 0),$$

$$F = -u + e^{qt}G(\eta,\omega), \quad \eta = x - kt, \omega = e^{-qt}u_x,$$

$$A^2_{3.9} = \left\langle |t|^{\frac{1}{2}}\sin(\ln|t|)\partial_u, |t|^{\frac{1}{2}}\cos(\ln|t|)\partial_u, t\partial_t + x\partial_x + \left(\tfrac{1}{2} + q\right)u\partial_u\right\rangle$$

$$(q \neq 0), \quad F = -\tfrac{5}{4}t^{-2}u + |t|^{q-\frac{3}{2}}G(\xi,\omega),$$

$$\xi = tx^{-1}, \quad \omega = |t|^{\frac{1}{2}-q}u_x.$$

C. Complete group classification of Eq. (2.7)

The aim of this section is finalizing group classification of (2.7). The majority of invariant equations obtained in the preceding section contain arbitrary functions of one variable. So that we can utilize the standard Lie–Ovsyannikov approach in order to complete their group classification.

1. Equations depending on an arbitrary function of one variable

Note that equations belonging to the already investigated class of (6.1) are not considered. As our computations show, new results could be obtained for the equations,

$$u_{tt} = u_{xx} + uG(\omega), \quad \omega = u^{-1}u_x, \tag{6.3}$$

$$u_{tt} = u_{xx} + G(u_x), \tag{6.4}$$

only. Below we give (without proof) the assertions describing their group properties.

Assertion 6: Equation (6.3) admits wider symmetry group iff it is equivalent to the following equation:

$$u_{tt}=u_{xx}+mu^{-1}u_x^2 \quad (m\neq 0,-1). \tag{6.5}$$

The maximal invariance algebra of (6.5) is the four-dimensional Lie algebra,

$$A_4 \sim A_{3.5}\oplus A_1, \quad A_4=\langle \partial_t,\partial_x,t\partial_t+x\partial_x,u\partial_u\rangle.$$

Assertion 7: Equation (6.4) admits wider symmetry group iff it is equivalent to one of the following PDEs:

$$u_{tt}=u_{xx}+e^{u_x}, \tag{6.6}$$

$$u_{tt}=u_{xx}+m\ln|u_x|, \quad m>0, \tag{6.7}$$

$$u_{tt}=u_{xx}+|u_x|^k, \quad k\neq 0,1. \tag{6.8}$$

The maximal invariance algebras of the above equations are five-dimensional solvable Lie algebras listed below,

$$A_5^2=\langle \partial_t,\partial_x,\partial_u,t\partial_u,t\partial_t+x\partial_x+(u-x)\partial_u\rangle,$$

$$A_5^3=\left\langle \partial_t,\partial_x,\partial_u,t\partial_u,t\partial_t+x\partial_x+\left(2u+\tfrac{1}{2}mt^2\right)\partial_u\right\rangle,$$

$$A_5^4=\left\langle \partial_t,\partial_x,\partial_u,t\partial_u,t\partial_t+x\partial_x+\frac{k-2}{k-1}u\partial_u\right\rangle.$$

Analyzing the remaining equations containing arbitrary functions of one variable we come to conclusion that one of them can admit wider invariance groups iff either

(1) it is equivalent to PDE of the form (6.1), or
(2) it is equivalent to PDE of the form (6.5).

To finalize the procedure of group classification of Eqs. (2.7) we need to consider invariant equations obtained in the preceding section that contain arbitrary functions of two variables.

2. Classification of equations with arbitrary functions of two variables

In the case under study the standard Lie–Ovsyannikov method is inefficient and we apply our classification algorithm. In order to do this we perform extension of three-dimensional solvable Lie algebras to all possible realizations of four-dimensional solvable Lie algebras. The next step will be to check which of the obtained realizations are symmetry algebras of nonlinear equations of the form (2.7). In what follows we use the results of Ref. 41, where all inequivalent (within the action of inner automorphism group) four-dimensional solvable abstract Lie algebras are given.

The computation details can be found in Ref. 35. Here we summarize the obtained results as follows:

(1) If the functions contained in the equations under study are arbitrary, then the corresponding realizations are their maximal invariance algebras, and
(2) Except for Eq. (6.4), all the equations in question do not allow for extension of their symmetry.

Below we give the complete list of PDEs (2.7) invariant under four-dimensional solvable Lie algebras that are obtained through group analysis of equations with arbitrary functions of two variables.

$A_{2.2}\oplus 2A_1$-invariant equations,

$$(1)\ \langle \partial_x, \partial_t + u\partial_u, e^t\partial_u, e^{-t}\partial_u\rangle, \quad F = u + e^t G(\omega), \quad \omega = u^{-t}u_x,$$

$$(2)\ \left\langle \frac{1}{2k}(\partial_t + k\partial_x), e^{x+kt}\partial_u, e^{\eta}\partial_u, \partial_x + u\partial_u \right\rangle \quad (k > 0, \eta = x - kt),$$

$$F = (k^2-1)u + e^{\eta}G(\omega), \quad \omega = e^{-\eta}(u_x - u).$$

$2A_{2.2}$-invariant equations,

$$(1)\ \langle \partial_t + \epsilon u\partial_u, \partial_x, e^{x+kt}\partial_u, e^{x-kt}\partial_u\rangle \quad (\epsilon = 0,1; k > 0),$$

$$F = (k^2-1)u + e^{\epsilon t}G(\omega), \quad \omega = e^{-\epsilon t}(u_x - u),$$

$$(2)\ \langle \alpha\partial_x - u\partial_u, \partial_t + k\partial_x, e^{-t}\partial_u, e^t\partial_u\rangle \quad (k \geqslant 0, \alpha > 0),$$

$$F = u + \exp(-\alpha^{-1}\eta)G(\omega), \quad \eta = x - kt, \quad \omega = \exp(\alpha^{-1}\eta)u_x.$$

$A_{3.3} \oplus A_1$-invariant equations,

$$(1)\ \langle \partial_t, \partial_x, \partial_u, t\partial_u\rangle, \quad F = G(u_x).$$

$A_{3.4} \oplus A_1$-invariant equations

$$(1)\ \langle \partial_u, \partial_x, t\partial_t + x\partial_x + (u+x)\partial_u, t\partial_u\rangle,$$

$$F = t^{-1}G(\omega), \quad \omega = u_x - \ln|t|,$$

$$(2)\ \langle \partial_t + u\partial_u, \partial_x, t\partial_u, \partial_u\rangle, \quad F = e^t G(\omega), \omega = e^{-t}u_x,$$

$$(3)\ \langle x^{-1}\partial_u, \partial_x - x^{-1}(u + \ln|x|)\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u\rangle,$$

$$F = 2x^{-1}u_x + x^{-2} + t^{-1}x^{-1}G(\omega), \quad \omega = xu_x + u - \ln|tx^{-1}|.$$

$A_{3.5} \oplus A_1$-invariant equations,

$$(1)\ \langle \partial_x, \partial_u, t\partial_t + x\partial_x + u\partial_u, t\partial_u\rangle, \quad F = t^{-1}G(u_x),$$

$$(2)\ \langle x^{-1}\partial_u, \partial_x - x^{-1}u\partial_u, t\partial_t + x\partial_x, tx^{-1}\partial_u\rangle,$$

$$F = -2x^{-2}u + 2t^{-1}(u_x + x^{-1}u)\ln|t(u_x + x^{-1}u)| + t^{-1}(u_x + x^{-1}u)G(\omega), \quad \omega = xu_x + u.$$

$A_{3.6} \oplus A_1$-invariant equations,

$$(1)\ \langle \partial_x, t\partial_u, t\partial_t + x\partial_x, \partial_u\rangle, \quad F = t^{-2}G(\omega), \quad \omega = t^{-1}u_x,$$

$$(2)\ \langle \partial_t, \partial_x, e^t\partial_u, e^{-t}\partial_u\rangle, \quad F = u + G(u_x).$$

$A_{3.7} \oplus A_1$-invariant equations,

$$(1)\ \left\langle \exp\left(-\tfrac{1}{2}(1-q)t\right)\partial_u, \exp\left(\tfrac{1}{2}(1-q)t\right)\partial_u, \partial_t + \tfrac{1}{2}(1+q)u\partial_u, \partial_x\right\rangle$$

$$(q \neq 0, \pm 1), \quad F = \tfrac{1}{4}(1-q)^2u + \exp\left(\tfrac{1}{2}(1+q)t\right)G(\omega),$$

$$\omega = \exp\left(-\tfrac{1}{2}(1+q)t\right)u_x,$$

$$(2)\ \left\langle \partial_x, |t|^{\frac{1}{2}(1-q)}\partial_u, |t|^{\frac{1}{2}(1+q)}\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}(1+q)u\partial_u \right\rangle$$

$$(q \neq 0, \pm 1), \quad F = \tfrac{1}{4}(q^2-1)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(\omega),$$

$$\omega = |t|^{\frac{1}{2}(1-q)}u_x,$$

$$(3)\ \left\langle |t|^{-1/q}|\xi|^{(q+1)/2q}\partial_u, \partial_x - \frac{1+q}{2q}x^{-1}u\partial_u, -q(t\partial_t + x\partial_x), |\xi|^{(1+q)/2q}\partial_u \right\rangle$$

$$(q \neq 0, \pm 1), \quad F = \left[\frac{1-q^2}{4q^2}(t^{-2}+x^{-2})\right]u + \frac{1+q}{q}x^{-1}u_x + t^{-2}|\xi|^{(1+q)/2q}G(\omega),$$

$$\xi = tx^{-1}, \quad \omega = |\xi|^{(q-1)/2q}\left[xu_x + \frac{q+1}{2q}u\right].$$

$A_{3.8} \oplus A_1$-invariant equations,

$$(1)\ \langle \sin t\partial_u, \cos t\partial_u, \partial_t, \partial_x\rangle, \quad F = -u + G(u_x).$$

$A_{3.9} \oplus A_1$-invariant equations,

$$(1)\ \langle \sin t\partial_u, \cos t\partial_u, \partial_t + qu\partial_u, \partial_x\rangle \quad (q > 0),$$

$$F = -u + e^{qt}G(\omega), \quad \omega = e^{-qt}u_x.$$

$A_{4.1}$-invariant equations,

$$(1)\ \langle \partial_u, -t\partial_u, \partial_x, \partial_t - tx\partial_u\rangle, \quad F = G(\omega), \quad \omega = u_x + \tfrac{1}{2}t^2,$$

$$(2)\ \left\langle \partial_u, -t\partial_u, \alpha\partial_x + \tfrac{1}{2}t^2\partial_u, \partial_t + kx\partial_x \right\rangle\ (k \geqslant 0, \alpha > 0),$$

$$F = \alpha^{-1}(x-kt) + G(u_x).$$

$A_{4.2}$-invariant equations,

$$(1)\ \left\langle |t|^{1-\frac{1}{2}q}\partial_u, |t|^{\frac{1}{2}q}\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\left(1+\tfrac{1}{2}q\right)u + x|t|^{\frac{1}{2}q}\right]\partial_u \right\rangle$$

$$(q \neq 0,1), \quad F = \tfrac{1}{4}q(q-2)t^{-2}u + |t|^{\frac{1}{2}(q-3)}G(\omega),$$

$$\omega = |t|^{\frac{1}{2}(1-q)}u_x - 2|t|^{\frac{1}{2}},$$

$$(2)\ \left\langle \partial_x, \sqrt{|t|}\partial_u, \sqrt{|t|}\ \ln|t|\partial_u, t\partial_t + x\partial_x + \left(q+\tfrac{1}{2}\right)u\partial_u \right\rangle$$

$$(q \neq 0), \quad F = -\tfrac{1}{4}t^{-2}u + |t|^{q-\frac{3}{2}}G(\omega), \quad \omega = |t|^{\frac{1}{2}-q}u_x.$$

$A_{4.3}$-invariant equations,

$$(1)\ \left\langle \partial_x, |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, t\partial_t + x\partial_x + \tfrac{1}{2}u\partial_u \right\rangle,$$

$$F = -\tfrac{1}{4}t^{-2}u + |t|^{-\frac{3}{2}}G(\omega), \quad \omega = |t|^{\frac{1}{2}}u_x,$$

$$(2)\ \langle \partial_x, t\partial_u, \partial_u, t\partial_t + x\partial_x \rangle, \quad F = t^{-2}G(\omega), \omega = tu_x,$$

$$(3)\ \langle e^{kt}\partial_u, \partial_t + ku\partial_u, \beta\partial_x + te^{kt}\partial_u, e^{-kt}\partial_u \rangle \quad (k \neq 0, \beta > 0),$$

$$F = k^2u + 2k\beta^{-1}xe^{kt} + e^{kt}G(\omega), \quad \omega = e^{-kt}u_x,$$

$$(4)\ \langle e^{x+kt}\partial_u, e^{\eta}\partial_u, \alpha(\partial_x + u\partial_u) + 2kte^{\eta}\partial_u, -(1/2k)(\partial_t + k\partial_x) \rangle$$

$$(\alpha \neq 0,\ k > 0), \quad F = (k^2 - 1)u - 4k^2\alpha^{-1}\eta e^{\eta} + e^{\eta}G(\omega),$$

$$\omega = e^{-\eta}(u_x - u), \quad \eta = x - kt.$$

$A_{4.4}$-invariant equations,

$$(1)\ \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u, \partial_x, t\partial_t + x\partial_x + \left[\tfrac{3}{2}u - x|t|^{\frac{1}{2}}\ln|t|\right]\partial_u \right\rangle,$$

$$F = \tfrac{1}{4}t^{-2}u + |t|^{-\frac{1}{2}}G(\omega), \omega = |t|^{-\frac{1}{2}}u_x + \tfrac{1}{2}\ln^2|t|.$$

$A_{4.5}$-invariant equations,

$$(1)\ \langle \partial_x, |t|^{m-\alpha}\partial_u, |t|^{1-m+\alpha}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle$$

$$\left(m \neq \tfrac{1}{2}(1+\alpha), \tfrac{1}{2} + \alpha; \alpha \neq 0\right),$$

$$F = (m-\alpha)(m-\alpha-1)t^{-2}u + |t|^{m-2}G(\omega), \quad \omega = |t|^{1-m}u_x.$$

$A_{4.6}$-invariant equations,

$$(1)\ \left\langle \partial_x, |t|^{\frac{1}{2}}\sin(q^{-1}\ln|t|)\partial_u, |t|^{\frac{1}{2}}\cos(q^{-1}\ln|t|)\partial_u, qt\partial_t + qx\partial_x\left(\tfrac{1}{2}q + p\right)u\partial_u \right\rangle \quad (q \neq 0, p \geqslant 0),$$

$$F = -\left(\tfrac{1}{4} + q^{-2}\right)t^{-2}u + |t|^{q^{-1}\left(p - \frac{3}{2}q\right)}G(\omega), \quad \omega = |t|^{q^{-1}\left(\frac{1}{2}q - p\right)}u_x.$$

$A_{4.7}$-invariant equations,

$$(1)\ \left\langle \partial_u, -t\partial_u, \partial_t + k\partial_x, t\partial_t + x\partial_x + \left(2u - \tfrac{1}{2}t^2\right)\partial_u \right\rangle \quad (k \geqslant 0),$$

$$F = -\ln|\eta| + G(\omega), \quad \omega = \eta^{-1}u_x, \quad \eta = x - kt.$$

$A_{4.8}$-invariant equations,

$$(1)\ \langle \partial_t + \epsilon u\partial_u, \partial_x, e^x\partial_u, te^x\partial_u \rangle \quad (\epsilon = 0; 1),$$

$$F = -u + e^{\epsilon t}G(\omega), \quad \omega = e^{-\epsilon t}(u_x - u),$$

$$(2)\ \langle |x|^{m-q}\partial_u, \partial_t, t|x|^{m-q}\partial_u, t\partial_t + x\partial_x + mu\partial_u \rangle \quad (q \neq 0,\ m \in \mathbb{R}),$$

$$F = -(m-q)(m-q-1)x^{-2}u + |x|^{m-2}G(\omega),$$

$$\omega = |x|^{1-m}[u_x - (m-q)x^{-1}u],$$

$$(3)\ \langle \partial_t + k\partial_x, \partial_u, t\partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle \quad (k > 0,\ q \in \mathbb{R}),$$

$$F = |\eta|^{q-2}G(\omega), \quad \omega = |\eta|^{1-q}u_x, \quad \eta = x - kt,$$

$$(4)\ \langle x^{-1}\partial_u, \partial_t + \partial_x - x^{-1}u\partial_u, tx^{-1}\partial_u, t\partial_t + x\partial_x \rangle,$$

$$F = 2x^{-1}u_x + x^{-1}(t-x)^{-1}G(\omega), \quad \omega = xu_x + u,$$

$$(5)\ \langle \partial_u, -t\partial_u, \partial_t + k\partial_x + u\partial_u, \alpha\partial_x + u\partial_u \rangle \quad (\alpha \neq 0,\ k \geqslant 0),$$

$$F = \exp(\alpha^{-1}\eta + t)G(\omega), \quad \omega = \exp(-\alpha^{-1}\eta - t)u_x, \quad \eta = x - kt.$$

$A_{4.10}$-invariant equations,

$$(1)\ \langle \sin t\partial_u, \cos t\partial_u, \partial_x + u\partial_u, \partial_t + k\partial_x \rangle \quad (k \geqslant 0),$$

$$F = -u + e^{\eta}G(\omega), \quad \omega = e^{-\eta}u_x, \quad \eta = x - kt.$$

In the above formulas $G=G(\omega)$ is an arbitrary function satisfying the condition $F_{u_x u_x} \neq 0$.

CONCLUDING REMARKS

Let us briefly summarize the results obtained in this paper.

We prove that the problem of group classification of the general quasilinear hyperbolic type equation (1.1) reduces to classifying equations of more specific forms,

$$(\mathrm{I})\ u_{tt} = u_{xx} + F(t,x,u,u_x), \quad F_{u_x u_x} \neq 0,$$

$$(\mathrm{II})\ u_{tt} = u_{xx} + g(t,x,u)u_x + f(t,x,u), \quad g_u \neq 0,$$

$$(\mathrm{III})\ u_{tx} = g(t,x)u_x + f(t,x,u), \quad g_x \neq 0, \quad f_{uu} \neq 0,$$

$$(\mathrm{IV})\ u_{tx} = f(t,x,u), \quad f_{uu} \neq 0.$$

If we denote as $\mathcal{DE}$ the set of PDEs (II)–(III), then the results of application of our algorithm for group classification of equations (I)–(IV) can be summarized as follows.

(1) We perform complete group classification of the class $\mathcal{DE}$. We prove that the Liouville equation has the highest symmetry properties among equations from $\mathcal{DE}$. Next, we prove that the only equation belonging to this class and admitting the four-dimensional invariance algebra is the nonlinear d'Alembert equations. It is established that there are 12 inequivalent equations from $\mathcal{DE}$ invariant under three-dimensional Lie algebras. We give the lists of all inequivalent equations from $\mathcal{DE}$ that admit one- and two-dimensional symmetry algebras.

(2) We have studied the structure of invariance algebras admitted by nonlinear equations from the class (I). It is proved, in particular, that their invariance algebras are necessarily solvable.

(3) We perform complete group classification of nonlinear equations from the class of PDEs (I). We prove that the highest symmetry algebras admitted by those equations are five dimen-

sional and construct all inequivalent classes of equations invariant with respect to five-dimensional Lie algebras. We also construct all inequivalent equations of the form (I) admitting one-, two-, three-, and four-dimensional Lie algebras.

In one of our future papers we intend to exploit the obtained classification results to construct exact solutions of nonlinear wave equations (I)–(IV).

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