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# On stability, superstability and strong superstability of classical systems of Statistical Mechanics.

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#### Abstract

A detailed analysis of conditions on 2-body interaction potential, which ensure stability, superstability or strong superstability of statistical systems is given. There has been given the connection between conditions of superstability (strong superstability) and the problem of minimization of Riesz energy in the bounded volumes.

Keywords : Continuous classical system; superstable interaction; minimal Riesz energy.

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## 1 Introduction

Stability (S) of the interaction is a necessary condition for the correct thermodynamic description of infinite statistical systems. This condition can be formulated by infinite system of inequalities on the interaction energy of an arbitrary finite subsystem, consisting of N particles, which are situated in the points  $x_1, ..., x_N$  of the space  $\mathbb{R}^d$ .

(S) Stability. There exists  $B \ge 0$  such that

$$U(x_1, \dots, x_N) \ge -BN \tag{1.1}$$

for any  $N \ge 2$  and  $\{x_1, ..., x_N\}$ .

In the present paper we consider an infinite system, which consists of identical point particles interacting via 2-body potential

$$V_2(x,y) = \Phi(|x-y|), \tag{1.2}$$

where |x - y| means Euclidean distance between points  $x, y \in \mathbb{R}^d$ . In this case

$$U(x_1, ..., x_N) = \sum_{1 \le i < j \le N} \Phi(|x - y|).$$
(1.3)

One of the most important conditions is the *condition of integrability at the infinity*. This means that for any R > 0

$$\int_{|x|\ge R} \Phi(|x|) \, dx < +\infty. \tag{1.4}$$

The conditions (1.1) and (1.4) are sufficient for the construction of Gibbs measure of an infinite system of particles in the area of small values of parameters  $\beta = \frac{1}{k_BT}$  and z, where T is a temperature of a system and z is a chemical activity, which is directly connected with a density of the system of particles(see for example [25], ch.4). In order to solve the problem of construction of Gibbs state(Gibbs measure) of an infinite system for all positive values of parameters  $\beta$  and z, it is necessary to impose more restrictive conditions on the interaction. Such a condition is the condition of superstability (SS)(see [8], [26]). At first we give several necessary definitions.

For each  $\lambda \in \mathbb{R}_+$  one can define the partition  $\overline{\Delta_{\lambda}}$  of the space  $\mathbb{R}^d$  into cubes  $\Delta$  with a rib  $\lambda$  and center in  $r \in \mathbb{Z}^d$ :

$$\Delta = \Delta_{\lambda}(r) := \left\{ x \in \mathbb{R}^d \mid \lambda \left( r^i - 1/2 \right) \le x^i < \lambda \left( r^i + 1/2 \right) \right\}.$$
(1.5)

Let  $\Gamma$  be a phase space of an infinite statistical system of identical point particles. In the case of an equilibrium system  $\Gamma$  coincides with the space of *configurations* (in our situation coordinates of particles)  $\gamma$  which are locally finite subsets of  $\mathbb{R}^d$ . In other words

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \, | \, |\gamma \cap \Lambda| < \infty, \, \text{for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}, \tag{1.6}$$

where  $\mathcal{B}_c(\mathbb{R}^d)$  is a set of all bounded Borel subsets of  $\mathbb{R}^d$ , and |X| is the cardinality of a set  $X \in \mathbb{R}^d$ . Let us define also the subset  $\Gamma_0$  of all finite configurations:

$$\Gamma_0 = \prod_{n \in \mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{ \gamma \in \Gamma \mid |\gamma| = n \}, \ \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \}.$$

$$(1.7)$$

Besides, let

$$\gamma_{\Lambda} := \gamma \cap \Lambda, \ \gamma \in \Gamma, \ \Lambda \in \mathcal{B}_c(\mathbb{R}^d).$$
(1.8)

**(SS)** Superstability. There exist  $A > 0, B \ge 0$  and the partition  $\overline{\Delta_{\lambda}}$  such that for any  $\gamma = \{x_1, \ldots, x_N\} \in \Gamma_0$  the following holds:

$$U(\gamma) \ge \sum_{\Delta \in \overline{\Delta_{\lambda}}} \left[ A |\gamma_{\Delta}|^2 - B |\gamma_{\Delta}| \right].$$
(1.9)

**Remark 1.1.** A slightly different definition was introduced by Ginibre (see [8]): An interaction is superstable if there exist two real constants  $B \ge 0$  and  $A_1 \ge 0$  such that for any  $\gamma \in \Gamma_0$  the following is true:

$$U(\gamma) \ge A_1 \frac{|\gamma|^2}{\xi^d} - B|\gamma|, \qquad (1.10)$$

where  $\xi = \max_{\{x,y\} \subset \gamma} |x-y|$ . Let us consider a box  $\Lambda$  with a volume  $V = vol(\Lambda)$  such that  $\gamma \subset \Lambda$ . Then the condition (1.10) can be rewritten in the following form:

$$U(\gamma) \ge A_{\Lambda} \frac{|\gamma|^2}{V} - B|\gamma|, \qquad (1.11)$$

where the constant  $A_{\Lambda}$  does not depend on the volume V for the given shape, but it may be shape dependent. It is easy to notice, that if we consider the box  $\Lambda$  as a union of the cubes  $\Delta$ , defined by (1.5) and containing at least one point of the configuration  $\gamma$ , then, using Cauchy-Schwarz inequality, one can write the following inequality:

$$|\gamma|^2 = \left(\sum_{\Delta \in \overline{\Delta_{\lambda}}} |\gamma_{\Delta}|\right)^2 \le \sum_{\Delta \in \overline{\Delta_{\lambda}} \cap \gamma} 1 \cdot \sum_{\Delta \in \overline{\Delta_{\lambda}}} |\gamma_{\Delta}|^2 = \frac{V}{\lambda^d} \sum_{\Delta \in \overline{\Delta_{\lambda}}} |\gamma_{\Delta}|^2.$$

So, the condition (1.11) follows directly from (1.9) with  $A_{\Lambda} = A\lambda^d$ .

There is a stronger condition on the interaction than (1.9).

(SSS) Strong superstability. There exist  $A > 0, B \ge 0, p \ge 2$  and the partition  $\overline{\Delta_{\lambda_0}}$  such that for any  $\gamma = \{x_1, \ldots, x_N\} \in \Gamma_0$  the following holds:

$$U(\gamma) \ge \sum_{\Delta \in \overline{\Delta_{\lambda}}} \left[ A |\gamma_{\Delta}|^p - B |\gamma_{\Delta}| \right].$$
(1.12)

for any  $\lambda \leq \lambda_0$ 

V. M. Park (see [19]) was the first, who used the condition (1.12) with p > 2 for the proof of bounds on exponentials of local number operators of quantum systems of interacting Bose gas.

In connection with the conditions (1.1), (1.9), (1.12) there is a problem to describe the behavior of interaction potentials, which ensure the stability, superstability or strong superstability of the statistical systems. Putting in (1.1)-(1.3) N = 2, we deduce that the function  $\Phi$  must be bounded from below:

$$\Phi(|x|) \ge -2B. \tag{1.13}$$

In addition to this, R.L.Dobrushin (see [6]) proposed a *necessary* condition of stability of interaction in the form:

$$\int_{\mathbb{R}^d} \Phi(|x|) \, dx \ge 0. \tag{1.14}$$

Consequently, a positive part of interaction must be big enough. As a rule, for neutral physical systems, the potential with the behavior as on the Figure 1 is considered.



A behavior of the potentials at the infinity  $(|x| \to \infty)$  is determined by the condition (1.4), but the behavior near the initial point depends on the chosen model and as we will see later, it actually defines **(S)**, **(SS)**, **(SSS)** type of interaction. D. Ruelle was the first, who introduced the conditions, which ensure the estimate (1.11) for the systems of particles, which are situated in the cube  $\Lambda$  with a volume V (see [24]). He proposed the potential  $\Phi$  in the following form:

$$\Phi(|x|) = \Phi_1(|x|) + \Phi_2(|x|), \qquad (1.15)$$

where  $\Phi_1$  is Lebesgue measurable function with values in the closed interval  $[0; \infty]$  and satisfies the condition (1.4);  $\Phi_2$  is a continuous function of positive type and:

$$\widetilde{\Phi}_{2}(0) = \int_{\mathbb{R}^{d}} \Phi_{2}(x) \, dx > 0.$$
(1.16)

The above mentioned conditions and their direct consequence - the inequality (1.11) were used in works [24] for the proof of existence of a thermodynamic limit ( $\Lambda \nearrow \mathbb{R}^d$ ) for a free energy (canonical ensemble) and a pressure (grand canonical ensemble). Later M. Fisher noticed(see remarks in [24]) that these results can be proved using less restrictive assumptions on the potential  $\Phi$ :

$$\Phi(|x|) \ge \frac{c}{|x|^{d+\varepsilon}} \text{ for } |x| < a_1,$$
(1.17)

$$\Phi(|x|) \ge -w \text{ for } a_1 \le |x| \le a_2,$$
 (1.18)

$$\Phi(|x|) \ge -\frac{c'}{|x|^{d+\varepsilon'}} \text{ for } |x| > a_2, \qquad (1.19)$$

where  $a_1, a_2, c, c', w, \varepsilon, \varepsilon'$  are some positive constants. See also the article [7] for the systems of particles with different species and "charged" systems. As the authors pointed out, the conditions (1.17) - (1.19) ensure **(S)** stability of a system, in other words the condition (1.1) holds. In fact, these conditions guarantee also superstability of interaction. But at that time such a notion was not yet introduced.

Independently, and at the same time A.Ya. Povzner(communication at the Moscow State University seminar on Statistical Mechanics (1963)) found the conditions on the potential, which ensure the existence of the estimate (1.1)(and even (1.9)). One can find his arguments in [28] where they have been refined for the analysis of stability of the classical statistical systems with highly singular potentials. Later R.L.Dobrushin proposed more general condition on the potential  $\Phi$ , which in contrast to (1.17) included also integrable at the origin potentials (see [6], formula (1.17)). Having modified Povzner conditions he proved, that stability and an existence of limit values of thermodynamic potentials follow from these conditions. In order to complete this short survey we have to mention the criterion of stability, which was proposed by Basuev [2]. Note that it is rather close to the Povzner's conditions (see also [20]).

In terms of usage of the conditions (1.9), (1.11) it is important to obtain the optimal values of the constants A, B. In this area we have to mention the article [17] in which for continuous  $L^1(\mathbb{R}^d)$  potentials of positive type, which satisfy the condition (1.16), the inequality (1.11) was proved with

$$A = \frac{1}{2} \left( \widetilde{\Phi}(0) - \varepsilon \right), B = \frac{1}{2} \Phi(0), \text{ and } V = V(\varepsilon)$$

for any small  $\varepsilon > 0$ . The constants A, B are best possible.

The purpose of the present article is not only to make an overview of the previous results, but to obtain some new sufficient conditions on the 2-body interaction potential, which make a system stable, superstable or strong superstable. It is important to notice that the remark about the possible behavior of singular potentials, which ensures the condition (1.12) for p > 2 was firstly proposed by D.Ruelle (see [25], ch.3, formula(2.28)). It seems to be just an intuitive assumption, which one can guess on the physical level of rigor, if we accept the following hypothesis: the configuration that minimizes energy of N particles, which are situated in the cube with a volume V is uniformly distributed. It means, that all particles are situated in the sites of a lattice on the distances  $\sim \left(\frac{V}{N}\right)^{\frac{1}{d}}$ . Implicitly such estimate of the energy was calculated also by Dobrushin (see [5] formulas (4.1), (3.2)). Therefore, the present work can be considered as a new proof of Ruelle's conjecture [25]. We used rigorous results, that have been obtained during last several years(see [10], [9], [11], [3] ) and some facts of the classical potential theory (see, for example [13]). Besides, exact values of the constants A and B in the Eqs. (1.9), (1.12) are established.

## 2 Notations and main results

Following [13] let us propose several new notations, some of them will be denoted in accordance with the chapter 1 of the present article. Let K be a compact in  $\mathbb{R}^d$ . For any configuration  $\gamma_K(|\gamma_K| = N)$  in K we define the Riesz s-energy:

$$E_s^{(N)}(\gamma_K) := \sum_{\{x,y\} \subset \gamma_K} \frac{1}{|x-y|^s}, \ s > 0.$$
(2.1)

In the case s < d consider the energy integral

$$I_s(\mu; K) := \frac{1}{2} \iint_{K \times K} \frac{1}{|x - y|^s} \,\mu(dx) \mu(dy), \tag{2.2}$$

w. r. t. some probability measure  $\mu$ , support of which is K ( $\mu(K) = 1$ ).

One of the most important problems in modern potential theory is to find a measure  $\mu^*$  that minimizes the integral (2.2).

There is a fact (see [13], ch. 2) that if the configuration  $\gamma_K^{min} = \{\xi_1, \ldots, \xi_N\}$  minimizes the energy (2.2), then a sequence of measures:

$$\mu_N(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{\xi_i}(\cdot), \qquad (2.3)$$

where  $\delta_{\xi_i}$  is a point Dirac measure, converges in the weak-star topology to the measure  $\mu^*$  (minimizing measure of the integral (2.2)).

A sequence  $e_{s,K}^{(N)} = \frac{E_s^{(N)}(\gamma_K^{min})}{N^2}$  is monotonically increasing and:

$$\lim_{N \to \infty} e_{s,K}^{(N)} = \lim_{N \to \infty} \frac{E_s^{(N)}(\gamma_K^{min})}{N^2} = I_s(\mu^*) < \infty.$$
(2.4)

There are two different behaviors of the minimizing configurations in the limit  $N \to \infty$ : 1) if  $s \leq d-2$ , then  $\operatorname{supp} \mu_N \subset \partial K$ ,  $\operatorname{supp} \mu^* \subset \partial K$ , where  $\partial K$  is a border of a compact K; 2 for  $d-2 < s < d \operatorname{supp} \mu^* \subset K$ .

Let  $K = \mathcal{B}^d(0; r)$  be a d-dimensional ball with a radius r and  $\partial K = S^d(0; r)$  be a surface of the corresponding sphere. Than for the case 1) for  $s \leq d-2$  minimizing measure is distributed uniformly on the surface of the ball  $\mathcal{B}^d(0; r)$  and:

$$\mu^*(dx; \mathcal{B}^d(0; r)) = \frac{m(dx)|_{S^d(0; r)}}{m(S^d(0; r))}, \ m(S^d(0; r)) = \frac{2\pi^{\frac{a}{2}}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1};$$
(2.5)

2) for d - 2 < s < d

$$\mu^*(dx; \mathcal{B}^d(0; r) = \frac{A(d; s)}{(r^2 - x^2)^{\frac{d-s}{2}}} m(dx), \ A(d; s) = \frac{\Gamma\left(1 + \frac{s}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(1 - \frac{d-s}{2}\right)},\tag{2.6}$$

where  $m(\cdot)$  is the Lebesgue measure in  $\mathbb{R}^d$ . Corresponding values of the energy integral (2.2) are:

1) for  $s \leq d-2$ 

$$I_s(\mu^*; \mathcal{B}^d(0; r)) = \frac{1}{r^s} \frac{2^{d-s-3} \Gamma\left(\frac{d-s-1}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(d-1-\frac{s}{2}\right)},$$
(2.7)

2) if d - 2 < s < d

$$I_s(\mu^*; \mathcal{B}^d(0; r)) = \frac{1}{r^s} \frac{\Gamma\left(1 + \frac{s}{2}\right) \Gamma\left(\frac{d-s}{2}\right)}{2 \Gamma\left(1 + \frac{d}{2}\right)}.$$
(2.8)

See for details [13].

The cases s = d and s > d are essentially different from the case s < d, which is considered in the classical potential theory. The construction of the minimizing measure and the estimates for the minimal energy of the configuration if  $s \ge d$  are proposed in [9] - [11] (see, also, [3]). Let us formulate the most important points:

1) the energy integral  $I_s(\mu) = +\infty$  for all probability measures on the compact  $K \subset \mathbb{R}^d$ ; 2) for any arbitrary compact  $K \subset \mathbb{R}^d$  the following is true:

$$\mu_N(\cdot) \to \frac{m(\cdot)|_K}{m(K)}.$$
(2.9)

or in other words point particles are asymptotically uniformly distributed; 3) if s = d the following holds:

$$C_d = \lim_{N \to \infty} \frac{E_s^N\left(\gamma_K^{min}\right)}{N^2 \ln N} = \frac{\varphi_0}{\lambda^s} \frac{\pi^{\frac{d}{2}}}{d \cdot \Gamma\left(\frac{d}{2}\right)};$$
(2.10)

4) if s > d then:

$$\lim_{N \to \infty} \frac{E_s^N\left(\gamma_K^{min}\right)}{N^{1+\frac{s}{d}}} = \frac{\varphi_0}{\lambda^s} \frac{C_{s,d}}{2}.$$
(2.11)

In the case d = 1 and K = [0,1]  $C_{s,1} = 2\xi(s)$ , where  $\xi(s)$  is a classical Riemann zetafunction;

5) let K be a d-dimensional cube with a rib  $\lambda$ , then if s > d the following holds:

$$E_s^N(\gamma_K) \ge \frac{\varphi_0}{\lambda^s} \frac{1}{2^{2s+1}} \left( \frac{2\pi^{\frac{d}{2}}}{d \cdot \Gamma\left(\frac{d}{2}\right)} \right)^{\frac{d}{d}} N^{1+\frac{s}{d}}.$$
(2.12)

(A): Assumption on the interaction potential. In this article we consider a general type of potentials  $\Phi$ , which are continuous on  $\mathbb{R}_+ \setminus \{0\}$ , and for which there exists  $\lambda > 0, R > \lambda, \varphi_0 > 0, \varphi_1 > 0$ , and  $\epsilon > 0$  such that:

1) 
$$\Phi(|x|) \equiv \Phi^{-}(|x|) \ge -\frac{\varphi_1}{|x|^{d+\epsilon}} \text{ for } |x| \ge R,;$$
 (2.13)

2) 
$$\Phi(|x|) \equiv \Phi^+(|x|) \ge \frac{\varphi_0}{|x|^s}, \ s \ge 0 \text{ for } |x| \le \lambda.$$
 (2.14)

where

$$\Phi^+(|x|) := \max\{0, \Phi(|x|)\}, \ \Phi^-(|x|) := \min\{0, \Phi(|x|)\}.$$
(2.15)

In contrast to [9], [13] we consider also the case s = 0, which looks probably trivial from the point of view of potential theory, but it will take place also in our description(see Remark 2.1 below).

Now we can formulate the following theorems.

**Theorem 2.1.** Let interaction potential satisfy the conditions (A). Then for  $0 \le s < d$ any  $\gamma \in \Gamma_0$  and sufficiently small  $\varepsilon > 0$  there exists constant  $B = B(\varepsilon)$  such that the following inequality holds :

$$U(\gamma) \ge \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| \ge 2}} \left( I_s(\mu^*; \Delta) \varphi_0 - \frac{v_0}{2} - \varepsilon \right) |\gamma_{\Delta}|^2 - B|\gamma|,$$
(2.16)

where

$$v_0 = v_0(\lambda) := \sup_{x \in \mathbb{R}^d} \sum_{\Delta \in \overline{\Delta_\lambda}} \sup_{y \in \Delta} \left| \Phi^-(|x - y|) \right|.$$
(2.17)

**Corollary 2.1.** In the case:  $0 \le s < d$  the potential  $\Phi$  yields the condition **(SS)** (see (1.9) if the following holds:

$$I_s(\mu^*; \Delta)\varphi_0 > \frac{v_0}{2}.$$
(2.18)

**Remark 2.1.** The condition (2.18) can be rewritten in simpler form if we consider that minimal Riesz energy of a configuration with fixed number of particles  $|\gamma_{\Delta}|$  in the cube  $\Delta \in \overline{\Delta_{\lambda}}$  is always bigger than minimal Riesz energy of a configuration with the same number of particles in the described ball with a radius  $r = \frac{\sqrt{d\lambda}}{2}$ . Consequently, one can substitute the formulas (2.7), (2.8) with  $r = \frac{\sqrt{d\lambda}}{2}$  for  $I_s(\mu^*; \Delta)$  in the l.h.s of (2.18) (cases:  $s \leq d-2, d-2 < s < d$  respectively). The r.h.s of (2.18) can be changed by  $\frac{C}{\lambda^d}$ , where a constant  $C \approx \int_{\mathbb{R}^d} |\Phi^-(|x|)| dx$  for sufficiently small  $\lambda$ . Then for the given configuration in the d-dimensional space and for the potential, which satisfies the condition (2.14) the system is superstable if there exists such  $\lambda$  (in other words such a partition  $\overline{\Delta_{\lambda}}$  of the space  $\mathbb{R}^d$ ), that the condition (2.18) holds. The set of potentials, which satisfy the condition (SS), is not empty, as one can choose sufficiently big  $\varphi_0$ , in order to make (2.18) true for any fixed  $\lambda > 0$ . For the case s = 0  $I_0(\mu^*; \Delta) = 1/2$ .

**Theorem 2.2.** Let interaction potential satisfy conditions (A). Then for s = d, any  $\gamma \in \Gamma_0$  and sufficiently small  $\varepsilon > 0$  there exists constant  $B = B(\varepsilon)$  such that the following inequality holds :

$$U(\gamma) \ge \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| \ge 2}} \left( C_d \ln |\gamma_{\Delta}| - \frac{v_0}{2} - \varepsilon \ln |\gamma_{\Delta}| \right) |\gamma_{\Delta}|^2 - B|\gamma|,$$
(2.19)

where (see [9])

$$C_d = \frac{1}{\lambda^d} \frac{\pi^{\frac{d}{2}}}{d\Gamma\left(\frac{d}{2}\right)} \varphi_0. \tag{2.20}$$

**Theorem 2.3.** Let interaction potential satisfy conditions (A). Then for s > d any  $\gamma \in \Gamma_0$  there exists constant  $B = B(\varepsilon)$  such that the following inequality holds:

$$U(\gamma) \ge \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| \ge 2}} \left( C_{s,d} \left| \gamma_{\Delta} \right|^{1+\frac{s}{d}} - \frac{v_0}{2} \left| \gamma_{\Delta} \right|^2 \right) - B|\gamma|,$$
(2.21)

where (see [9])

$$C_{s,d} = \frac{1}{\lambda^s} \frac{1}{2^{2s+1}} \left( \frac{2\pi^{\frac{d}{2}}}{d\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{s}{d}} \varphi_0.$$
(2.22)

**Remark 2.2.** In the case s = d the system of particles is superstable (SS) for all partitions  $\overline{\Delta_{\lambda}}$  since for any  $\varepsilon > 0$  and  $v_0$  one can find  $N_0 \ge 2$  and  $B = B(N_0)$  such that for  $N > N_0$ 

$$C_d \ln N > \frac{v_0}{2}.$$
 (2.23)

In the case s > d system of particles is strong superstable (SSS), since one can always choose sufficiently small  $\lambda > 0$  and some  $A = A(\lambda)$  such that:

$$C_{s,d}|\gamma_{\Delta}|^{1+\frac{s}{d}} - \frac{v_0}{2}|\gamma_{\Delta}|^2 \ge A|\gamma_{\Delta}|^{1+\frac{s}{d}}$$

$$(2.24)$$

for  $|\gamma_{\Delta}| \geq 2$ .

# 3 Proof of the results

#### 3.1 Proof of Theorem 2.1

We have for any  $\gamma \in \Gamma_0$  and any partition  $\overline{\Delta_{\lambda}}$ :

$$U(\gamma) = \sum_{\{x,y\}\subset\gamma} \Phi(|x-y|) = \sum_{\Delta\in\overline{\Delta}_{\lambda}:|\gamma_{\Delta}|\geq 2} U(\gamma_{\Delta}) + \sum_{\{\Delta,\Delta'\}\subset\overline{\Delta}_{\lambda}} \sum_{\substack{x\in\gamma_{\Delta}\\y\in\gamma_{\Delta'}}} \Phi(|x-y|).$$
(3.1)

Taking into account the assumptions (A) on the interaction potential , definitions (2.1), (2.17) and the inequality  $|\gamma_{\Delta}| |\gamma_{\Delta'}| \leq \frac{1}{2} (|\gamma_{\Delta}|^2 + |\gamma_{\Delta'}|^2)$  we obtain from (3.1):

$$U(\gamma) \ge \sum_{\Delta \in \overline{\Delta_{\lambda}}: |\gamma_{\Delta}| \ge 2} \left[ E_s^{(N_{\Delta}(\gamma))}(\gamma_{\Delta}^{\min})\varphi_0 - \frac{v_0}{2} |\gamma_{\Delta}|^2 \right] - \frac{v_0}{2} |\gamma|, \ N_{\Delta}(\gamma) = |\gamma_{\Delta}|.$$
(3.2)

For the fixed  $\varepsilon > 0$  let's define  $N_0$  such that  $I_s(\mu^*; \Delta) - e_{s,\Delta}^{(N)} > \varepsilon$  if  $N < N_0$ and  $I_s(\mu^*; \Delta) - e_{s,\Delta}^{(N)} < \varepsilon$  if  $N \ge N_0$  (see (2.4)). Let's also define a sequence:

$$B_N = \begin{cases} \left( e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(N)} \right) \cdot N_0, \ N \le N_0; \\ 0, \ N > N_0. \end{cases}$$
(3.3)

For  $N \leq N_0$ :  $e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)} \leq 0$  and  $N^2 \leq N N_0$ . As a result we have: 1) if  $N \leq N_0$ :  $\left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N^2 > \left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N N = R_0 N$ 

$$\left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N^2 \ge \left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N_0 N = -B_N N;$$

2) if  $N > N_0$ :

Then for any  $N \ge 2$ :

$$e_{s,\Delta}^{(N)} \cdot N^2 \ge e_{s,\Delta}^{(N_0)} \cdot N^2 - B_N \cdot N, \qquad (3.4)$$

Because of  $B_2 > B_N$  for any  $N \ge 2$  we deduce from (3.4) that for all  $N \ge 2$ :

$$e_{s,\Delta}^{(N)} N^{2} \ge e_{s,\Delta}^{(N_{0})} N^{2} - B_{2}N =$$
  
=  $I_{s}(\mu^{*}, \Delta)N^{2} + \left(e_{s,\Delta}^{(N_{0})} - I_{s}(\mu^{*}, \Delta)\right) \cdot N^{2} - B_{2}N \ge$   
 $\ge (I_{s}(\mu^{*}, \Delta) - \varepsilon) \cdot N^{2} - B_{2}N.$  (3.5)

The inequality (3.5) proves the Theorem (2.1) for the partition  $\overline{\Delta_{\lambda}}$  such that for the given  $\gamma \in \Gamma_0$  there exists at least one cube with  $|\gamma_{\Delta}| \geq 2$ . In this case:

$$B = B_2(\varepsilon) = \left(e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(2)}\right) \cdot N_0, \ N_0 = N_0(\varepsilon).$$
(3.6)

For  $\gamma \in \Gamma_0$  with  $|\gamma_{\Delta}| = 1$  or 0 it is clear that  $B_2 = v_0/2$ . So, one can choose:

$$B = \max\left\{ \left( e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(2)} \right) \cdot N_0; \frac{v_0}{2} \right\}.$$
 (3.7)

The end of the proof.

#### 3.2 Proof of Theorem 2.2 and Theorem 2.3

In our case K is a d-dimensional cube  $\Delta$  with a rib  $\lambda$ . As in the previous case we start from (3.1), (3.2). For the fixed  $\varepsilon > 0$  let us define  $N_0$  such that  $\left|C_d - \frac{E_s^N(\gamma_K^{min})}{N^2 \ln N}\right| > \varepsilon$  if  $N < N_0$  and  $\left|C_d - \frac{E_s^N(\gamma_K^{min})}{N^2 \ln N}\right| < \varepsilon$  if  $N \ge N_0$  (the constant  $C_d$  is taken from (2.10)). Using (3.1), (3.2), (2.10) and neglecting in (3.1) the part of interaction energy  $\sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| < N_0}} U(\gamma_{\Delta})$  one

can write an estimate for the total energy of the system in the following form:

$$U(\gamma) \ge \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| \ge 2}} \left[ C_d \ln |\gamma_{\Delta}| - \frac{v_0}{2} - \varepsilon \ln |\gamma_{\Delta}| \right] |\gamma_{\Delta}|^2 - \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| = 1}} \frac{v_0}{2} |\gamma_{\Delta}|^2 - \sum_{\substack{\Delta \in \overline{\Delta_{\lambda}}, \\ |\gamma_{\Delta}| = i}} \left[ C_d - \varepsilon \right] |\gamma_{\Delta}|^2 \ln |\gamma_{\Delta}|.$$

$$(3.8)$$

Number of cubes with  $|\gamma_{\Delta}| = i$  is not more than  $\frac{|\gamma|}{i}$ . That's why:

$$\sum_{i=2}^{N_0-1} \sum_{\substack{\Delta \in \overline{\Delta}_{\lambda}, \\ |\gamma_{\Delta}|=i}} \left[ C_d - \varepsilon \right] |\gamma_{\Delta}|^2 \ln |\gamma_{\Delta}| \le |\gamma| \sum_{i=2}^{N_0-1} \frac{\left[ C_d - \varepsilon \right] i^2 \ln i}{i}.$$
(3.9)

As a result, we can put:

$$B = \frac{v_0}{2} + \sum_{i=2}^{N_0 - 1} \left[ C_d - \varepsilon \right] i \ln i.$$
 (3.10)

The end of the proof.

**Remark 3.1.** The proof of the Theorem 2.3 is very similar to the previous proof of the Theorem 2.2. In this case according to (2.14) the minimal energy of  $N_{\Delta}(\gamma)$  particles which are situated in the d-dimensional cube  $\Delta \in \overline{\Delta}_{\lambda}$  can be estimated from below by the inequality (2.12) (see [10] and [3]). Substituting this inequality in (3.2) we obtain Eq.(2.21) with  $B = \frac{v_0}{2}$ .

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