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Quasilattice approximation of statistical systems with strong superstable interactions: Correlation functions

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A continuous infinite system of point particles interacting via two-body strong superstable potential is considered in the framework of classical statistical mechanics. We define some kind of approximation of main quantities, which describe macroscopical and microscopical characteristics of systems, such as grand partition function and correlation functions. The pressure of an approximated system converges to the pressure of the initial system if the parameter of approximation $a \to 0$ for any values of an inverse temperature $\beta > 0$ and a chemical activity z. The same result is true for the family of correlation functions in the region of small z. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3081054]

I. INTRODUCTION

The main achievements of mathematical physics in research of critical phenomena are connected first of all with studying infinite lattice systems. However, one can see totally another situation concerning continuous systems. The mathematical results have been obtained in the majority of cases only for the small values of parameters $\beta = 1/kT$ (where T is a temperature) and a chemical activity z. The research of continuous systems in the area of critical values of these parameters is restricted to some artificial models such as the Widom–Rowlinson model[25] or with field theory of type Hamiltonian,[9] and the methods of investigation are copied from lattice systems (see, e.g., Refs. 22 and 10, using Peierls' argument[2] and using the Pirogov–Sinai theory, or Refs. 3 and 4, using random cluster expansion). Another type of arguments was invented by Gruber and Griffiths[5] and used in Refs. 19 and 6 to prove the existence of orientational ordering transitions in the continuous-spin models of ferrofluid.

Some important characteristics of critical phenomena can be also described by using lattice approximation of continuous systems. It was especially successful to apply lattice approximation to research of the models of quantum field theory (see, e.g., Ref. 23 and references therein). Substantial progress was also reached in studying models of lattice gas.[20] However, the main disadvantage of the last example is that it does not contain the parameter that ensures the transition to the classical continuous gas.

On the other hand the main mathematical problems in the research of infinite continuous systems appear because it is necessary to take into account all possible configurations of particles, even if the probability of their occurrence is rather small. One of possible ways to solve this problem is to introduce hard-core potentials. It helps to avoid mathematical difficulties, which is connected with an accumulation of many numbers of particles in the small volume, but at the same time it leads to some new problems, which is connected with interpretation of physical results and application of some mathematical methods.

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In the present article we propose some intermediate approximation of several main quantities, which describe macroscopical and microscopical characteristics of systems, such as grand partition function and correlation functions. The main idea is in the following: we split the space $\mathbb{R}^d$ into nonintersecting hypercubes with a volume a^d and define approximated grand partition function and the family of approximated correlation functions in such a way that they take into account only such configurations of particles in $\mathbb{R}^d$, when there is not more than one particle in each cube.

It was shown in this work that for the potentials which have nonintegrable singularity in the neighborhood of the origin [strong superstable (SSS) potentials] the pressure of the approximated system converges to the pressure of the initial system if $a \to 0$ for any value of an inverse temperature $\beta > 0$ and a chemical activity z. The same result is true for the family of correlation functions in the region of small z.

II. NOTATIONS AND MAIN RESULTS

A. Configuration space

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. The set of positions $\{x_i\}_{i\in\mathbb{N}}$ of identical particles is considered to be a locally finite subset in $\mathbb{R}^d$ and the set of all such subsets creates the configuration space

$$\Gamma = \Gamma_{\mathbb{R}^d} := \{\gamma \subset \mathbb{R}^d \| \gamma \cap \Lambda| < \infty \ \text{ for all } \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\},$$

where $|A|$ denotes the cardinality of the set A and $\mathcal{B}_c(\mathbb{R}^d)$ denote the systems of all bounded Borel sets in $\mathbb{R}^d$. We also need to define the space of finite configurations Γ_0:

$$\Gamma_0 = \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{\eta \subset \mathbb{R}^d \| \eta| = n\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

For every $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one can define a mapping $N_\Lambda : \Gamma \to \mathbb{N}_0$ of the form

$$N_\Lambda(\gamma) := |\gamma \cap \Lambda| = |\gamma_\Lambda|.$$

The Borel σ-algebra $\mathfrak{B}(\Gamma)$ is equal to $\sigma(N_\Lambda | \Lambda \in \mathfrak{B}_c(\mathbb{R}^d))$ and additionally one may introduce the following filtration:

$$\mathfrak{B}_\Lambda(\Gamma) := \sigma(N_{\Lambda'} | \Lambda' \in \mathcal{B}_c(\mathbb{R}^d), \quad \Lambda' \subset \Lambda),$$

see Refs. 11, 12, and 1 for details. We need also to define

$$\Gamma_\Lambda := \{\eta \in \Gamma_0 | \eta \subset \Lambda\}.$$

By $\mathfrak{B}(\Gamma_\Lambda)$ we denote the corresponding σ-algebra on Γ_Λ. For the given intensity measure σ [in this context σ is Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$] and any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered as a measure on

$$\widetilde{(\mathbb{R}^d)^n} = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n | x_k \neq x_l \ \text{ if } \ k \neq l\}$$

and hence as a measure $\sigma^{(n)}$ on $\Gamma^{(n)}$ through the map

$$\mathrm{sym}_n : \widetilde{(\mathbb{R}^d)^n} \ni (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma^{(n)}.$$

Define the Lebesgue–Poisson measure $\lambda_{z\sigma}$ on $\mathfrak{B}(\Gamma_0)$ by the formula

$$\lambda_{z\sigma} := \sum_{n\geq 0} \frac{z^n}{n!} \sigma^{(n)}. \tag{2.1}$$

The restriction of $\lambda_{z\sigma}$ to $\mathfrak{B}(\Gamma_\Lambda)$ we also denote by $\lambda_{z\sigma}$. For a more detailed structure of the configuration spaces Γ, Γ_0, Γ_Λ see Ref. 1.

As in Ref. 16 define two additional configuration spaces: a space of *dilute* configurations and a space of *dense* configurations. Let $a>0$ be arbitrary. Following Ref. 21 for each $r\in\mathbb{Z}^d$ we define an elementary cube with an edge a and a center r,

$$\Delta_a(r) := \{x\in\mathbb{R}^d | a(r^i-1/2)\le x^i < a(r^i+1/2)\}. \tag{2.2}$$

We will write Δ instead of $\Delta_a(r)$ if a cube Δ is considered to be arbitrary and there is no reason to emphasize that it is centered at the concrete point $r\in\mathbb{Z}^d$. Let $\bar{\Delta}_a$ be the partition of $\mathbb{R}^d$ into cubes $\Delta_a(r)$. Without loss of generality consider only that $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ which is a union of cubes $\Delta_a(r)$. Then for any $X\subseteq\Lambda$ which is a union of cubes $\Delta\in\bar{\Delta}_a$ define

$$\Gamma_X^{\mathrm{dil}} := \{\gamma\in\Gamma_X | N_\Delta(\gamma)=0\vee 1 \text{ for all } \Delta\subset X\} \tag{2.3}$$

and

$$\Gamma_X^{\mathrm{den}} := \{\gamma\in\Gamma_X | N_\Delta(\gamma)\ge 2 \text{ for all } \Delta\subset X\}. \tag{2.4}$$

B. Definition of the system

The energy of any configuration $\gamma\in\Gamma_\Lambda$ or $\gamma\in\Gamma_0$ is defined by the following formula:

$$U_\phi(\gamma)=U(\gamma) := \sum_{\{x,y\}\subset\gamma}\phi(|x-y|), \tag{2.5}$$

where $\{\cdot,\cdot\}$ means sum over all possible different couples of particles from the configuration γ, $\phi(|x-y|)$-pair interaction potential. Define also interaction energy between configurations $\eta,\gamma\in\Gamma_0$ by

$$W(\eta;\gamma) := \sum_{\substack{x\in\eta\\ y\in\gamma}}\phi(|x-y|). \tag{2.6}$$

We introduce three kinds of interactions, which will be used in this article.

Definition 1: *Interaction is called*

(a) *stable (S) if there exists $B>0$ such that*

$$U(\gamma)\ge -B|\gamma| \quad \text{for any } \gamma\in\Gamma_0; \tag{2.7}$$

(b) *superstable (SS) if there exist $A(a)>0$, $B(a)\ge 0$, and $a>0$ such that*

$$U(\gamma)\ge A(a)\sum_{\Delta\in\bar{\Delta}_a:|\gamma_\Delta|\ge 2}|\gamma_\Delta|^2 - B(a)|\gamma| \quad \text{for any } \gamma\in\Gamma_0; \tag{2.8}$$

(c) *SSS if there exist $A(a)>0$, $B(a)\ge 0$, $m\ge 2$, and $a_0>0$ such that*

$$U(\gamma)\ge A(a)\sum_{\Delta\in\bar{\Delta}_a:|\gamma_\Delta|\ge 2}|\gamma_\Delta|^m - B(a)|\gamma| \quad \text{for any } \gamma\in\Gamma_0 \tag{2.9}$$

for any $a\le a_0$.

In the above conditions constants $A(a)$, $B(a)$ depend on $\bar{\Delta}_a$ and consequently on a. In accordance with these definitions there is a problem of describing the necessary conditions on two-body potential, which ensure stability, superstability, or strong superstability of an infinite statistical system. For the latest review and some new results on this problem see Refs. 18 and 24 for the many-body case.

(A) Assumption on the interaction potential: *In this article we consider a general type of potentials ϕ, which are continuous on $\mathbb{R}_+\setminus\{0\}$ and for which there exist $r_0>0$, $R>r_0$, $\varphi_0>0$, $\varphi_1>0$, $s\ge d$, and $\varepsilon_0>0$ such that*

(1)

$$\phi(|x|) \equiv -\phi^{-}(|x|) \geq -\frac{\varphi_1}{|x|^{d+\varepsilon_0}} \quad \text{for } |x| \geq R, \tag{2.10}$$

(2)

$$\phi(|x|) \equiv \phi^{+}(|x|) \geq \frac{\varphi_0}{|x|^{s}} \quad \text{for } |x| \leq r_0, \tag{2.11}$$

where

$$\phi^{+}(|x|) := \max\{0, \phi(|x|)\}, \quad \phi^{-}(|x|) := -\min\{0, \phi(|x|)\}. \tag{2.12}$$

Note that in Eq. (2.9) the constant $a_0 \leq r_0$. For the interaction potentials which satisfy assumption (A) define two important characteristics (for any $\Delta \in \bar{\Delta}_a$ with $a < r_0$):

(1)

$$v_0(a) := \sum_{\Delta' \in \bar{\Delta}_a} \sup_{x\in\Delta} \sup_{y\in\Delta'} \phi^{-}(|x-y|), \tag{2.13}$$

(2)

$$b(a) := \inf_{\{x,y\}\subset\Delta} \phi^{+}(|x-y|). \tag{2.14}$$

Due to the translation invariance of the two-body potential v_0 and b do not depend on the position of Δ. The following statement is true.

Proposition 2.1: *Let potential ϕ satisfy assumption (A). Then the interaction is SSS and the energy U satisfies inequality (2.9) with some $0 < a < a_0$ and if $s > d$ then*

$$m = 2, \quad A = A(a) = \frac{b - 2v_0}{4} > 0, \quad B = B(a) = \frac{v_0}{2}. \tag{2.15}$$

Proof: For any $\gamma \in \Gamma_0$ and any $a > 0$

$$U(\gamma) = \sum_{\{x,y\}\subset\gamma} \phi(|x-y|) = \sum_{\Delta\in\bar{\Delta}_a:|\gamma_\Delta|\geq 2} \sum_{\{x,y\}\subset\gamma_\Delta} \phi(|x-y|) + \sum_{\{\Delta,\Delta'\}\subset\bar{\Delta}_a} \sum_{\substack{x\in\gamma_\Delta \\ y\in\gamma_{\Delta'}}} \phi(|x-y|)$$

$$\geq \sum_{\Delta\in\bar{\Delta}_a:|\gamma_\Delta|\geq 2} \frac{1}{2}|\gamma_\Delta|(|\gamma_\Delta|-1)b - \sum_{\{\Delta,\Delta'\}\subset\bar{\Delta}_a:|\gamma_\Delta|\geq 2,|\gamma_{\Delta'}|\geq 2} |\gamma_\Delta||\gamma_{\Delta'}| \sup_{x\in\gamma_\Delta} \sup_{y\in\gamma_{\Delta'}} \phi^{-}(|x-y|) - \frac{v_0}{2}|\gamma|$$

$$\geq \sum_{\Delta\in\bar{\Delta}_a:|\gamma_\Delta|\geq 2} |\gamma_\Delta|^2\left(\frac{b}{4} - \frac{v_0}{2}\right) - \frac{v_0}{2}|\gamma|.$$

We use definitions (2.12)–(2.14) and the inequality

$$|\gamma_\Delta||\gamma_{\Delta'}| \leq \tfrac{1}{2}(|\gamma_\Delta|^2 + |\gamma_{\Delta'}|^2).$$

In the case $s = d$ the following statement is true (see Ref. 18 for details): for any sufficiently small $\varepsilon > 0$ there exists a constant $B = B(\varepsilon, a)$ such that the following inequality holds:

$$U(\gamma) \geq \sum_{\substack{\Delta \in \Delta_a, \\ |\gamma_\Delta| \geq 2}} \left(C_d \log|\gamma_\Delta| - \frac{v_0}{2} - \varepsilon \log|\gamma_\Delta| \right) |\gamma_\Delta|^2 - B|\gamma|, \tag{2.16}$$

where (see Ref. 7)

$$C_d = \frac{1}{a^d} \frac{\pi^{d/2}}{d\Gamma\left(\frac{d}{2}\right)} \varphi_0. \tag{2.17}$$

$\Gamma(\cdot)$ is a classical gamma function.

The system of particles is SSS because for any $\varepsilon > 0$ one can find such numbers $N_0 \geq 2$ and $B = B(N_0; \varepsilon, a)$ that for any $|\gamma_\Delta| > N_0$,

$$C_d \log|\gamma_\Delta| > \frac{v_0}{2}. \tag{2.18}$$

It follows from (2.16)–(2.18) that if $s = d$ we can set

$$A(a) = K_s(\varepsilon) v_0, \quad B(a) = L_s(\varepsilon) v_0 + M_s(\varepsilon),$$

where $K_s(\varepsilon)$, $L_s(\varepsilon)$, $M_s(\varepsilon)$ do not depend on the parameter a.

In the sequel we will use the estimates (2.15) of the constants $A(a)$ and $B(a)$ because the proof of the main results is the same for both cases. ■

Proposition 2.2: *For the potentials which satisfy conditions (2.10)–(2.12) inequality (2.7) holds with*

$$B = \left(\frac{2^{2d-s} d^{sd/2} \phi_0^s}{\varphi_0^d} \right)^{1/(s-d)}, \tag{2.19}$$

where the constant ϕ_0 is close to $\int_{\mathbb{R}^d} \phi^-(|x|) dx$ for small $a > 0$.

Proof: We can set $a = a_m$ in such a way that $b(a_m) = 2v_0(a_m)$. From definitions (2.13) and (2.14) it is clear that

$$b(a_m) \geq \frac{\varphi_0}{d^{s/2} a_m^s}$$

and $v_0(a_m) = (1/a_m^d)\phi_0$ as $\lim_{a\to 0} a^d v_0(a) = \int_{\mathbb{R}^d} \phi^-(|x|) dx$. As a result

$$a_m \geq \frac{\left(\frac{\varphi_0}{2\phi_0} \right)^{1/(s-d)}}{d^{s/2(s-d)}}. \tag{2.20}$$

Estimate (2.19) of the constant B directly follows from (2.20) and (2.15). This ends the proof. ■

Remark 2.1: *It is important to stress that the constant B in (2.19) does not depend on the partition $\bar{\Delta}_a$ and depends only on the potential ϕ and dimension of the space.*

Remark 2.2: *Indeed, for the potentials which satisfy assumption (A) inequality (2.9) holds with $m = 1 + s/d$ (see Ref. 18). However, for our purpose it is sufficient to apply (2.9) with (2.15) and (2.7) with (2.19).*

C. Partition functions, corresponding pressure, and correlation functions

The main characteristics of Gibbs states are correlation functions.[8] A family of finite volume correlation functions with empty boundary conditions for the grand canonical ensemble is defined by the following formula:

$$\rho_{\Lambda}(\eta;z,\beta) := \frac{z^{|\eta|}}{Z_{\Lambda}(z,\beta)}\int_{\Gamma_{\Lambda}} e^{-\beta U(\eta\cup\gamma)}\lambda_{z\sigma}(d\gamma), \quad \eta\in\Gamma_{\Lambda}, \tag{2.21}$$

where

$$Z_{\Lambda}(z,\beta) := \int_{\Gamma_{\Lambda}} e^{-\beta U(\gamma)}\lambda_{z\sigma}(d\gamma) \tag{2.22}$$

is the grand partition function which plays the role of normalizing constant in the definition of the Gibbs measure. Besides it has independent important physical meaning for the definition of the thermodynamic function–pressure:

$$p(z,\beta) = \lim_{|\Lambda|\to\infty} p_{\Lambda}(z,\beta) = \frac{1}{\beta}\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\log Z_{\Lambda}(z,\beta), \tag{2.23}$$

The existence of this limit for the above defined system of particles is well known (see, e.g., Ref. 21).

To define the above mentioned approximation let us introduce the following family of correlation functions:

$$\rho_{\Lambda}^{(-)}(\eta;z,\beta,a) := \frac{z^{|\eta|}}{Z_{\Lambda}^{(-)}(z,\beta,a)}\int_{\Gamma_{\Lambda}} e^{-\beta U(\eta\cup\gamma)}\prod_{\Delta\in\bar{\Delta}_a\cap\Lambda}\chi_{-}^{\Delta}(\eta\cup\gamma)\lambda_{z\sigma}(d\gamma), \quad \eta\in\Gamma_{\Lambda}, \tag{2.24}$$

$$Z_{\Lambda}^{(-)}(z,\beta,a) := \int_{\Gamma_{\Lambda}^{dil}} e^{-\beta U(\gamma)}\lambda_{z\sigma}(d\gamma) = \int_{\Gamma_{\Lambda}} e^{-\beta U(\gamma)}\prod_{\Delta\in\bar{\Delta}_a\cap\Lambda}\chi_{-}^{\Delta}(\gamma)\lambda_{z\sigma}(d\gamma). \tag{2.25}$$

where we introduced $\mathfrak{B}_{\Delta}(\Gamma_{\Lambda})$ measurable function χ_{-}^{Δ} by the formula

$$\chi_{-}^{\Delta}(\gamma) = \begin{cases} 1 & \text{for } \gamma \text{ with } N_{\Delta}(\gamma) = |\gamma_{\Delta}| = 0 \vee 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

Remark 2.3: *By definition* $\rho_{\Lambda}^{(-)}(\eta;z,\beta;a)=0$ *for* $\eta\notin\Gamma_{\Lambda}^{(dil)}$.

One can define the corresponding pressure:

$$p^{(-)}(z,\beta,a) = \lim_{|\Lambda|\to\infty} p_{\Lambda}^{(-)}(z,\beta,a) = \frac{1}{\beta}\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\log Z_{\Lambda}^{(-)}(z,\beta,a). \tag{2.27}$$

Remark 2.4: *The main point of this approximation consists in that in expressions for the basic characteristics of the system integration is carried out not over all space of configurations* Γ_{Λ} *but only over those configurations which contain for the given partition* $\bar{\Delta}_a$ *not more than one particle in each cube* $\Delta\in\bar{\Delta}_a$*. That fact is surprising as for an infinite system the set of such configurations in* Γ *is the set of measure zero with respect to the Poisson measure and the Gibbs measure. Nevertheless, as we shall see in following section, the basic characteristics of the approximated system (even in a thermodynamic limit* $\Lambda\nearrow\mathbb{R}^d$*) can be somehow close to the corresponding characteristics of the initial system.*

D. Main results

We prove the results for the infinite volume characteristics; so let us define the sequence of bounded Lebesgue measurable regions of $\Lambda_l\subset\mathbb{R}^d$:

$$\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda_n\subset\cdots, \quad \bigcup_l \Lambda_l = \mathbb{R}^d. \tag{2.28}$$

We consider only such $\Lambda_l\in\mathcal{B}_c(\mathbb{R}^d)$ which is union of cubes $\Delta_a(r)$ defined by (2.2).

Theorem 2.1: *Let the interaction potential $\phi(|x|)$ satisfy assumption (A). Then the limits*

$$p(z,\beta) = \frac{1}{\beta}\lim_{l\to\infty}\frac{1}{|\Lambda_l|}\log Z_{\Lambda_l}(z,\beta), \tag{2.29}$$

$$p^{(-)}(z,\beta,a) = \frac{1}{\beta}\lim_{l\to\infty}\frac{1}{|\Lambda_l|}\log Z^{(-)}_{\Lambda_l}(z,\beta,a) \tag{2.30}$$

are finite and for any $\varepsilon>0$ there exists $a_1=a_1(z,\varepsilon)>0$ such that

$$|p(z,\beta) - p^{(-)}(z,\beta,a)| < \varepsilon \tag{2.31}$$

holds for all positive z, β and $a\in(0,a_1(z,\varepsilon))$.

The proof of limit (2.29) is well known (see Ref. 21). One can find the proof of (2.30) and (2.31) in Ref. 17. However, for completeness of the presentation we give a sketch of the proof in the next section.

To formulate a similar result for correlation functions note that for any configuration $\eta\in\Gamma_0$ and any sequence (2.28), such that $\eta\subset\Lambda_1$, there exists subsequence (Λ'_k) of (Λ_l), such that

$$\lim_{k\to\infty}\rho_{\Lambda'_k}(\eta;z,\beta) = \rho(\eta;z,\beta) < \infty \tag{2.32}$$

for all positive z, β uniformly on $\mathfrak{B}_c(\Gamma_0)$. This result follows from the uniform bounds of the family $\{\rho_\Lambda:\Lambda\in\mathfrak{B}_c(\mathbb{R}^d)\}$ (see Refs. 21, 16, and 14).

It is also clear that the same uniform bounds hold for the family of $\{\rho^{(-)}_\Lambda:\Lambda\in\mathfrak{B}_c(\mathbb{R}^d)\}$. So, there exists subsequence (Λ''_m) of the sequence (Λ'_k) such that one can define

$$\rho^{(-)}(\eta;z,\beta,a) = \lim_{m\to\infty}\rho^{(-)}_{\Lambda''_m}(\eta;z,\beta,a) < \infty. \tag{2.33}$$

In the case of small values of a chemical activity z there exists the unique limit $\rho(\eta;z,\beta)$ that is a solution of Kirkwood–Salzburg (KS) equations in the space E_ξ (see Ref. 20). In the next chapter we will show that similar equations can be easily written for the functions $\rho^{(-)}(\eta;z,\beta)$ that is a unique solution of these equations for sufficiently small values of parameters z or β.

Theorem 2.2: *Let the interaction potential $\phi(|x|)$ satisfy assumption (A). Then for any $\varepsilon>0$, sufficiently small z, and any configuration $\eta\in\Gamma_0$ there exists $a_1=a_1(z,\beta,\varepsilon)>0$ such that*

$$|\rho(\eta;z,\beta) - \rho^{(-)}(\eta;z,\beta,a)| < \varepsilon \tag{2.34}$$

holds for all $a\in(0,a_1(z,\varepsilon))$.

Corollary 2.1: *Inequalities (2.31) and (2.34) ensure the existence of limits*

$$\lim_{a\to 0}p^{(-)}(z,\beta,a) = p(z,\beta) \tag{2.35}$$

for any positive $z,\beta>0$, $\eta\in\Gamma_0$ and

$$\lim_{a\to 0}\rho^{(-)}(\eta;z,\beta,a) = \rho(\eta;z,\beta) \tag{2.36}$$

for small positive z, any $\beta>0$, and $\eta\in\Gamma_0$.

III. PROOF OF THEOREM 2.1

The proof is based on the expansion which was proposed in Ref. 16. In order to arrange this expansion let us define also an indicator of a dense configuration in any cube $\Delta\in\bar{\Delta}_a$ as

$$\chi_+^\Delta(\gamma) := 1 - \chi_-^\Delta(\gamma).$$

Then we use the following partition of the unity for any $\gamma \in \Gamma_\Lambda$:

$$1 = \prod_{\Delta\subset\Lambda} [\chi_-^\Delta(\gamma) + \chi_+^\Delta(\gamma)] = \sum_{n=0}^{N_\Lambda} \sum_{\{\Delta_1,\ldots,\Delta_n\}\subset\Lambda} \prod_{i=1}^{n} \chi_+^{\Delta_i}(\gamma) \prod_{\Delta\subset\Lambda\setminus\cup_{i=1}^n\Delta_i} \chi_-^\Delta(\gamma) = \sum_{X\subseteq\Lambda} \tilde{\chi}_+^X(\gamma)\tilde{\chi}_-^{\Lambda\setminus X}(\gamma), \tag{3.1}$$

where $N_\Lambda = |\Lambda|/a^d$ (here the symbol $|\cdot|$ means Lebesgue measure of the set Λ) is the number of cubes Δ in the volume Λ, and

$$\tilde{\chi}_\pm^X(\gamma) := \prod_{\Delta\subset X} \chi_\pm^\Delta(\gamma). \tag{3.2}$$

Inserting (3.1) into (2.22) we obtain

$$Z_\Lambda(z,\beta) = \sum_{X\subseteq\Lambda} \int_{\Gamma_\Lambda} e^{-\beta U(\gamma)}\tilde{\chi}_+^X(\gamma)\tilde{\chi}_-^{\Lambda\setminus X}(\gamma)\lambda_{z\sigma}(d\gamma). \tag{3.3}$$

It is obvious that the first term in (3.3) (at $X=\varnothing$) coincides with $Z_\Lambda^{(-)}(z,\beta,a)$ [see (2.25)]. Using infinite divisible property of the Lebesgue–Poisson measure [see, for example, (2.5) in Ref. 15] one deduce that

$$Z_\Lambda(z,\beta) = Z_\Lambda^{(-)}(z,\beta,a)\left[1 + \sum_{\varnothing\neq X\subseteq\Lambda} \int_{\Gamma_X} \tilde{\rho}_{\Lambda\setminus X}^{(-)}(\gamma;a)\tilde{\chi}_+^X(\gamma)\lambda_{z\sigma}(d\gamma)\right] := Z_\Lambda^{(-)}(z,\beta,a)Z_\Lambda^{(+)}(z,\beta,a), \tag{3.4}$$

where

$$\tilde{\rho}_{\Lambda\setminus X}^{(-)}(\gamma_X;a) = \frac{e^{-\beta U(\gamma_X)}}{Z_\Lambda^{(-)}(z,\beta,a)} \int_{\Gamma_{\Lambda\setminus X}} e^{-\beta W(\gamma_X|\gamma')-\beta U(\gamma')}\tilde{\chi}_-^{\Lambda\setminus X}(\gamma')\lambda_{z\sigma}(d\gamma'). \tag{3.5}$$

We also define $p^{(+)}(z,\beta,a)$ in the same way as in (2.30),

$$p^{(+)}(z,\beta,a) = \lim_{l\to\infty} p_{\Lambda_l}^{(+)}(z,\beta,a) = \frac{1}{\beta}\lim_{l\to\infty}\frac{1}{|\Lambda_l|}\log Z_{\Lambda_l}^{(+)}(z,\beta,a). \tag{3.6}$$

Consequently, in order to prove Theorem 2.1 we have to estimate the value of $p^{(+)}(z,\beta,a)$. Using Proposition 2.1 [Eqs. (2.9) and (2.15)] one can obtain

$$e^{-\beta U(\gamma_X)} \le \prod_{\Delta\in\overline{\Delta_a}\cap X} e^{-\beta A(a)|\gamma_\Delta|^2+\beta B(a)|\gamma_\Delta|}, \quad A(a) = \frac{b-2v_0}{4}, \quad B(a) = \frac{v_0}{2}. \tag{3.7}$$

Taking into account assumption (A) [Eq. (2.10)] and (2.13) we obtain

$$e^{-\beta W(\gamma_X|\gamma')} \le \prod_{\Delta\in\overline{\Delta_a}\cap X} e^{\beta v_0|\gamma_\Delta|}. \tag{3.8}$$

Using the infinite divisible property of the measure $\lambda_{z\sigma}$ and using (3.7) and (3.8) we have

$$\int_{\Gamma_X} \tilde{\rho}_{\Lambda\setminus X}^{(-)}(\gamma_X;a)\tilde{\chi}_+^X(\gamma)\lambda_{z\sigma}(d\gamma) \le \frac{Z_{\Lambda\setminus X}^{(-)}(z,\beta,a)}{Z_\Lambda^{(-)}(z,\beta,a)} \prod_{\Delta\in\overline{\Delta_a}\cap X} \int_{\Gamma_\Delta} e^{-\beta A|\gamma_\Delta|^2+\beta B|\gamma_\Delta|+\beta v_0|\gamma_\Delta|}\chi_+^\Delta(\gamma_\Delta)\lambda_{z\sigma}(d\gamma_\Delta).$$

As a result, using the definition of the Lebesgue–Poisson measure [see (2.1)] one can obtain the following estimate:

$$\int_{\Gamma_\Delta} e^{-\beta A|\gamma_\Delta|^2+\beta(B+v_0)|\gamma_\Delta|}\chi_+^\Delta(\gamma_\Delta)\lambda_{z\sigma}(d\gamma_\Delta) = \sum_{n=2}^{\infty}\frac{(a^d z)^n}{n!}e^{-1/4\beta(b-2v_0)n^2+3/2\beta v_0 n} \le \varepsilon_1(a), \tag{3.9}$$

with

$$\epsilon_1(a) = \tfrac{1}{2}z^2 a^{2d} e^{-\beta(b-5v_0)} \exp\{z a^d e^{-\beta(b-3v_0)}\}. \tag{3.10}$$

Now from the definition of N_Λ, $Z_\Lambda^{(+)}(z,\beta,a)$ [see (3.4)] and the above estimates we have

$$\log Z_\Lambda^{(+)}(z,\beta,a) \le \log\left[1 + \sum_{\emptyset\neq X\subseteq\Lambda}\epsilon_1(a)^{N_X}\right] = \log\left[1 + \sum_{k=1}^{N_\Lambda}\frac{N_\Lambda!}{k!\,(N_\Lambda-k)!}\epsilon_1(a)^k\right] = \log[1+\epsilon_1(a)]^{N_\Lambda}$$
$$= \frac{|\Lambda|}{a^d}\log[1+\epsilon_1(a)].$$

As a result

$$p^{(+)}(z,\beta;a) \le \frac{1}{\beta a^d}\log[1+\epsilon_1(a)].$$

It is important for the proof of the theorem to find the asymptotic behavior of $\varepsilon_1(a)$ at $a\to 0$. It follows from Eq. (3.10) and the corresponding behavior of b and v_0 (see [(2.10)–(2.14)]. As a result we have

$$\epsilon_1(a) \sim a^{2d}e^{-1/a^s}, \quad s \ge d. \tag{3.11}$$

So,

$$\lim_{a\to 0} p^{(+)}(z,\beta;a) = 0.$$

This ends the proof. ■

IV. PROOF OF THEOREM 2.2

Using definitions (2.1) and (2.26) we can rewrite definition (2.24) for the family of correlation functions $\rho_\Lambda^{(-)}(\cdot;z,\beta,a)$ in the following form:

$$\rho_\Lambda^{(-)}(\eta;z,\beta,a) = \frac{z^{|\eta|}}{Z_\Lambda^{(-)}(z,\beta,a)}e^{-\beta U(\eta)}\prod_{\Delta\in\Lambda_\eta}\chi_-^\Delta(\eta_\Delta)\left[1 + \sum_{k=1}^{N_{\Lambda\setminus\Lambda_\eta}} z^k \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset(\Lambda\setminus\Lambda_\eta)\cap\bar{\Delta}_a}\right.$$
$$\left.\times\int_{\Delta_1}\cdots\int_{\Delta_k} e^{-\beta W(\eta;\{y_1,\ldots,y_k\})}e^{-\beta U(\{y_1,\ldots,y_k\})}dy_1\cdots dy_k\right], \tag{4.1}$$

where Λ_η is a union of cubes of $\bar{\Delta}_a$ which contain points from the configuration η (and in the sequel we will use such a notation) and summation is taken over all possible sets of cubes from $\bar{\Delta}_a$ that belong to the area $\Lambda\setminus\Lambda_\eta$. We prove the theorem using KS equations for the functions $\rho(\eta;z,\beta)$ and $\rho^{(-)}(\eta;z,\beta,a)$. Remind that KS equations for the functions $\rho(\eta;z,\beta)$ can be written in the form of a one operator equation (see Ref. 20),

$$\rho = z\tilde{K}\rho + z\delta, \tag{4.2}$$

where operator $\tilde{K}$ acts on an arbitrary function φ according with the rule

$$(\tilde{K}\varphi)(\{x_1\}) = \sum_{k=1}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{i=1}^{k}(e^{-\beta\phi(|y_i-x_1|)}-1)\times\varphi(\{y_1,\dots,y_k\})dy_1\cdots dy_k \quad \text{if}|\eta| = 1(\eta=\{x_1\}), \tag{4.3}$$

$$(\tilde{K}\varphi)(\eta) = \sum_{x\in\eta}\tilde{\pi}(x;\eta\setminus\{x\})e^{-\beta W(x;\eta\setminus\{x\})}\Bigg[\varphi(\eta\setminus\{x\}) + \sum_{k=1}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} \times\prod_{i=1}^{k}(e^{-\beta\phi(|y_i-x|)}-1)\varphi(\eta\setminus\{x\}\cup\{y_1,\dots,y_k\})dy_1\cdots dy_k\Bigg] \quad \text{if}|\eta|\geq 2, \tag{4.4}$$

where

$$\tilde{\pi}(x;\eta\setminus\{x\}) = \frac{\pi_W(x;\eta\setminus\{x\})}{\sum_{y\in\eta}\pi_W(y;\eta\setminus\{y\})}, \quad \pi_W(x;\eta\setminus\{x\}) = \begin{cases}1 & \text{if } W(x;\eta\setminus\{x\}) \geq -2B\\ 0 & \text{otherwise},\end{cases} \tag{4.5}$$

$$\rho := \{\rho(\eta;z,\beta)\}_{\eta\in\Gamma_0}. \tag{4.6}$$

$\delta(\eta)=1$ if $|\eta|=1$ and $\delta(\eta)=0$ otherwise.

Remark 4.1: *Operator $\tilde{K}=\Pi K$ in Ruelle's notation*[20] *and (4.4) and (4.5) are exact realization of the operator Π.*

Operator $\tilde{K}$ is a bounded operator in Banach space of measurable bounded functions $E_\xi(\xi > 0)$ with the norm

$$\|\varphi\|_\xi = \sup_{\eta\in\Gamma_0}|\varphi(\eta)|\xi^{-|\eta|}. \tag{4.7}$$

The solution of Eq. (4.2) can be represented in the form of convergent in E_ξ (and point convergent for any fixed $\eta\in\Gamma_0$) series,

$$\rho(\eta;z,\beta) = \sum_{n=0}^{\infty} z^{n+1}(\tilde{K}^n\delta)(\eta;z,\beta), \tag{4.8}$$

if

$$|z| \leq e^{-2\beta B-1}C(\beta)^{-1}, \quad C(\beta) = \int_{\mathbb{R}^d}|e^{-\beta\phi(|x|)}-1|dx \tag{4.9}$$

and the interaction satisfies the conditions (2.7), (2.10), and (2.11).

One can write similar equations for the functions $\rho_\Lambda^{(-)}(\eta;z,\beta,a)$. It can be easily done in the way like it was shown in Ref. 13 for the case of lattice gas. Let us proceed with several new notations that correspond to the notations in the space of configurations in the lattice gas system (see Ref. 13). Define the space $C=C_{\bar{\Delta}_a}$ of configurations of cubes from $\bar{\Delta}_a$. Let $s=\{\Delta_\eta^1,\dots,\Delta_\eta^{|\eta|}\}$ be the finite configuration of $|\eta|$ cubes from $\bar{\Delta}_a$ with all points from the configuration $\eta\in\Gamma_0$ and $s'=s\setminus\{\Delta_\eta^1\}$. Let us denote by $C_{\bar{\Delta}_a}^{\text{fin}}$ a space of all finite configurations of cubes from C (see also Ref. 13) and $c=\{\Delta_1,\dots,\Delta_k\}\in C_{\bar{\Delta}_a}^{\text{fin}}$ be any finite configuration of k cubes from $\bar{\Delta}_a$ (if $k=0$ $c=\{\varnothing\}$).

For technical reason we also introduce a new potential,

$$\hat{\phi}(x,y) = \phi(|x-y|) + \phi^{\rm cor}_{\bar{\Delta}_a}(x,y),$$

where

$$\phi^{\rm cor}_{\bar{\Delta}_a}(x,y) = \begin{cases} +\infty & \text{if } x,y \in \Delta \in \bar{\Delta}_a \\ 0 & \text{if } x \in \Delta,\ y \in \Delta',\ \text{and } \Delta \neq \Delta'. \end{cases} \tag{4.10}$$

As in definition (4.1) all points of the configurations η, γ are situated in different cubes we can put the potential $\hat{\phi}$ instead of the potential ϕ in definitions (2.24) and (2.25). Let us define also a potential $\hat{\phi}(\Delta,\Delta')$ as the family of potentials $\hat{\phi}(x,y)$:

$$\hat{\phi}(\Delta,\Delta') = \{\hat{\phi}(x,y) | x \in \Delta, y \in \Delta'\},$$

$$\hat{\phi}(\Delta,\Delta) = +\infty \quad \text{for any } \Delta \in \bar{\Delta}_a. \tag{4.11}$$

Remark 4.2: *For $c=\{\Delta_1,\ldots,\Delta_k\}$, $s=\{\Delta^1_\eta,\ldots,\Delta^m_\eta\}$, $m=|\eta|$ the functions $U(c)$, $W(s;c)$, $\rho^{(-)}_\Lambda(s;z,\beta,a)$, $\rho^{(-)}(s;z,\beta,a)$ are the families [see (4.10)] of the corresponding $U(\gamma)$, $W(\eta;\gamma)$, $\rho^{(-)}_\Lambda(\eta;z,\beta,a)$, $\rho^{(-)}(\eta;z,\beta,a)$ with $\gamma=\{\gamma_{\Delta_1},\ldots,\gamma_{\Delta_k}\}$, $\eta=\{\eta_{\Delta^1_\eta},\ldots,\eta_{\Delta^m_\eta}\}$ and at $a\to 0$ every cube shrinks in the corresponding point so that $c\to\gamma$, $s\to\eta$.*

Configuration $\eta\in\Gamma_0$ in the definition of the function $\rho(\eta;z,\beta)$ is fixed and coordinates of cubes $\Delta^1_\eta,\ldots,\Delta^{|\eta|}_\eta$ in $\mathbb{R}^d$ change, but Lebesgue measure of Λ_η tends to zero [mess $\Lambda_\eta(a)\to 0$].

The energy $U(\gamma)$ of the configuration $\gamma\in\Gamma_X$, $X\subseteq\Lambda$ in these notations is

$$U(c) = \sum_{1\le i<j\le|c|} \hat{\phi}(\Delta_i,\Delta_j). \tag{4.12}$$

The energy of interaction between configurations of cubes $s,c\in C^{\rm fin}_{\bar{\Delta}_a}$ is

$$W(s;c) = \sum_{\Delta\in s,\Delta'\in c} \hat{\phi}(\Delta,\Delta'). \tag{4.13}$$

Then definition (4.1) for the functions $\rho^{(-)}_\Lambda(\eta;z,\beta,a)$ takes the form

$$\rho^{(-)}_\Lambda(s;z,\beta,a) = \frac{1}{Z^{(-)}_\Lambda(z,\beta,a)} \sum\!\!\!\!\!\!\int_{c\subseteq\Lambda\setminus s} z^{|s\cup c|} e^{-\beta U(s\cup c)}, \tag{4.14}$$

where we introduce the new notation

$$\sum\!\!\!\!\!\!\int_{c\subseteq X} f(c) = \sum_{k=0}^{|X|/a^d} \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset\bar{\Delta}_a\cap X} \int_{\Delta_1}\cdots\int_{\Delta_k} f(x_1,\ldots,x_k)dx_1\cdots dx_k. \tag{4.15}$$

Following standard procedure (see Ref. 13) one can rewrite (4.14) in the form of the KS equation for the family of correlation functions $\rho^{(-)}_\Lambda(s;z,\beta,a)$:

$$\rho^{(-)}_\Lambda(s;z,\beta,a) = z e^{-\beta W(\Delta^1_\eta;s')} \left\{ \rho^{(-)}_\Lambda(s';z,\beta,a) + \sum\!\!\!\!\!\!\int_{\substack{Q\subset\bar{\Delta}_a, Q\neq\emptyset \\ Q\cap s=\emptyset}} \prod_{\Delta'\in Q} (e^{-\beta\hat{\phi}(\Delta^1_\eta;\Delta')}-1) \rho^{(-)}_\Lambda(s'\cup Q;z,\beta,a) \right\}. \tag{4.16}$$

Like in the case of functions ρ_Λ and ρ Eq. (4.16) can be modified and rewritten in the form of a one operator equation,

$$\rho_\Lambda^{(-)} = z\tilde{K}_\Lambda^{(-)}\rho_\Lambda^{(-)} + z\delta_\Lambda, \tag{4.17}$$

and for the limit correlation functions $\rho^{(-)}$ we obtain

$$\rho^{(-)} = z\tilde{K}^{(-)}\rho^{(-)} + z\delta. \tag{4.18}$$

Operator $\tilde{K}^{(-)}$ acts on an arbitrary function $\varphi \in C_{\bar{\Delta}_a}^{\text{fin}}$ according with the rule

$$(\tilde{K}^{(-)}\varphi)(\{\Delta_\eta^1\};z,\beta,a) = \sum\!\!\!\!\!\!\int_{Q\subset\bar{\Delta}_a,Q\neq\varnothing} \prod_{\Delta'\in Q}(e^{-\beta\hat{\phi}(\Delta_\eta^1;\Delta')}-1)\varphi(Q;z,\beta,a) \tag{4.19}$$

for $|s|=1$ and

$$(\tilde{K}^{(-)}\varphi)(s;z,\beta,a) = \sum_{\Delta\in s}\tilde{\pi}(\Delta;s')e^{-\beta W(\Delta;s')} \times\left\{\varphi(s') + \sum\!\!\!\!\!\!\int_{\substack{Q\subset\bar{\Delta}_a,Q\neq\varnothing\\ Q\cap s=\varnothing}}\prod_{\Delta'\in Q}(e^{-\beta\hat{\phi}(\Delta,\Delta')}-1)\varphi(s'\cup Q)\right\} \quad \text{for } |s|\geq 2. \tag{4.20}$$

Proof of existence of the solutions of Eqs. (4.17) and (4.18) in the form of the convergent series

$$\rho_\Lambda^{(-)}(\cdot;z,\beta,a) = \sum_{n=0}^{\infty} z^{n+1}((\tilde{K}_\Lambda^{(-)})^n\delta)(\cdot;z,\beta,a), \tag{4.21}$$

$$\rho^{(-)}(\cdot;z,\beta,a) = \sum_{n=0}^{\infty} z^{n+1}((\tilde{K}^{(-)})^n\delta)(\cdot;z,\beta,a) \tag{4.22}$$

and the equality

$$\lim_{\Lambda\nearrow\mathbb{R}^d}\rho_\Lambda^{(-)}(s;z,\beta,a) = \rho^{(-)}(s;z,\beta,a), \quad s\in C_{\bar{\Delta}_a}^{\text{fin}}, \tag{4.23}$$

for z, β that yield conditions (4.9) can be done in a similar way as in Ref. 13. So, we have to show that solution (4.22) of Eq. (4.18) converges to the solution of KS equation (4.2) if $a\to 0$.

In the sequel in the expressions for the operators $\tilde{K}$, $\tilde{K}^{(-)}$ we will consider only the case $|s|\geq 2$, as the case $|s|=1$ is rather similar.

Due to the convergence of the series (4.21) and (4.22) uniformly in a it is sufficient to prove the point convergence $(\tilde{K}^{(-)})^n\delta\to\tilde{K}^n\delta$ for any $n\geq 1$. It implies obviously $\rho^{(-)}(\cdot;z,\beta,a)\to\rho(\cdot;z,\beta,a)$ if $a\to 0$ for sufficiently small values of a chemical activity z. To prove this statement let us use the method of mathematical induction. Let us set $n=1$ (base of induction). We have from (4.3), (4.4), (4.19), and (4.20)

$$(\tilde{K}^{(-)}\delta)(s) = \begin{cases} \displaystyle\int_{\mathbb{R}^d\setminus\Delta_\eta^1}(e^{-\beta\phi(|y-x_1|)}-1)dy & \text{if}|s|=1\\ \left(\displaystyle\prod_{\Delta\in\bar{\Delta}_a}\chi_-^\Delta(\eta)\right)e^{-\beta\phi(|x_2-x_1|)} & \text{if}|s|=2\\ 0 & \text{if}|s|>2, \end{cases}$$

$$(\widetilde{K}\delta)(\eta) = \begin{cases} \int_{\mathbb{R}^d} (e^{-\beta\phi(|y-x_1|)} - 1)dy & \text{if} |\eta| = 1 \\ e^{-\beta\phi(|x_2-x_1|)} & \text{if} |\eta| = 2 \\ 0 & \text{if} |\eta| > 2. \end{cases}$$

It is clear that $\widetilde{K}^{(-)}\delta \to \widetilde{K}\delta$ if $a \to 0$ in the sense of point convergence. It is useful to notice that $((\widetilde{K}^{(-)})^n\delta)(s) = (\widetilde{K}^n\delta)(\eta) = 0$ if $|s| > n+1$ $(|\eta| > n+1)$. Let us make the step of induction. Let $(\widetilde{K}^{(-)})^n\delta \to \widetilde{K}^n\delta$ in the sense of point convergence. Using this assumption we have to prove that $(\widetilde{K}^{(-)})^{n+1}\delta \to \widetilde{K}^{n+1}\delta$ in the same sense. It follows from (4.4) and (4.20) that $(|\eta| = |s| \geq 2)$

$$((\widetilde{K}^{(-)})^{n+1}\delta)(s) = \sum_{\Delta \in s} \widetilde{\pi}(\Delta; s')e^{-\beta W(\Delta; s')} \prod_{\Delta \in \bar{\Delta}_a} \chi_-^{\Delta}(\eta) \Bigg\{ ((\widetilde{K}^{(-)})^n\delta)(s') + \sum_{k=1}^{n-|s|+2} \sum_{\substack{Q \subset \bar{\Delta}_a, Q \neq \emptyset \\ Q \cap s = \emptyset, |Q| = k}} \prod_{\Delta' \in Q} (e^{-\beta\hat{\phi}(\Delta\Delta')} - 1)((\widetilde{K}^{(-)})^n\delta)(s' \cup Q) \Bigg\}, \tag{4.24}$$

$$(\widetilde{K}^{n+1}\delta)(\eta) = \sum_{x \in \eta} \widetilde{\pi}(x; \eta \setminus \{x\})e^{-\beta W(\{x_1\}; \eta \setminus \{x_1\})} \Bigg\{ (\widetilde{K}^n\delta)(\eta \setminus \{x_1\}) + \sum_{k=1}^{n-|\eta|+2} \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \prod_{1 \leq i \leq k} (e^{-\beta\phi(|y_i - x_1|)} - 1)(\widetilde{K}^n\delta)(\eta \setminus \{x_1\} \cup \{y_1, \ldots, y_k\})dy_1 \cdots dy_k \Bigg\}. \tag{4.25}$$

Note that $(\widetilde{K}^n\delta)(\eta \setminus \{x_1\} \cup \{y_1, \ldots, y_k\})$ and $((\widetilde{K}^{(-)})^n\delta)(s' \cup Q)$ are measurable bounded functions as operators $\widetilde{K}$, $\widetilde{K}^{(-)}$ are bounded in the spaces E_ξ with some $\xi > 0$. Besides because of stability condition (2.7): $\prod_{1 \leq i \leq k} (e^{-\beta\phi(|y_i - x_1|)} - 1) \leq |e^{2\beta B} - 1|^k < +\infty$. Then the proof of the theorem is based on one technical lemma.

Lemma 4.1: *Let $F_-(\cdot; a), F(\cdot) \in L^1(\mathbb{R}^{dk})$ be symmetric bounded functions of its variables and $\lim_{a \to 0} F_-(x_1, \ldots, x_k; a) = F(x_1, \ldots, x_k)$ for any $(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k$. Then the following equality is true:*

$$\lim_{a \to 0} \sum_{\{\Delta_1, \ldots, \Delta_k\} \subset \bar{\Delta}_a \setminus \Lambda_\eta} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k F_-(x_1, \ldots, x_k; a) = \frac{1}{k!} \int_{(\mathbb{R}^d)^k} F(x_1, \ldots, x_k)dx_1 \cdots dx_k. \tag{4.26}$$

Proof: See the Appendix.

The step of induction follows directly from (4.15), (4.24), and (4.25) and the statement of the lemma. The theorem is proven. ■

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APPENDIX: PROOF OF THE LEMMA 4.1

We have to prove that for any $\varepsilon > 0$ there exists a_ε that for any $a < a_\varepsilon$ the following estimate holds:

$$\left| \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset\bar{\Delta}_a\setminus\Lambda_\eta} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k F_-(x_1,\ldots,x_k;a) - \frac{1}{k!}\int_{(\mathbb{R}^d)^k} F(x_1,\ldots,x_k)dx_1\cdots dx_k \right| < \varepsilon. \tag{A1}$$

From the integrability conditions of the functions F_-, F one can obtain that for any $\varepsilon > 0$ there exists bounded $\Lambda_\varepsilon \subset \mathbb{R}^d$, such that

$$\Big| \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset\bar{\Delta}_a\setminus\Lambda_\eta} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k F_-(x_1,\ldots,x_k;a) - \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset(\bar{\Delta}_a\setminus\Lambda_\eta)\cap\Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k F_-(x_1,\ldots,x_k;a)\Big| < \frac{\varepsilon}{3} \tag{A2}$$

and

$$\left| \frac{1}{k!}\int_{(\mathbb{R}^d)^k} F(x_1,\ldots,x_k)dx_1\cdots dx_k - \frac{1}{k!}\int_{\Lambda_\varepsilon^k} F(x_1,\ldots,x_k)dx_1\cdots dx_k \right| < \frac{\varepsilon}{3}. \tag{A3}$$

Using (A1)–(A3) it is easy to notice that the proof of the lemma can be reduced to verification of the fact that for any $\varepsilon > 0$ there exists $a_\varepsilon = f(\varepsilon) > 0$ such that for any $a < a_\varepsilon$ the following estimate is true:

$$R = \left| \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset(\bar{\Delta}_a\setminus\Lambda_\eta)\cap\Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k F_-(x_1,\ldots,x_k;a) - \frac{1}{k!}\int_{\Lambda_\varepsilon^k} F(x_1,\ldots,x_k)dx_1\cdots dx_k \right| < \frac{\varepsilon}{3}. \tag{A4}$$

Dividing each integral over Λ_ε into the sum of integrals over $\Delta \in \bar{\Delta}_a \cap \Lambda_\varepsilon$ one can arrange two terms in (A4) into three ones to get the estimates

$$R \le R_1 + R_2 + R_3,$$

$$R_1 = \sum_{j=1}^{k-1} \sum_{\substack{\{k_1,\ldots,k_j\},\\ k_1+\cdots+k_j=k}} \frac{1}{k_1!\cdots k_j!} \sum_{\pi\in P_j}{}' \sum_{\{\Delta_1,\ldots,\Delta_j\}\subset\bar{\Delta}_a\cap\Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_1} dx_{k_{\pi(1)}} \cdots \int_{\Delta_j} dx_{k-k_{\pi(j)}+1} \cdots \times \int_{\Delta_j} |F(x_1,\ldots,x_k)| dx_k, \tag{A5}$$

$$R_2 = \sum_{\{\Delta_1,\ldots,\Delta_k\}\subset(\bar{\Delta}_a\setminus\Lambda_\eta)\cap\Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k |F_-(x_1,\ldots,x_k;a) - F(x_1,\ldots,x_k)|, \tag{A6}$$

$$R_3 = \sum_{\substack{\{\Delta_1,\ldots,\Delta_k\}\subset\bar{\Delta}_a\cap\Lambda_\varepsilon,\\ \{\Delta_1,\ldots,\Delta_k\}\cap\Lambda_\eta\neq\emptyset}} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k |F(x_1,\ldots,x_k)|, \tag{A7}$$

where P_j is a set of all permutations of numbers $\{1,\ldots,j\}$, but the sum $\Sigma'_{\pi\in P_j}$ means that we consider only different permutations of numbers $\{k_1,\ldots,k_j\}$ (for example, if $k_i = k_j$ the permutation of numbers k_i, k_j is considered only once). Then for R_1 we have

$$R_1 < \sum_{j=1}^{k-1} \frac{1}{j!} \sum_{\substack{\{k_1,\ldots,k_j\},\\ k_1+\cdots+k_j=k}} \frac{1}{k_1 ! \cdots k_j!} \sum_{\pi \in P_j}{}' \sum_{\Delta_1 \subset \bar{\Delta}_a \cap \Lambda_\varepsilon, \ldots, \Delta_j \subset \bar{\Delta}_a \cap \Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_1} dx_{k_{\pi(1)}} \cdots \int_{\Delta_j} dx_{k-k_{\pi(j)}+1} \cdots$$

$$\times \int_{\Delta_j} |F(x_1, \ldots, x_k)| dx_k$$

$$< \sum_{j=1}^{k-1} \frac{a^{dk}}{j!} \sum_{\substack{\{k_1,\ldots,k_j\},\\ k_1+\cdots+k_j=k}} \frac{1}{k_1 ! \cdots k_j!} \sum_{\pi \in P_j}{}' \sum_{\Delta_1 \subset \bar{\Delta}_a \cap \Lambda_\varepsilon, \ldots, \Delta_j \subset \bar{\Delta}_a \cap \Lambda_\varepsilon} \sup_{\{x_1,\ldots,x_k\} \in (\mathbb{R}^d)^k} |F(x_1, \ldots, x_k)|$$

$$< \sum_{j=1}^{k-1} \frac{a^{d(k-j)}}{j!} |\Lambda_\varepsilon|^j \sum_{\substack{\{k_1,\ldots,k_j\},\\ k_1+\cdots+k_j=k}} \frac{1}{k_1 ! \cdots k_j!} \sum_{\pi \in P_j}{}' \sup_{\{x_1,\ldots,x_k\} \in (\mathbb{R}^d)^k} |F(x_1, \ldots, x_k)| \to 0 \quad \text{if } a \to 0. \tag{A8}$$

For R_2:

$$R_2 < \frac{1}{k!} \sum_{\Delta_1 \subset \bar{\Delta}_a \cap \Lambda_\varepsilon, \ldots, \Delta_k \subset \bar{\Delta}_a \cap \Lambda_\varepsilon} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k |F_{-}(x_1, \ldots, x_k; a) - F(x_1, \ldots, x_k)|$$

$$< \frac{|\Lambda_\varepsilon|^k}{k!} \sup_{\{x_1,\ldots,x_k\} \in (\mathbb{R}^d)^k} |F_{-}(x_1, \ldots, x_k; a) - F(x_1, \ldots, x_k)| \to 0 \quad \text{if } a \to 0, \tag{A9}$$

and for R_3:

$$R_3 = \sum_{i=1}^{\min(|\eta|;k)} \sum_{\{\Delta_1,\ldots,\Delta_i\} \subset \Lambda_\eta} \sum_{\{\Delta_{i+1},\ldots,\Delta_k\} \subset \overline{\Delta_a} \cap \Lambda_\varepsilon \setminus \Lambda_\eta} \int_{\Delta_1} dx_1 \cdots \int_{\Delta_k} dx_k |F(x_1, \ldots, x_k)|$$

$$< \sum_{i=1}^{\min(|\eta|;k)} \frac{|\Lambda_\eta|^i}{i!} \frac{|\Lambda_\varepsilon|^{k-i}}{(k-i)!} \sup_{\{x_1,\ldots,x_k\} \in (\mathbb{R}^d)^k} |F(x_1, \ldots, x_k)| \to 0 \tag{A10}$$

If $a \to 0$ as $[\text{mess } \Lambda_\eta(a) \to 0]$ (see Remark 4.2). Estimate (A1) is a consequence of (A8)–(A10). The lemma is proven.

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