GIBBS STATE UNIQUENESS FOR ANHARMONIC QUANTUM CRYSTAL WITH A NONPOLYNOMIAL DOUBLE-WELL POTENTIAL

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Abstract

We construct the Gibbs state for ν-dimensional quantum crystal with site displacements from $I\!R^d$, $d \geq 1$, and with a one-site *non-polynomial* double-well potential, which has *harmonic* asymptotic growth at infinity. We prove the uniqueness of the corresponding *Euclidean Gibbs measure* (EGM) in the *light-mass regime* for the crystal particles. The corresponding state is constructed via a cluster expansion technique for an arbitrary temperature $T \geq 0$. We show that for all $T \geq 0$ the Gibbs state (correlation functions) is analytic with respect to external field conjugated to displacements provided that the mass of particles m is less than a certain value $m_* > 0$. The high temperature regime is also discussed.

Keywords : quantum crystal model, Gibbs state, Euclidean Gibbs measures, quantum fluctuations, light-mass regime, cluster expansions.

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1 Introduction

It is generally excepted that for investigation of different physical phenomena in quantum crystals one can consider an infinite system of interacting anharmonic oscillators, which are situated in the sites of ν-dimensional lattice $\mathbb{Z}^\nu$. The heuristic Hamiltonian of such a system (in the case of 2-body interaction) has the following form:

$$H = -\frac{\hbar^2}{2m} \sum_{j \in \mathbb{Z}^\nu} \Delta_j + \sum_{j \in \mathbb{Z}^\nu} W(q_j) + \frac{J}{2} \sum_{i,j \in \mathbb{Z}^\nu, i \neq j} Q(q_i, q_j) - \sum_{j \in \mathbb{Z}^\nu} h \cdot q_j, \quad (1.1)$$

where m is the mass of particles, the operator Δ_j corresponds the kinetic energy of the system and, in fact, is d-dimensional Laplace operator in the one-particle Hilbert space $L^2(\mathbb{R}^d, dq)$, where dq is the Lebesgue measure on $\mathbb{R}^d$, $q_j \in \mathbb{R}^d$ for $j \in \mathbb{Z}^\nu$ is displacement of a particle from its position in the site $j \in \mathbb{Z}^\nu$. For general case $d \leq \nu \geq 1$. But most results which where obtained earlier are for $d = 1$. The particles are confined near their sites by potential $W(q_j)$. A *harmonic* one-site potential $W_{harm}(q_j) = \frac{1}{2}aq_j^2$, $a > 0$ together with a *harmonic* two-particle interaction $Q(\cdot, \cdot)$ define a well-known harmonic crystal model (1.1). To produce a model describing a (ferroelectric) structural phase transition one usually takes for $W(\cdot)$ a *double-well anharmonic* potential, keeping $Q(\cdot, \cdot)$ harmonic (see e.g. [1,19,24]). For example,

$$W(q) = \frac{1}{4}b(q^2)^2 - \frac{1}{2}aq^2, \quad a, b \in \mathbb{R}^1_+,$$

or semibounded from below polynomials of higher degree with more than two equal minima. Here $q^2 \equiv q \cdot q$ is the scalar square of displacement vector $q \in I\!R^d$. The proof of existence of phase transition in such kind of systems for *harmonic* interaction $Q(q_i, q_j)$ were obtained in [12,15,26,34,40,47,53] for the case

$$Q(q_i, q_j) = Q_{ij}(q_i - q_j)^2, \qquad (1.2)$$

where $\hat{Q} = (Q_{ij})_{i,j \in Z\!\!\!Z^\nu}$ is a matrix of non-negative harmonic-force constants. The term $h \cdot q_j = \sum_{\alpha=1}^d h^\alpha q_j^\alpha$ break the symmetry in the direction of the external field h. The non-uniqueness of Gibbs states for $h = 0$ is proved when the mass of the particles m is sufficiently large and the temperature T is sufficiently low, or $\beta = (kT)^{-1}$ is sufficiently large.

From the other side, another interesting phenomenon, the *suppression* of the long-range order by strong quantum fluctuations in such systems was experimentally observed (see, e.g. [55]) and was discussing long time ago from the physical point of view, see [51], or the books [1,19]. A rigorous study of this phenomenon was given in the series of papers. For the $d = 1$ displacement and all $\nu \geq 1$ the suppression of the order parameter for all temperatures including $T = 0$ (as soon as the particle mass is smaller than some threshold: $m < m_*$) was proved in [58]. Later it was shown in [3] for $d = 1$ and in [37-39] for $d \in I\!N$, that not only the long-range order but also any critical behaviour of displacements of particles from the equilibrium positions are suppressed at all temperatures $T > 0$ if the model is "strongly quantum", i.e., if the mass m is small enough. But a more important problem is to prove the *uniqueness* of the Gibbs state for such systems for sufficiently small mass of particles. It would imply the absence of all critical anomalies in the regime of "strong quantumness". The first step in this direction was made in [44] for $d = 1$, where for small values of m the convergence of corresponding cluster expansions was proved for all values of temperature $T \geq 0$. See also [8], where the cluster expansion is constructed by another and simpler technique but for small m, which depends on the positive $T > 0$. The result of [44] has been extended to the multi-dimensional spin in [27]. This in particular implies that the *order parameter* is always zero if $m < m_*$ for any values of $T \geq 0$ [43]. It gives, also, *uniqueness* of the EGM (we define it in the next section, see, also, [2,31,33] and [7] for detailed description of the Euclidean approach) at least for so-called "compact" boundary conditions for displacements. In particular, this is true if displacements are reduced to a compact space (so-called *compact spins*). This resumes the results of [44] and [43] about the *suppression* of the displacement phase transition by the large quantum fluctuations in *small*

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mass domain $m < m_*$. For models with compact spin the uniqueness mass was proved in [9]. Then in [4] the uniqueness of EGM was proved for $d = 1$ and for every fixed inverse temperature $\beta < \infty$ if the mass is less than some threshold: $m < m_*(\beta)$. Finally, the *dependence* of the mass threshold on the temperature was eliminated in recent papers [5],[6] for the set of *tempered* EGM.

Remark 1.1 *These results were obtained for the case of polynomial anharmonic one-site potential of the form:*

$$W(q) = v(q^2) = \sum_{p=1}^{s} a_{2p}(q^2)^p, \ \ a_{2s} > 0, \ \ s > 1, \ \ s \in \mathbb{N}. \qquad (1.3)$$

with additional assumption on $W(q)$ that $v(\cdot)$ is convex on $\mathbb{R}_+^1$. For the proof of these results the fact of smallness of the parameter $m^{(s-1)/(s+1)}$ for small m, i.e. $s > 1$, is crucial. Recall that in this case, after an appropriate unitary transformation of the Hamiltonian, the parameter $m^{(s-1)/(s+1)}$ stands in front of the harmonic interaction term Q. So the light-mass regime corresponds to the one-site energy domination and the weak interaction of anharmonic oscillators defined by W, that implies the uniqueness of the EGM.

On the other hand, let us define $s \geq 0$ by

$$s = \inf_{\alpha \geq 0} \{ \, \alpha : \limsup_{q^2 \to \infty} v(q^2)/(q^2)^{\alpha} < \infty \}. \qquad (1.4)$$

If $s \leq 1$, then in the *light-mass domain* ("*strong quantumness*") the parameter $m^{(s-1)/(s+1)}$ is not small, i.e. the approach developed for $s > 1$ does not work. So, the rate of the growth of anharmonic potential is an important parameter distinguishing different regimes, see discussion in [43] and [58].

Remark 1.2 *If $s < 1$, then in the light-mass limit the harmonic interaction term dominates the one-site anharmonic potential, i.e. the system has to behave close to a Debye harmonic crystal (not proved). In the marginal case $s = 1$ the both harmonic interaction and harmonic part of W dominate the anharmonicity. From the physical point of view the large-distant behaviour of the one-site anharmonic potential is not well-justified. So, a priori all above scenario are possible.*

For example, the paper [54] proposed a model with the one-site non-polynomial anharmonic potential of the form:

$$W(q) \ = \ \frac{1}{2}aq^2 + be^{-\delta q^2/2}, \quad a, b, \delta \in \mathbb{R}_+^1. \qquad (1.5)$$

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corresponding to $s = 1$. It is easy to see that for $a < b\delta$ the minimum of the potential (1.5) is not unique and the system may undergo a displacement phase transition. For the case of so-called *self-consistent phonon approximation* [1], or for the *displacement quantum spherical model* it was proved in [48] for (1.3) and in [54,58] for (1.5). In general case this follows from [15], see also [47]. On the other hand, in [48] and [54,58] it is shown that there is a *non-zero* threshold mass m_* such that the phase transition in the *displacement quantum spherical model* is suppressed by the "strong quantumness" as soon as $m < m_*$.

The aim of the present paper is study the case of the *quadratic growth*, $s = 1$, of the double-well anharmonic one-site potential W (1.5). We prove that (similar to the case $s > 1$) there is a *non-zero* mass m_* such that in the *light-mass* domain: $m < m_*$ the critical behaviour of the quantum anharmonic system (1.1) for $h = 0$ is suppressed for arbitrary dimension of particles displacements $d \geq 1$ and $\nu \geq 3$. More precisely, we prove the *uniqueness* of the corresponding Gibbs state. Our result is based on the proof of convergence of the *cluster expansions* for the quantum Euclidean Gibbs Measure in the thermodynamic limit for arbitrary temperature $T \geq 0$ and sufficiently small mass $m < m_*$ of particles (uniformly in temperature). Notice that since for $s = 1$ the interaction in the light-mass limit $m < m_*$ is not negligible (Remark 1.2), our strategy of the proof the uniqueness is different to that for $s > 1$, cf [44]. In that sense our result supplements the recent proof of the Gibbs state uniqueness for $s > 1$, see [5,6].

Now there are few other remarks in order to outline *our strategy* of the proof the uniqueness for the case $s = 1$.

The anharmonic perturbation in (1.5) can be re-written in the following form

$$V(q) \ = \ be^{-\delta q^2/2} = b \int d\mu(\alpha)e^{i\sqrt{\delta}\alpha \cdot q}, \tag{1.6}$$

where $d\mu(\alpha) = (2\pi)^{-d/2}e^{-\alpha^2/2}d\alpha$ is the Gaussian measure on $\mathbb{R}^d$. So anharmonic term (1.6) is a *gentle* perturbation of the quantum harmonic system in the sense of Albeverio–Høegh-Krohn [2]. Unfortunately, our situation is similar to [2] only at the first glance. In fact, there are some essential differences. In [2] the bounded measure $d\mu$ has to have a *compact* support. Under this condition the authors managed to apply the *Kirkwood–Salsburg Equation* (KSE) technique to prove the existence of the corresponding correlation functions in thermodynamic limit as well as their analyticity in coupling constant λ (in our case $\lambda \equiv b$) in some domain $|\lambda| < \lambda_0$ for all temperatures.

We can not apply this technique, since of the Gaussian measure in (1.6) has infinite support. By consequence it is impossible to find a Banach space E_ξ for correlation functions and the corresponding KSE. As it was indicated

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in [2], Remark 2, for extension of the results to the measure with unbounded support one needs a modification of the Banach space. But even that is not essential, since the main difficulty for $s = 1$ is connected with the parameter of analyticity. Instead of λ we have as the small parameter the particle mass m, which is involved in the Hamiltonian via the kinetic-energy term, see (1.1).

We overcome these difficulties by a *cluster expansions* technique. The choice of the cluster expansions is defined by the dependence of the parameters in the Hamiltonian on the mass m after an appropriate rescaling, see (2.12). In the case of anharmonic potential with harmonic asymptotics $s = 1$ the dependence on the mass m remains only in the *rescaled* temperature and in the *rescaled* anharmonic potential (1.6). Therefore, the expansions should be arranged *with respect to* (w.r.t.) the anharmonic part of the *one-site potential*. (Recall that for $s > 1$ the corresponding cluster expansions are constructed w.r.t. two-body harmonic *interaction*.)

To prove the uniqueness of the corresponding EGM we profit from Theorem 3.1 of [6] and we use the cluster property of the EGM moments, which is a consequence of the cluster expansion convergence established in the present paper. Similar to [43] we prove the analyticity of the state w.r.t. the external field h. As a consequence this immediately implies that the displacement *order parameter* (in the region of convergence of the cluster expansions) is equal to zero for all temperatures $T \geq 0$ as soon as $m < m_*$.

A brief contents of this paper is the following. In Section 2 we describe our system (1.5) in more details and we formulate the main results. In Section 3 we construct the cluster expansions w.r.t. the anharmonic part of the one-site potential, and we give the proof of main theorems. In Section 4 we provide all necessary estimates. Section 5 is devoted to the limiting Gibbs state analyticity w.r.t. to the external field, and in Section 6 we briefly discuss the proof of the convergence of cluster expansions and the uniqueness in the case of high temperatures.

2   Description of the system and main results

We consider a $\nu(\geq 3)$-dimensional quantum crystal model with $d(\geq 1)$-component site displacements (or d-dimensional *continuous infinite-spin* variables) on the cubic lattice $\mathbb{Z}^\nu$. With each site $j \in \mathbb{Z}^\nu$ we associate a one-particle physical Hilbert space $L^2(\mathbb{R}^d, dq)$, where dq is the Lebesgue measure on $\mathbb{R}^d$. Then

$$\mathcal{H}_\Lambda = \bigotimes_{j \in \Lambda} L^2(\mathbb{R}^d, dq_j) \simeq L^2(\mathbb{R}^{d|\Lambda|}, dq_\Lambda),$$

$$dq_\Lambda = \prod_{j \in \Lambda} dq_j,$$

is the Hilbert space related to some given bounded set $\Lambda \subset \mathbb{Z}^\nu$, with number of sites $|\Lambda| < \infty$.

For a finite set $\Lambda \subset \mathbb{Z}^\nu$ we consider in $\mathcal{H}_\Lambda$ the Hamiltonian:

$$H_\Lambda = H_0^\Lambda + V^\Lambda. \qquad (2.1)$$

Here the harmonic part of the Hamiltonian has the form:

$$H_0^\Lambda = -\frac{1}{2m} \sum_{j \in \Lambda} \Delta_j + \frac{1}{2} a \sum_{j \in \Lambda} q_j^2 + \frac{J}{4} \sum_{\langle i,j \rangle \subset \Lambda} (q_i - q_j)^2 + C_\Lambda, \qquad (2.2)$$

where m is the mass of particles (we put $\hbar = 1$), Δ_j is the Laplace operator in $L^2(\mathbb{R}^d, dq_j)$, the sum is extended over all pairs $\langle i, j \rangle \subset \Lambda$ for which $\|i-j\| = 1$ and $J > 0$. We consider a nearest-neighbors interaction only for simplicity. The *constant* C_Λ we will be chosen later.

We put a one-site potential $W(q)$ in (1.1) to be equal to (1.5) for $q^2 = q \cdot q$, i.e., for $q \in \mathbb{R}^d$. So the anharmonic part of the one-site potential in (2.1) has the form

$$V(q) = b \, e^{-\delta q^2/2}, \quad b \geq 0, \delta \geq 0, \quad q \in \mathbb{R}^d. \qquad (2.3)$$

To prove an existence of the Gibbs states in the light-mass regime for small temperatures, including $T = 0$, we need a supplementary argument based on the *cluster expansions*. Below we do this first for the *periodic boundary conditions* (*p.b.c.*), then we complete our construction by the case of the *empty* boundary conditions. The last we use later to pass to *tempered non-empty* boundary conditions. Notice that the cluster expansions for *p.b.c.* prove also the clustering property of the corresponding EGM for $m < m_*$ and for *any* $T \geq 0$. Therefore, simultaneously with existence we get *uniqueness* of EGM for all temperature $T > 0$, as soon as the particle mass is *light enough*: $m < m_*$. We also discuss the case $T = 0$. So, first we assume in (2.1), (2.2) the *p.b.c.*:

$$\Lambda = \{j \in \mathbb{Z}^\nu \mid -N_\mu/2 < j^{(\mu)} \leq N_\mu/2, \mu = 1, ..., \nu; N_\mu/2 \in \mathbb{N}\}.$$

Thus, $|\Lambda| = N_1 \cdots N_\nu$ and

$$q_{(j^{(1)}, ..., j^{(\mu)}+N_\mu, ..., j^{(\nu)})}^{(\alpha)} = q_j^{(\alpha)}, \quad \alpha = 1, ..., d, \quad \mu = 1, ..., \nu. \qquad (2.4)$$

Then, to complete the proof of the *uniqueness* of the EGM, in the next section we consider also a general boundary conditions generated by some

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fixed configuration $\bar{\omega}$ from the set $\tilde{\Omega}^t_{\tilde{\beta}}$, which includes so-called *tempered configurations*, see [6,13,14] and [46].

Usually, equilibrium quantum Gibbs states are defined (according to [18]) on $C^\star$-algebra $\mathfrak{A}$ of quasi-local observables, which satisfy the KMS conditions with respect to one-parameter group of automorphism $\alpha_t, t \in \mathbb{R}$, which defines time evolution in $\mathfrak{A}$. Unfortunately, for the considered systems such definition can not be used. Therefore, one can (see, for example, [2, 44]) treat it in the following way.

First we define the family of local Hamiltonians (2.1)-(2.2) for $|\Lambda| < \infty$, which are defined as self-adjoint operators in $\mathcal{H}_\Lambda$. Then, one can construct on the algebra of bounded linear operators $\mathcal{L}(\mathcal{H}_\Lambda)$ the group of local automorphism

$$\alpha^\Lambda_\tau(B) = e^{-i\tau\hat{H}_\Lambda} B e^{i\tau\hat{H}_\Lambda}, \quad B \in \mathcal{L}(\mathcal{H}_\Lambda)$$

and define local Gibbs state for finite temperature $\beta > 0$ by the formula

$$\rho^\beta_\Lambda(A) \;=\; \frac{\mathrm{Tr}(A e^{-\beta H_\Lambda})}{Z_\beta(\Lambda)}, \quad \beta = \frac{1}{k_B T}, \tag{2.5}$$

where $Z_\beta(\Lambda) = \mathrm{Tr} e^{-\beta H_\Lambda}$ and $A \in \mathcal{L}(\mathcal{H}_\Lambda)$.

Note, that the algebras $\mathcal{L}(\mathcal{H}_\Lambda)$ (for different Λ) are isometrically embedded into $\mathcal{L}(\mathcal{H}_{\Lambda'})$ for $\Lambda \subset \Lambda'$. Using this fact we can define the inductive limit

$$\mathfrak{A}_0 \;=\; \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \mathcal{L}(\mathcal{H}_\Lambda),$$

which is a normed $*-$ algebra of local observables. The closure of this algebra in the operator norm forms the algebra of bounded quasi-local observables

$$\mathfrak{A} \;=\; \bar{\mathfrak{A}}_0.$$

The definition (2.5) is correct because operator $A e^{-t H_\Lambda}$ with $t > 0$ is of the *trace class* (see [59]). Now, it would be reasonable to define (see e.g. [2]) the temperature quantum Gibbs state as

$$\rho^\beta(A) \;=\; \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \rho^\beta_\Lambda(A). \tag{2.6}$$

However, it is necessary to stress that it is impossible to construct this limit in the framework of the operator technique. To overcome these difficulties funtional-integral approach was developed in [32, 2]. This approach is very close to the Euclidean approach in investigating of models of the quantum field theory. The main point of this approach is in the following.

For every bounded measurable function $\mathcal{A}(x_\Lambda)$ on $I\!\!R^{d|\Lambda|}$ we consider a bounded operator A_0 on $\mathcal{H}_\Lambda$ defined as :

$$(A_0 f)(x_\Lambda) \; = \; \mathcal{A}(x_\Lambda)f(x_\Lambda), \tag{2.7}$$

and for any $t > 0$ we consider the operator

$$A_t \; = \; e^{-tH_\Lambda} A_0 e^{tH_\Lambda}. \tag{2.8}$$

Then for every set of bounded functions $\mathcal{A}^{(0)}, \ldots, \mathcal{A}^{(n)}$ and for increasing sequence of the "time" moments $0 = t_0 \leq t_1 \leq \ldots \leq t_n \leq t_{n+1} = \beta$, we define the operator

$$A \; = \; \prod_{l=0}^{n} A_{t_l}^{(l)} \; = \; A_0^{(0)} e^{-t_1 H_\Lambda} A_0^{(1)} e^{-(t_2-t_1)H_\Lambda} \ldots e^{-(t_n-t_{n-1})H_\Lambda} A_0^{(n)} e^{t_n H_\Lambda} \tag{2.9}$$

Then by (2.5) one gets:

$$\rho_\Lambda^\beta(A) \; = \; \frac{1}{Z_\beta(\Lambda)} \mathrm{Tr}(A_0^{(0)} e^{-t_1 H_\Lambda} A_0^{(1)} e^{-(t_2-t_1)H_\Lambda} \ldots e^{-(t_n-t_{n-1})H_\Lambda} A_0^{(n)} e^{-(\beta-t_n)H_\Lambda}). \tag{2.10}$$

The state (2.10) is obviously analytic in the domain

$$0 < \mathrm{Re}\ t_1 < \mathrm{Re}\ t_2 < ... < \mathrm{Re}\ t_n < \beta$$

with boundary values at $\mathrm{Re}\ t_i = 0, i = 0, ..., n$, and it is uniformly bounded by the generalized Ginibre–Gruber inequality [23]:

$$\mathrm{Tr}\Big(\prod_{l=0}^{n} \alpha_{Imt_l}^\Lambda (A_0^{(l)}) e^{-\beta H_\Lambda} \Big) \leq \| \prod_{l=0}^{n} A_0^{(l)} \| \mathrm{Tr}\Big(e^{-\beta H_\Lambda} \Big).$$

The r.h.s. of (2.10) defines so-called Matsubara functions (temperature Euclidean Green's functions). It was proved (see [32,7] for a detailed description) that, for a given $\Lambda, |\Lambda| < \infty$, any operator $B \in \mathcal{L}(\mathcal{H}_\Lambda)$ may be approximated strongly by linear combinations of operators of the form $\alpha_{t_1}(A_0^{(1)}) \cdots a_{t_1}(A_0^{(n)})$ and the Matsubara functions constructed from $\rho_\Lambda^\beta(A)$ with help the operators (2.7)-(2.9) already determine completely the state ρ_Λ^β.

The main technical tool to construct thermodynamic limit $\Lambda \nearrow \mathbb{Z}^\nu$ for $\rho_\Lambda^\beta(A)$ is the representation (2.10) by moments of some probability measure:

$$\rho_\Lambda^\beta(\prod_{l=0}^{n} A_{t_l}^{(l)}) = \int \prod_{l=0}^{n} \mathcal{A}^{(l)}(\phi_\Lambda(t_l)) d\mu^\Lambda(\phi) := \langle \prod_{l=0}^{n} \mathcal{A}^{(l)}(\cdot) \rangle_{\mu^\Lambda}. \tag{2.11}$$

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Following [2] we will use the Feynman-Kac formula to get (2.11). But at first we fulfil some preliminary constructions. Since we are going to consider domain of small masses, the following standard *light-mass rescaling* of variables (appeared for the first time in this context in [58]) is relevant:

$$q_j = \alpha x_j, \ \alpha = m^{-\frac{1}{4}}. \tag{2.12}$$

This change of variables induces the unitary map :

$$U : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_\Lambda$$
$$(Uf)(x_\Lambda) = \alpha^{|\Lambda|/2} f((\alpha x)_\Lambda), \ (\alpha x)_\Lambda = \{\alpha x_j, \ j \in \Lambda\}.$$

One can check that

$$U H_\Lambda U^{-1} = m^{-\frac{1}{2}} \hat{H}_\Lambda,$$

where

$$\hat{H}_\Lambda = \hat{H}_0^\Lambda + \hat{V}^\Lambda, \tag{2.13}$$

$$\hat{H}_0^\Lambda = -\frac{1}{2} \sum_{j \in \Lambda} \Delta_{x_j} + \frac{1}{2} a \sum_{j \in \Lambda} x_j^2 + \frac{J}{4} \sum_{\langle i,j \rangle \subset \Lambda} (x_i - x_j)^2 + C_m^\Lambda, \tag{2.14}$$

$$C_m^\Lambda = C_\Lambda m^{1/2}, \tag{2.15}$$

$$\hat{V}^\Lambda = b_m \sum_{j \in \Lambda} e^{-\delta_m x_j^2/2}, \quad b_m = b m^{1/2}, \quad \delta_m = \delta m^{-1/2}. \tag{2.16}$$

We also define a *rescaled* inverse temperature

$$\hat{\beta} = \beta m^{-\frac{1}{2}}, \tag{2.17}$$

which yields

$$U \beta H_\Lambda U^{-1} = \hat{\beta} \hat{H}_\Lambda.$$

Then the Gibbs state (2.10) takes the form:

$$\rho_\Lambda^\beta(A) = \hat{\rho}_\Lambda^{\hat{\beta}}(\hat{A}), \ \hat{A} = U A U^{-1},$$

where

$$\hat{\rho}_\Lambda^{\hat{\beta}}(\cdot) = \hat{Z}_{\hat{\beta}}^{-1}(\Lambda) \mathrm{Tr} \left\{ \cdot \exp(-\hat{\beta} \hat{H}_\Lambda) \right\}. \tag{2.18}$$

To represent the state (2.10) in the form of the functional integral we rewrite the operator $\hat{H}_0^\Lambda$ as follows

$$\hat{H}_0^\Lambda = \sum_{\alpha=1}^d \hat{H}_0^{(\alpha),\Lambda}, \tag{2.19}$$

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$$\hat{H}_0^{(\alpha),\Lambda} = -\frac{1}{2}\Delta_\Lambda^{(\alpha)} + \frac{1}{2}(x_\Lambda^{(\alpha)}, B_\Lambda^2 x_\Lambda^{(\alpha)}) - \frac{1}{2}\text{Tr}B_\Lambda, \qquad (2.20)$$

where

$$\Delta_\Lambda^{(\alpha)} = \sum_{j\in\Lambda} \frac{\partial^2}{\partial x_j^{(\alpha)^2}}, \quad B_\Lambda^2 = -J\hat{\Delta}_\Lambda + a\mathbb{1}_\Lambda, \qquad (2.21)$$

$x_\Lambda^{(\alpha)} = (x_j^{(\alpha)})_{j\in\Lambda} \in I\!\!R^{|\Lambda|}$, and B_Λ^2 is a periodic $|\Lambda| \times |\Lambda|$ matrix, defined by the elements:

$$(-\hat{\Delta}_\Lambda)_{jk} = \begin{cases} 2\nu, & \text{for } j = k, \\ -1, & \text{for } |j - k| = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad (2.22)$$

and

$$(\mathbb{1})_{jk} = \delta_{jk} = \delta_{j^{(1)}k^{(1)}} \cdots \delta_{j^{(\nu)}k^{(\nu)}}.$$

Now we fix the choice of the *constant* C_Λ in (2.2),(2.14) by

$$C_\Lambda = \frac{d}{2\sqrt{m}}\text{Tr}B_\Lambda. \qquad (2.23)$$

This choice is connected with the possibility to calculate spectrum of the operator $\hat{H}_0^{(\alpha),\Lambda}$ and the correspondent trace of the operator $\exp[-\beta\hat{H}_0^{(\alpha),\Lambda}]$ by the eigenvalues of the operator B_Λ (see [32] for details). By virtue of *p.b.c.* the lattice operator B_Λ is *cyclic* (see [30, Sec.16.4]), so we can calculate its eigenvalues explicitly:

$$\lambda_k = \sqrt{\varepsilon(k)}, \quad \varepsilon(k) = a + 4J\sum_{\mu=1}^\nu \sin^2(k^{(\mu)}/2), \quad k \in \Lambda^*, \qquad (2.24)$$

where the set Λ^* is *dual* to Λ, i.e.

$$\Lambda^* = \{k \in I\!\!R^\nu \mid k = \{k^{(\mu)}\}_{\mu=1}^\nu, k^{(\mu)} = \frac{2\pi}{N_\mu}n^{(\mu)}, n^{(\mu)} = 0, \pm1, ..., \pm N_\mu - 1/2, N_\mu/2\}. \qquad (2.25)$$

Now the *Feynman-Kac* formula (see e.g. [50],[52]) gives

$$\text{Tr}e^{-\hat{\beta}\hat{H}_\Lambda} = \int_{I\!\!R^{d|\Lambda|}} dx_\Lambda \left(e^{-\hat{\beta}\hat{H}_\Lambda}\right)(x_\Lambda; x_\Lambda) =$$

$$= \int_{I\!\!R^{d|\Lambda|}} dx_\Lambda \int_{\Omega_{\hat{\beta},\Lambda}} W_{x_\Lambda;x_\Lambda}^{\hat{\beta}}(d\omega_\Lambda)e^{-\int_0^{\hat{\beta}} d\tau\hat{U}(\omega_\Lambda(\tau))}, \qquad (2.26)$$

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with

$$\hat{U}(x_\Lambda) = \frac{1}{2}(x_\Lambda, B_\Lambda^2 x_\Lambda) - \frac{d}{2}\text{Tr}B_\Lambda + \hat{V}(x_\Lambda), \tag{2.27}$$

where we put $\hat{V}(x_\Lambda) = \hat{V}^\Lambda$, see (2.16), and

$$W_{x_\Lambda;x_\Lambda}^{\hat{\beta}}(d\omega_\Lambda) = \prod_{j\in\Lambda} W_{x_j;x_j}^{\hat{\beta}}(d\omega_j),$$

here $W_{x_j;x_j}^{\hat{\beta}}(d\omega_j)$ is the conditional Wiener measures on the space of continuous periodic trajectories

$$\Omega_{\hat{\beta},\Lambda} = \{\omega_\Lambda(\cdot)|\omega_\Lambda : S_{\hat{\beta}} \to I\!R^{d|\Lambda|}\}, \quad S_{\hat{\beta}} := [0,\hat{\beta}], \tag{2.28}$$

$$\omega_\Lambda(\cdot) = \{\omega_j(\cdot), j \in \Lambda | \omega_j \in \Omega_{\hat{\beta}}, \Omega_{\hat{\beta}} := C(S_{\hat{\beta}} \to I\!R^d)\}.$$

Now, following [2] the *unperturbed measure* in the right-hand side of (2.26) (i.e. $\hat{V}^\Lambda = 0$) can be rewritten with the help of the measure on the space of continuous periodic functions $\Omega_{\hat{\beta},\Lambda}$ obtained from the Gaussian distribution indexed by the real Hilbert space

$$\mathcal{H}_{\hat{\beta},\Lambda} := \{\omega_\Lambda \in (\Omega_{\hat{\beta},\Lambda}, \Sigma_{\hat{\beta},\Lambda}) \mid \|\omega_\Lambda\|^2 < \infty\}. \tag{2.29}$$

Here $\Sigma_{\hat{\beta},\Lambda}$ is the standard σ-algebra of $\Omega_{\hat{\beta},\Lambda}$-subsets generated by Borel cylinder subsets and norm is defined by

$$\|\omega_\Lambda\|^2 = \int_0^{\hat{\beta}} d\tau \Big[\Big(\frac{d\omega_\Lambda(\tau)}{d\tau}, \frac{d\omega_\Lambda(\tau)}{d\tau}\Big) + \big(\omega_\Lambda(\tau), B_\Lambda^2 \omega_\Lambda(\tau)\big) \Big],$$

where $(\cdot, \cdot)$ is the scalar product in $l_2(I\!R^{d|\Lambda|})$. Then for any real continuous function F defined on $\Omega_{\hat{\beta}}$ one can get:

$$\int_{I\!R^{d|\Lambda|}} dx_\Lambda \int_{\Omega_{\hat{\beta},\Lambda}} W_{x_\Lambda;x_\Lambda}^{\hat{\beta}}(d\omega_\Lambda) e^{-\frac{1}{2}\int_0^{\hat{\beta}} d\tau(\omega_\Lambda(\tau), B_\Lambda^2 \omega_\Lambda(\tau)) + \frac{d}{2}TrB_\Lambda} F(\omega_\Lambda) :=$$

$$= C_0 \int d\mu_0^\Lambda(\phi_\Lambda \mid p) F(\phi_\Lambda) = C_0 \int d\mu_0^\Lambda(\phi_\Lambda) F(\phi_\Lambda), \tag{2.30}$$

where C_0 is a normalization constant and $d\mu_0^\Lambda(\phi_\Lambda \mid p)$ is the Gaussian measure which corresponds to *p.b.c.*. We skip the index p in the following. By (2.26) and (2.30) it is clear that

$$C_0 = Z_{\hat{\beta}}^0(\Lambda) = \text{Tr}e^{-\hat{\beta}\hat{H}_0^\Lambda}. \tag{2.31}$$

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Hence the right-hand side of (2.30) is the expectation:

$$\langle F \rangle_{\mu_0^\Lambda} = \int d\mu_0^\Lambda(\phi_\Lambda) F(\phi_\Lambda) \tag{2.32}$$

with respect to the Gaussian measure on the space of continuous periodic functions generated by the corresponding Gaussian process with the zero mean and the covariance G_0^Λ:

$$G_{0;jk}^{\Lambda;\alpha\alpha'}(\tau - \tau') = \int d\mu_0^\Lambda(\phi_\Lambda) \phi_j^{(\alpha)}(\tau) \phi_k^{(\alpha')}(\tau') = \delta_{\alpha\alpha'} G_{0;jk}^\Lambda(\tau - \tau').$$

Here $\alpha, \alpha' \in \{1, 2, ..., d\}$, and $G_{0;jk}^\Lambda(\tau - \tau')$ is the Green function of the operator $\left(-\frac{d^2}{d\tau^2} \right) \mathbb{1}_\Lambda + B_\Lambda^2$ on $L_2(\Lambda \otimes S_{\hat\beta}; \mathbb{R}^d)$ with $p.b.c.$ More explicitly [50]:

$$G_{0;jk}^\Lambda(\tau - \tau') = \frac{1}{|\Lambda|} \sum_{l \in \Lambda^*} e^{i(j-k)\cdot l} \quad \frac{1}{\hat\beta} \sum_{n \in \mathbb{Z}^1} \frac{\cos(2\pi(\tau-\tau')n/\hat\beta)}{(4\pi^2 n^2/\hat\beta^2) + \varepsilon(l)} \ , \tag{2.33}$$

or taking the sum over n, we get (see e.g.[50])

$$G_{0;jk}^\Lambda(\tau - \tau') = \frac{1}{|\Lambda|} \sum_{l \in \Lambda^*} e^{i(j-k)\cdot l} \ \frac{e^{(\hat\beta - |\tau - \tau'|)\sqrt{\varepsilon(l)}} + e^{|\tau - \tau'|\sqrt{\varepsilon(l)}}}{2\sqrt{\varepsilon(l)}(e^{\hat\beta\sqrt{\varepsilon(l)}} - 1)} \ . \tag{2.34}$$

It is important to note that G_0^Λ depends on the mass m only via *rescaled* temperature $\hat\beta = \beta m^{-1/2}$.

Remark 2.1 *From (2.33)-(2.34) one gets that for $\Lambda \nearrow \mathbb{Z}^\nu$ the covariance $G_{0;jk}^\Lambda(\tau - \tau')$ converges pointwise to a translation invariant function $G_{0;|j-k|}(\tau - \tau')$ which is also a bounded continuous function and in fact it is the kernel of a positive self-adjoint operator*

$$-\frac{d^2}{dt^2} + B_{\mathbb{Z}^\nu}^2$$

in $L_2(\mathbb{Z}^\nu \otimes S_{\hat\beta}; \mathbb{R}^d)$. From this convergence it follows that the corresponding process(measure) converges weakly to a limit process, which is a homogeneous Gaussian process on $\mathbb{Z}^\nu$ and periodic on $S_{\hat\beta}$ with values in $\mathbb{R}^d$.

By virtue of (2.26) and (2.30) it is clear that

$$Z_\Lambda = \mathrm{Tr}_\Lambda e^{-\hat\beta \hat H_\Lambda} = Z_\Lambda^0 \int d\mu_0^\Lambda(\phi_\Lambda) e^{-\int_0^{\hat\beta} d\tau \hat V(\phi_\Lambda(\tau))}, \tag{2.35}$$

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Now, it is clear that the measure in (2.11) is the *Gibbsian modification* of the measure $d\mu_0^\Lambda(\phi_\Lambda)$:

$$d\mu^\Lambda(\phi_\Lambda) = Z_{\hat{\beta}}(\Lambda)^{-1} e^{-\int_0^{\hat{\beta}} d\tau \hat{V}(\phi_\Lambda(\tau))} d\mu_0^\Lambda(\phi_\Lambda). \qquad (2.36)$$

Following [7] we shall call this measure the *Local Euclidean Gibbs Measure* (LEGM), which corresponds to our particular model (2.1)-(2.3) in this context. According to [6], a probability measure μ on the measure space $(\Omega_{\hat{\beta}}, \Sigma_{\hat{\beta}, \Lambda})$ is said to be an EGM of the model (2.1)-(2.3) considered at the temperature $\hat{\beta}$ if it satisfies DLR equation. The measure $d\mu^\Lambda(\phi_\Lambda)$ corresponds to the model (2.1)-(2.3) with p.b.c..

To prove the uniqueness of the limit measure for *non-periodic* boundary conditions, we keep the DLR language (see [25]) and consider EGM, with some general *boundary conditions* ξ, from a class studied already in [6,7]. To this end we define the set of *tempered* configurations:

$$\tilde{\Omega}_{\hat{\beta}}^t := \bigcup_{\rho > 0} \tilde{\Omega}_{\hat{\beta}}^\rho = \bigcup_{\rho > 0} \left\{ \xi \in \Omega_{\hat{\beta}, \mathbb{Z}^\nu} \mid \sum_{l \in \mathbb{Z}^\nu} e^{-\rho|l|} \|\xi_l\|_{L^2[0, \hat{\beta}]} < \infty \right\}. \qquad (2.37)$$

Remark 2.2 *For tempered configurations $\Omega_{\hat{\beta}}^t$ defined in [6] one has: $\Omega_{\hat{\beta}}^t \subset \tilde{\Omega}_{\hat{\beta}}^t$, since we include also configurations that may have an exponential growth at infinity.*

First we define the harmonic measure with *zero* boundary conditions $\mu_0^\Lambda(\cdot \mid 0)$ in the same way as it was done in (2.30) for $\mu_0^\Lambda(\cdot \mid p)$. Then for any $\xi \in \tilde{\Omega}_{\hat{\beta}}^\rho$ with $\rho < \sqrt{a}$, where a is the one-site harmonic constant from (2.2), we define the perturbed measure with *non-zero* boundary conditions by

$$d\mu^\Lambda(\phi \mid \xi) := Z_{\hat{\beta}}(\Lambda \mid \xi)^{-1} e^{-\int_0^{\hat{\beta}} d\tau \hat{V}(\phi_\Lambda(\tau)) + \frac{J}{2} \sum_{\langle l, l' \rangle}^{\partial \Lambda} \int_0^{\hat{\beta}} d\tau \phi_l(\tau) \xi_{l'}(\tau)} d\mu_0^\Lambda(\phi \mid 0), \qquad (2.38)$$

where

$$\sum_{\langle l, l' \rangle}^{\partial \Lambda} (\ldots) := \sum_{l \in \Lambda, l' \in \Lambda^c, |l - l'| = 1} (\ldots). \qquad (2.39)$$

The main result of the present paper is the following theorem.

Theorem 2.1 *For the system of quantum particles with Hamiltonian (2.1)–(2.3) there exists a sufficiently small mass m_* such that for any $0 < m < m_*$*

the weak limit of measures (2.38)

$$\lim_{\Lambda \nearrow \mathbb{Z}^{\nu}} \mu^{\Lambda}(\cdot \mid \xi) \; = \; \mu, \tag{2.40}$$

exists and μ is a unique EGM on $(\Omega_{\hat{\beta}, \mathbb{Z}^{\nu}}, \Sigma_{\hat{\beta}, \mathbb{Z}^{\nu}})$ in the set of tempered Gibbs measures, for all temperatures $T > 0$. For $T = 0$ the limit (2.40) also exists, but it is not clear if it is EGM. This measure defines the corresponding quantum Gibbs state on the algebra $\mathfrak{A}_0$, which can be by continuity extended to the algebra $\mathfrak{A}$.

Remark 2.3 *To prove the existence of the limit measure we are going to apply a cluster expansions procedure to the perturbed measure μ^{Λ} (2.34), which is constructed by the Hamiltonian (2.1)-(2.3) with p.b.c.(2.4). But, in fact, one can prove the convergence (in the thermodynamic limit) of cluster expansions for some non-empty set of boundary configurations of tempered type (see (2.37)), and so to prove the existence of the limit measure in the class of measures supported on these tempered configurations (the set of tempered Gibbs measures).*

To prove the uniqueness of the limit measure in this set we use one elegant criterium of uniqueness due to [6] (Theorem 3.1).

Remark 2.4 *To prove the existence of the limit measure μ it is sufficient to prove the convergence of the averages*

$$\langle \mathcal{A}_B(\cdot) \rangle_{\mu} \; = \; \lim_{\Lambda \nearrow \mathbb{Z}^{\nu}} \langle \mathcal{A}_B(\cdot) \rangle_{\mu^{\Lambda}}, \tag{2.41}$$

for local observables

$$\mathcal{A}_B(\phi_B) = \prod_{j \in B} \mathcal{A}_j(\phi_{\tau_j}) \tag{2.42}$$

for any bounded subsets $B \subset \Lambda$, $\tau_j \in S_{\hat{\beta}}$ and linear span of infinitely differentiable functions

$$\mathcal{A}_j(\phi_{\tau_j}) = e^{i\alpha\phi_j(\tau_j)}, j \in B, \tau_j \in S_{\beta}, \alpha \in I\!\!R \tag{2.43}$$

3 Cluster expansions and proof of main Theorem

Remark 3.1 *Before we construct the cluster expansions, which we need for the proof of Theorem 2.2 we recall the difference between the case $s > 1$*

in (1.3) and our case (1.5) with harmonic growth at infinity (see Section 1 and discussion in [43,44]). Notice that for harmonic interaction (i.e. for polynomials of the second degree) the rescaling (2.12) implies, for $s > 1$, a week interaction between oscillators proportional to $m^{(s-1)/(s+1)}$. But for the harmonic growth of the one-site potential and for harmonic interaction the rescaling (2.12) leads to smallness of anharmonic perturbation $\hat{V}^\Lambda$, see (2.16).

Remark 3.2 *Notice that the cluster expansions developed in [8,36,44] for $s > 1$ are suited for small interaction between oscillators. In contrast to that the cluster expansions presented in the present paper are relevant for small one-site anharmonism and harmonic oscillators interaction. Therefore, it is suitable to arrange our reference measure in such a way it includes this interaction. Following the paper [2] it gives us the Gaussian measure representation for the state.*

The cluster expansions technique for the Gaussian integrals with non-diagonal covariance was developed in [29] (see also [30,49]). Then it was applied in [20,22] for the proof of the Debye screening in dilute charged particle systems. This type of cluster expansions includes derivatives acting on the Gibbs factor $\exp[-\hat{V}^\Lambda]$. Since the coefficient b_m is proportional to $m^{1/2}$, and the coefficient δ_m is proportional to $m^{-1/2}$ (see (2.10)), we have no analyticity at $m = 0$. Therefore, the estimates are delicate in the small-mass domain.

To prove the convergence of the measure $d\mu^\Lambda$ in the thermodynamic limit it is sufficient to prove the existence of limit state for observables $\mathcal{A}_B$ (2.35).

We would like to notice that similar to non-polynomial quantum field theory (see e.g. [30,49]) our cluster expansions correspond to a one-site perturbation, and not to the interaction between sites. To start the construction of cluster expansions we define new configuration space. First we consider $S_{\hat{\beta}}$ for large $\hat{\beta}$ including $\hat{\beta} = \infty$. So for a fixed m we consider sufficiently low temperature such that critical behavior (if any) might be in the interval $[0, \hat{\beta}]$. Then just for convenience we consider $\hat{\beta} \in \mathbb{N}$ to define the partition of the interval $[0, \hat{\beta}]$ into *unit* intervals. So, we define

$$\bar{\Delta}_{\hat{\beta}} := \left\{ \Delta_{\hat{\tau}} := [\hat{\tau}, \hat{\tau} + 1] \mid \hat{\tau} = 0, 1, ..., \hat{\beta} - 1, \quad \hat{\beta} \in \mathbb{N} \right\}, \qquad (3.1)$$

Then we define

$$t := (j, \tau), \quad j \in \mathbb{Z}^\nu, \quad \tau \in S_{\hat{\beta}}, \qquad (3.2)$$

and the box

$$T := \Lambda \otimes S_{\hat{\beta}} \subset \mathbb{Z}^\nu \otimes \mathbb{R}_+, \qquad (3.3)$$

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where Λ and $S_{\hat{\beta}}$ are defined in the previous section. With these notations we introduce

$$G_0^T(t, t') := G_{0;jj'}^{\Lambda}(\tau, \tau'). \tag{3.4}$$

For any $j \in \mathbb{Z}^{\nu}$ and $\Delta_{\hat{\tau}} \in \bar{\Delta}_{\hat{\beta}}$ we put

$$\Delta_{j,\hat{\tau}} := (j, \Delta_{\hat{\tau}}), \tag{3.5}$$

and consider T as finite union of "rods" $\Delta_{j,\hat{\tau}}, j \in \mathbb{Z}^{\nu}, \quad \hat{\tau} = 0, 1, ..., \hat{\beta} - 1$.

To construct the cluster expansions for the expectations of (2.41) we define (following [20]) a family of sets by the *inductive procedure*. Let for $n = 1$

$$X_1 = Y_1 = (B, \Delta_B), \quad \Delta_B = \bigcup_{j \in B} \Delta_j, \tag{3.6}$$

where Δ_j coincides with $\Delta_{\hat{\tau}}$ such that $\tau_j \in \Delta_{\hat{\tau}}$. For $n \geq 2$ we put

$$Y_n = (j, \Delta_{\hat{\tau}}), j \in \Lambda \setminus B, \quad \text{or} \quad \Delta_{\hat{\tau}} \in \bar{\Delta}_{\hat{\beta}} \setminus \Delta_B, \tag{3.7}$$

and define

$$X_n := X_{n-1} \bigcup Y_n, \quad X_n^c := T \setminus X_n. \tag{3.8}$$

So, Y_n for $n \geq 2$ are "rods" $\Delta_{j,\hat{\tau}}$, and X_n are their unions.

Now define the sequence of new covariances (for $0 \leq s_i \leq 1, i = 1, ..., n$):

$$G_0^T(t, t'; (s)_n) = G_0^T(t, t')p(t, t'; (s)_n), \tag{3.9}$$

where

$$p(t, t'; (s)_n) = \sum_{m=1}^{n+1} \mathbb{1}_{Y_m}(t)\mathbb{1}_{Y_m}(t') +$$

$$\sum_{1 \leq l < m \leq n+1} s_l s_{l+1} \cdots s_{m-1} \big[\mathbb{1}_{Y_l}(t)\mathbb{1}_{Y_m}(t') + \mathbb{1}_{Y_m}(t)\mathbb{1}_{Y_l}(t') \big]. \tag{3.10}$$

Here $\mathbb{1}_Y(t)$ is indicator of the "rod" Y, and the "rod" Y_{n+1} we identify with X_n^c.

One can treat $G_0^T(t, t'; (s)_n)$ as a kind of "interaction potential" in the space of "cluster" configurations $\Lambda \times \bar{\Delta}_{\hat{\beta}}$. Then the interpolation parameters s_i specify the intensity of this "interaction" between "rods" of X_i and of X_i^c. Putting $s_n = 0$ we get that there is no "interaction" between the "rods" of X_n and "rods" of X_n^c. Then let $\mu_{0;(s)_n}^T$ be the Gaussian measure with covariance $G_0^T(t, t'; (s)_n)$. This is the kernel of a positive-definite operator in $L_2(T; \mathbb{R}^d)$ because it is a linear combination with positive coefficients of operators of the form $\mathbb{1}_X G_0^T \mathbb{1}_X$, where X is some of the unions of Y_m, see [21], [30, Sec.18.2]

and Section 4 for details. It is clear from (3.10) that $\mu_{0;(1)_n}^T = \mu_0^\Lambda$. For $s_n = 0$ one gets that the measure $d\mu_{0;(s)_{n-1},0}^T$ becomes the product measure

$$d\mu_{0;(s)_{n-1},0}^T = d\mu_{0;(s)_{n-1}}^{\Lambda,X_n} d\mu_{0;0}^{\Lambda,X_n^c}, \tag{3.11}$$

where the covariance of $d\mu_{0;(s)_{n-1}}^{\Lambda,X_n}$ is

$$G_0^{\Lambda,X_n}(t,t';(s)_{n-1},0) = G_0^T(t,t';(s)_{n-1},0) - \mathbb{1}_{X_n^c}(t)G_0^T(t,t')\mathbb{1}_{X_n^c}(t'), \tag{3.12}$$

and the covariance of $d\mu_{0;0}^{\Lambda,X_n^c}$ is

$$G_0^{\Lambda,X_n}(t,t') = \mathbb{1}_{X_n^c}(t)G_0^T(t,t')\mathbb{1}_{X_n^c}(t'). \tag{3.13}$$

Notice, also, that $(s)_{n-1}\big|_{n=1} = \emptyset$ and $d\mu_{0;(s)_{n-1}}^{\Lambda,X_n}\big|_{n=1} =: d\mu_0^{\Lambda,X_1}$, and that each step in the expansion can be obtained by applying the Newton-Leibnitz formula:

$$\langle \mathcal{A}_B(\cdot)\rangle_{\mu^\Lambda} = Z_T^{-1} \int d\mu_{0;0}^T(\phi)\mathcal{A}_B(\phi_B)e^{-\hat{V}(\phi_T)}+$$

$$Z_T^{-1}\int_0^1 ds_1 \frac{d}{ds_1}\int d\mu_{0;s_1}^T(\phi)\mathcal{A}_B(\phi_B)e^{-\hat{V}(\phi_T)} =$$

$$\int d\mu_0^{\Lambda,X_1}(\phi_{X_1})\mathcal{A}_B(\phi_B)e^{-\hat{V}(\phi_{X_1})}\frac{Z_T(X_1^c)}{Z_T} + R_1^\Lambda, \tag{3.14}$$

where $Z_T := Z_{\hat{\beta}}(\Lambda)$ and

$$Z_T(X) = \int d\mu_0^T(\phi)e^{-\hat{V}(\phi_X)}. \tag{3.15}$$

Using integration by parts formula to calculate the derivative w.r.t. parameter s_1 (see, for example, [30, Sec.9.1]) we obtain

$$R_1^\Lambda = Z_T^{-1}\int_0^1 ds_1 \int d\mu_{0;s_1}^T(\phi)\Delta_\phi \mathcal{A}_B(\phi_B)e^{-\hat{V}(\phi_T)}, \tag{3.16}$$

where

$$\Delta_\phi = \sum_{\alpha=1}^d \Delta_\phi^{(\alpha)}, \tag{3.17}$$

$$\Delta_\phi^{(\alpha)} = \frac{1}{2}\sum_{j,k\in\Lambda}\int_0^{\hat{\beta}} d\tau \int_0^{\hat{\beta}} d\tau' G_{0;jk}^\Lambda(\tau-\tau')\frac{d}{ds_1}p\big((j,\tau),(k,\tau');s_1\big)\frac{\delta^2}{\delta\phi_j^{(\alpha)}(\tau)\delta\phi_k^{(\alpha)}(\tau')}$$

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$$=: \frac{1}{2} \int_T dt \int_T dt' G_0^T(t,t') \frac{d}{ds_1} p(t,t';s_1) \frac{\delta^2}{\delta\phi^{(\alpha)}(t)\delta\phi^{(\alpha)}(t')}$$

$$= \int_{Y_1} dt \int_{X_1^c} dt' G_0^T(t,t') \big(\frac{d}{ds_1} s_1\big) \frac{\delta^2}{\delta\phi^{(\alpha)}(t)\delta\phi^{(\alpha)}(t')}$$

$$= \sum_{Y_2 \in X_1^c} \big(\frac{d}{ds_1} s_1\big) \Delta_{1,2}^{(\alpha)}. \tag{3.18}$$

For definition of $\Delta_{p,p'}^{(\alpha)}$ ($p, p' \in I\!N$) see (3.24). Then, taking into account that

$$e^{-\hat{V}(\phi_T)} = e^{-\hat{V}(\phi_{X_2})} e^{-\hat{V}(\phi_{X_2^c})} \tag{3.19}$$

with $X_2 = X_1 \cup Y_2 = Y_1 \cup Y_2$, and putting (3.17)-(3.19) in to (3.16) we repeat the procedure in each term of the sum over Y_2 to carry out the second step of expansion. After $n_T = |T \setminus X_1|$ steps all "rods" in T will be exhausted and we get the identity:

$$\langle \mathcal{A}_B(\cdot)\rangle_{\mu^\Lambda} = \sum_{n=1}^{n_T} \sum_{\bar{Y} \subset T} \mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B) F_n^T(\bar{Y}), \tag{3.20}$$

where

$$\mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B) = \int (ds)^{n-1} f(\eta; (s)_{n-2}) I_n^T(\bar{Y}, \eta, \mathcal{A}_B; (s)_{n-1}), \tag{3.21}$$

$$I_n^T(\bar{Y}, \eta, \mathcal{A}_B; (s)_{n-1}) = \int d\mu_{0;(s)_{n-1}}^{\Lambda, X_n}(\phi_{X_n}) \Delta(\eta, \bar{Y}) \mathcal{A}_B(\phi_B) e^{-\hat{V}(\phi_{X_n})}, \tag{3.22}$$

with

$$\Delta(\eta, \bar{Y}) = \prod_{l=2}^n \Delta_{\eta(l),l}, \quad \Delta_{p,p'} = \sum_{\alpha=1}^d \Delta_{p,p'}^{(\alpha)}, \tag{3.23}$$

where $\eta := \{(\eta(l), l)_{l=2}^n\}$ is a sequence of couples with $\eta(l) < l$, which we use to construct the *trees* estimates according the standard procedure, see e.g. [21], or [42] Ch.II.4. Here

$$\Delta_{p,p'}^{(\alpha)} = \int_{Y_p} dt \int_{Y_{p'}} dt' G_0^T(t,t') \frac{\delta^2}{\delta\phi^{(\alpha)}(t)\delta\phi^{(\alpha)}(t')}, \tag{3.24}$$

$$f(\eta; (s)_{n-2}) = \prod_{2 \le m \le n} (s_{\eta(m)} s_{\eta(m)+1} \cdots s_{m-2}), \tag{3.25}$$

$$\eta(m) \leq m - 2, \quad s = (s_1, \ldots, s_{n-1}), \quad 0 \leq s_j \leq 1. \tag{3.26}$$

The second sum in (3.20) is over all possible sets $\bar{Y} = \{Y_2, ..., Y_n\}$ in T. The product in (3.25) is equal to 1 if $\eta(m) = m - 1$. Finally

$$F_n^T(\bar{Y}) = \frac{Z_T(X_n^c)}{Z_T}, \quad Z_T(X_n^c) = Z_T(T \setminus X_n). \tag{3.27}$$

(For more details about construction of this type of expansions see, for example the survey [49].)

To prove Theorem 2.2 we use the representation (3.20) and the following collection of lemmas.

Lemma 3.1 *With the same assumptions as in Theorem 2.2 there exists a sufficiently small value of the mass m_*, and constant C_B which does not depend on T and on m, such that for all $0 < m < m_*$*

$$\sum_{\bar{Y} \subset \mathbb{Z}^\nu \otimes S_{\hat{\beta}}} \mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B) \leq C_B \varepsilon^{n-1}, \tag{3.28}$$

where $\varepsilon = \varepsilon(m) \to 0$ as $m \to 0$ and

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B) = \mathcal{K}_{\bar{Y}}^\beta(\mathcal{A}_B). \tag{3.29}$$

Lemma 3.2 *With the same assumptions as in Theorem 2.2 there exists a constant c such that*

$$F_n^T(\bar{Y}) \leq e^{c(|B|+n-1)}. \tag{3.30}$$

The constant c does not depend on $\Lambda, \hat{\beta}, B, m$, and one has the limit:

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} F_n^T(\bar{Y}) = F_n^{\hat{\beta}}(\bar{Y}). \tag{3.31}$$

We also need the following corollary of Lemmas 3.1 and 3.2.

Corollary 3.1 *Let $B' \subset \Lambda$ and $B \cap B' = \emptyset$ and let $\tilde{B}' = B' \cup \Delta_{B'}$ ($\Delta_{B'}$ as in (3.6)) and the sum over $\bar{Y}$ be restricted so that $X_n \cap \tilde{B}' \neq \emptyset$. Then*

$$\sum_{n=1}^{\infty} \sum_{\bar{Y}, X_n \cap \tilde{B}' \neq \emptyset} |\mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B)| F_n^T(\bar{Y}) \leq C_B' e^{-\sqrt{a} dist(B,B')}, \tag{3.32}$$

where a is the constant of the one-site harmonic potential.

Now, using the standard "doubling measure" trick by J. Ginibre [28] (or [30, Sec.18.3]) we obtain from Corollary 3.1 the exponential clustering of correlations:

Lemma 3.3 *With the same assumptions as in Theorem 2.1 and for any measurable functions $\mathcal{A}_{B_1}, \mathcal{A}_{B_2}$ w.r.t. σ-algebras $\Sigma_{\hat{\beta}, B_1}$ and $\Sigma_{\hat{\beta}, B_2}$ respectively, where $B_1, B_2 \subset \Lambda$ and $B_1 \cap B_2 = \emptyset$, there exists a sufficiently small value of m_*, such that for $0 < m < m_*$ and any temperature one has the estimate*

$$\langle \mathcal{A}_{B_1}(\cdot) \mathcal{A}_{B_2}(\cdot) \rangle_{\mu^\Lambda} - \langle \mathcal{A}_{B_1}(\cdot) \rangle_{\mu^\Lambda} \langle \mathcal{A}_{B_2}(\cdot) \rangle_{\mu^\Lambda} \leq$$

$$\leq C_1(\mathcal{A}_{B_1}) C_2(\mathcal{A}_{B_2}) e^{-\sqrt{a} \, dist(B_1, B_2)}. \qquad (3.33)$$

uniformly in Λ.

The proofs of Lemmas 3.1-3.2 and Corollary 3.1 are presented in the next Section. They imply the proof of our Theorem 2.1 for the case of *p.b.c.*

We also need the following *auxiliary measure* $\tilde{\mu}^\Lambda(\cdot \mid y_\Lambda)$ which depends on some fixed trajectories $y_\Lambda \in \Omega_{\hat{\beta}, \Lambda}$:

$$d\tilde{\mu}^\Lambda(x_\Lambda \mid y_\Lambda) := \tilde{Z}_{\hat{\beta}}(\Lambda \mid y_\Lambda)^{-1} e^{-\int_0^{\hat{\beta}} d\tau \tilde{V}(x_\Lambda(\tau) \mid y_\Lambda(\tau))} d\mu_0^\Lambda(x_\Lambda \mid 0), \qquad (3.34)$$

where (see (2.16)):

$$\tilde{V}(x_\Lambda(\tau) \mid y_\Lambda(\tau)) = b_m \sum_{j \in \Lambda} \left[e^{-\delta_m (x_j(\tau) + y_j(\tau))^2/4} + e^{-\delta_m (x_j(\tau) - y_j(\tau))^2/4} \right]. \quad (3.35)$$

It is clear that with the measure $\tilde{\mu}^\Lambda(\cdot \mid y_\Lambda)$ one can construct the *same* cluster expansion as in (3.20), but with *additional* dependence on configurations y_Λ:

$$\langle \mathcal{A}_B(\cdot) \rangle_{\tilde{\mu}^\Lambda(\cdot \mid y_\Lambda)} = \sum_{n=1}^{n_T} \sum_{\bar{Y} \subset T} \mathcal{K}_{\bar{Y}}^T(\mathcal{A}_B \mid y_\Lambda) F_n^T(\bar{Y} \mid y_\Lambda), \qquad (3.36)$$

In the next Section we prove the following lemma:

Lemma 3.4 *Let $m < m_*$ as in Lemmas 3.1-3.3. Then for the measure $\tilde{\mu}^\Lambda(\cdot \mid y_\Lambda)$, defined by (3.34)-(3.35), the estimates (3.30)-(3.33) are true with the same constants which do not depend on configurations y_Λ.*

Proof of Theorem 2.1.
Existence. First we prove the existence of measures with *p.b.c.* and *zero* boundary conditions As we mentioned in Remark 2.1 the covariance $G_{0;jk}^\Lambda(\tau - \tau')$ converges pointwise to the translation invariant function $G_{0;|j-k|}(\tau - \tau')$ and the corresponding measures $d\mu_0$ and $d\mu_{0;(s)_{(n-1)}}^{X_n}$ are defined on $\Sigma_{\hat{\beta}}$. So,

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using Lemmas 3.1 and 3.2 we can rewrite the expansion (3.20) in the *thermodynamic limit*:

$$\langle \mathcal{A}_B(\cdot) \rangle_\mu = \sum_{n \geq 1} \sum_{\bar{Y} \subset \mathbb{Z}^\nu \otimes S_{\hat{\beta}}} \mathcal{K}_{\bar{Y}}^{\hat{\beta}}(\mathcal{A}_B) F_n^{\hat{\beta}}(\bar{Y}), \qquad (3.37)$$

with the same definitions of I_n and $\Delta(\eta, \bar{y})$ (see (3.22), (3.23)), but with G_0 and $\mu_{0;(s)_{n-1}}^{X_n}$ instead of G_0^T and $\mu_{0;(s)_{n-1}}^{\Lambda, X_n}$.

Hence, there exists probability measures μ_B indexed by $B \subset \mathbb{Z}^\nu$ such that

$$\langle \mathcal{A}_B \rangle = \int \mathcal{A}_B(\phi) \, d\mu_B(\phi).$$

All these measures are consistent and consequently by the Kolmogorov Theorem one can construct a unique limit measure μ on the σ-algebra $\Sigma_{\hat{\beta}, \mathbb{Z}^\nu}$. The fact that limit measure is just tempered EGM follows from arguments based on superstability (see [11]) which are true for our model. $\qquad \square$

Remark 3.3 *In the case of general boundary conditions $\xi \in \tilde{\Omega}_{\hat{\beta}}^\rho$ with $\rho < \sqrt{a}$, the factors $\mathcal{K}_{\bar{Y}}^{\hat{\beta}}$, $F_n^{\hat{\beta}}$, and the constants c, C_B in the Lemmas 3.1-3.2 will depend on ξ. This means that from convergence of the cluster expansions we can obtain only existence of the limit measure, but not uniqueness.*

Uniqueness. To prove the uniqueness we use the following

Proposition 3.1 *(see Theorem 3.1 of [6])*
Suppose that for every $l_0 \in \mathbb{Z}^\nu$ and $\tau_0 \in [0, \hat{\beta}]$, for every increasing sequence of boxes $\Lambda_n \subset \mathbb{Z}^\nu$ ($\Lambda_n \to \mathbb{Z}^\nu$), such that $l_0 \in \Lambda_n$ for all n, and for any two configurations $\xi, \eta \in \tilde{\Omega}_{\hat{\beta}}^\rho$, $\rho < \sqrt{a}$, one has the convergence:

$$\langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \xi)} - \langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \eta)} \to 0 \qquad (3.38)$$

as $n \to \infty$. Then there exist a unique limit measure μ which does not depend on configurations ξ, η.

This statement is a quantum version of some known previous results, namely, the monotonicity arguments [41] for the measures on the configurations in the Classical Statistical Mechanics and a priori estimates for EGM due to [10,11]. These arguments are true for the wide class of anharmonic potentials with *ferromagnetic* type of interaction. We use this statement as follows. Consider

$$\langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \xi)} - \langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \eta)} =$$

$$= \int_0^1 ds \frac{d}{ds} \langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \eta + s\zeta)}, \quad \zeta = \xi - \eta. \tag{3.39}$$

After differentiation we obtain:

$$|\langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \xi)} - \langle \phi_{l_0}(\tau_0) \rangle_{\mu^{\Lambda_n}(\cdot | \eta)}| \leq$$

$$\sup_{s \in [0,1]} \frac{J}{2} \sum_{\langle l, l' \rangle}^{\partial \Lambda} \int_0^{\hat{\beta}} d\tau K_{l l_0}^{\Lambda}(\tau, \tau_0 \mid \eta + s\zeta) \zeta_{l'}(\tau), \tag{3.40}$$

where

$$K_{l l'}^{\Lambda}(\tau, \tau' \mid \eta) = \langle \phi_l(\tau) \phi_{l'}(\tau') \rangle_{\mu^{\Lambda}(\cdot | \eta)} - \langle \phi_l(\tau) \rangle_{\mu^{\Lambda}(\cdot | \eta)} \langle \phi_{l'}(\tau') \rangle_{\mu^{\Lambda}(\cdot | \eta)}. \tag{3.41}$$

It is easy to see that the convergence (3.38) follows from the

Lemma 3.5 *For $m < m_*$ there are constants $C_i, i = 1, 2$ independent of $\tau, \tau', \Lambda, \eta$, such that*

$$K_{l l'}^{\Lambda}(i) \leq C_i \, e^{-\sqrt{a}|l - l'|}, \tag{3.42}$$

where

$$K_{l l'}^{\Lambda}(1) = K_{l l'}^{\Lambda}(\tau, \tau' \mid \eta), \quad and \quad K_{l l'}^{\Lambda}(2) = \int_0^{\hat{\beta}} d\tau' K_{l l'}^{\Lambda}(\tau, \tau' \mid \eta). \tag{3.43}$$

Finally, applying (similar to [6]) the Cauchy-Schwarz inequality in (3.40) and using (3.42), we get (3.38). $\qquad \square$

$\hfill \square$

4 Convergence of the cluster expansions

To prove the convergence of the cluster expansions (3.20) for $\Lambda \nearrow \mathbb{Z}^\nu$ we should first give demonstrations of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. To simplify the proof we omit indexes $\alpha = 1, ..., d$ and we take into account that the number of derivatives for each order of n should increase as d^{n-1}. To write down the general term for any fixed *tree η* (see (3.23)) let us introduce (following [16,17]) the numbers $d_\eta(k), k = 1, ..., n$ which are the characteristics of tree η, i.e. $d_\eta(k)$ is a number of lines, coming into the vertex k from vertices m $(m > k)$. It is clear that $n_1 = d_\eta(1)$, $n_k = d_\eta(k) + 1$ for $k = 2, ..., n$, are exactly the number of lines coming

from vertices $k = 1, ..., n$ and these numbers coincides with the numbers of derivatives supported in $Y_1, Y_2, ..., Y_n$. Hence,

$$\sum_{k=1}^{n} d_\eta(k) = n - 1 \quad \text{and} \quad \sum_{k=1}^{n} n_k = 2(n - 1). \tag{4.1}$$

Let us rewrite the operator $\Delta(\eta, \bar{Y})$ in the following form

$$\Delta(\eta, \bar{Y}) = \Big(\prod_{k=1}^{n} \prod_{l=1}^{n_k} \int_{Y_k} dt_k^{(l)} \Big) \prod_{k=2}^{n} G_0^T(t_{\eta(k)}^{(\eta_k)} - t_k^{(n_k)}) \times$$

$$\prod_{k=1}^{n} \frac{\delta^{n_k}}{\delta\phi(t_k^{(1)}) \cdots \delta\phi(t_k^{(n_k)})}. \tag{4.2}$$

Here we put

$$\eta_2 := 1, \quad \eta_k := 1 + \sum_{l=2}^{k-1} \delta_{\eta(l), \eta(k)}, \quad k \geq 3,$$

where $\delta_{\eta(l), \eta(k)}$ is the symbol of Kronecker. In the following we also put $t_k^{(n_k)} := t_k$.

The main technical estimate is the following

Proposition 4.1 *Let D_{n_k} be parts of operator $\Delta(\eta, \bar{Y})$ and factor $e^{-\hat{V}(\phi_{X_n})}$ corresponding to the k-th vertex on the tree η:*

$$D_{n_k} = D_{n_k}(t_{\eta(k)}^{(\eta_k)}; t_1, ..., t_{d_\eta(k)}) =$$

$$= \int_{(Y_k)^{n_k}} G_0^T(t_{\eta(k)}^{(\eta_k)} - t_k^{(n_k)}) G_0^T(t_k^{(1)} - t_1) \cdots G_0^T(t_k^{(d_\eta(k))} - t_{d_\eta(k)}) \times$$

$$\times \frac{\delta^{n_k}}{\delta\phi(t_k^{(1)}) \cdots \delta\phi(t_k^{(n_k)})} e^{-\hat{V}(\phi_{Y_k})}. \tag{4.3}$$

Then for $b_m < 1$ the following estimate is true:

$$D_{n_k}(t_{\eta(k)}^{(\eta_k)}; t_1, ..., t_{d_\eta(k)}) \leq 2^{n_k} b_m \delta_m^{n_k/2} n_k! \times$$

$$\times \sup_{(*)_{d_\eta(k)}} \int_{Y_k} dt_k G_0^T(t_{\eta(k)}^{(\eta_k)} - t_k^{(n_k)}) G_0^{*,T}(\cdot - t_1) \cdots G_0^{*,T}(\cdot - t_{d_\eta(k)}) e^{-\delta_m \phi(t_k)^2/4}, \tag{4.4}$$

here supremum is taken over all terms in the product, where one can choose $G_0^{*,T}(\cdot - t_i)$ equal either to $G_0^T(t_k - t_i)$, or $\int_{Y_k} dt' G_0^T(t' - t_i)$.

Proof. For simplicity we demonstrate this proposition for the case when instead of functional derivatives we have the ordinary derivatives, since exactly the same line of reasoning gives the proof for functional derivatives.

Let $X = -b_m e^{-\delta_m x^2/2}$ denote the *scalar prototype* of our anharmonic potential in (4.3), supported in Y_k. Then it easy to estimate by induction the value of n-th derivatives of X:

$$|X^{(n)}| \leq b_m 2^n \delta_m^{n/2} \sqrt{n!} e^{-\delta_m x^2/4}. \tag{4.5}$$

This inequality can be obtained by induction using the recursion relation:

$$I_n(x) = -\delta_m x I_{n-1}(x) - (n-1)\delta_m I_{n-2}(x), \quad I_n(x) = e^{\delta_m x^2/4} X^{(n)}(x), \quad (4.6)$$

and the inequality:

$$|x|^k e^{-\delta_m x^2/4} \leq 2^{k/2} \delta_m^{-k/2} \sqrt{k!}, \ .$$

Now using the Leibnitz formula for the product $X^{(1)} e^X$, where the first derivative $X^{(1)} = b_m \delta_m x e^{-\delta_m x^2/2}$, and the estimate (4.5), again by induction one obtains:

$$|\frac{d^n}{dx^n} e^X| \leq 2^n b_m \delta_m^{n/2} n! e^{-\delta_m x^2/4}. \tag{4.7}$$

We also used here the fact that $e^X \leq 1$.

Remark 4.1 *The inequality (4.7) explains the provenance of the coefficient in (4.4), since by the same way this coefficient appears when one passes to functional derivatives (as in (4.3)), instead of the ordinary ones.*

Remark 4.2 *Notice that we also should consider the terms in which some derivatives act on the factor $\mathcal{A}_B$. But, since $\mathcal{A}_B$ is supported only in $Y_1 = X_1$, then even in the case, when for a given tree η one has all $\eta(k) = 1$, for $k = 2, ..., n$, and $(n-1)$ derivatives act only on $\mathcal{A}_B$, the other $(n-1)$ derivatives in $\Delta(\eta, \bar{Y})$ act on the factor $\exp[-\check{V}_a(\phi_{X_n})]$. Hence, we obtain the powers of the small parameter b_m, that we need.*

This finishes the proof of Proposition. $\qquad\qquad\qquad\qquad\qquad$ $\square$

As a result we can write for I_n^T the following estimate:

$$I_n^T(\bar{Y}, \eta, \mathcal{A}_B; (s)_{n-1}) \leq C(\mathcal{A}_B)(8 d b_m \delta_m)^{n-1} \prod_{k=1}^{n-1} d_\eta(k)! \times \tag{4.8}$$

$$\times \prod_{k=2}^{n} \left(\sup_{t_k'} \int_{Y_k} dt_k G_0^T(t_k', t_k) \right) \int d\mu_{0;(s)_{n-1}}^{\Lambda, X_n}(\phi_{X_n}) \prod_{k=1}^{n} e^{-\delta_m \phi(t_k)^2/4}, \quad t_k \in Y_k.$$

The next step is to estimate the Gaussian integral in (4.8).

Proposition 4.2 *Let*

$$\tilde{I}_n = \int d\mu_{0,(s)_{n-1}}^{\Lambda,X_n}(\phi_{X_n}) \prod_{k=1}^{n} e^{-\delta_m \phi(t_k)^2/4}, \quad t_k \in Y_k, \quad n \geq 2, \qquad (4.9)$$

then there exists a constant c, which depends on parameters d, J, a such that

$$\tilde{I}_n \leq e^{c|X_n|} \quad \left(\frac{2\sqrt{a}}{2\sqrt{a} + \delta_m} \right)^{d|X_n|/4}. \qquad (4.10)$$

Proof. Write the product in (4.9) as new a Gaussian integral w.r.t. the measure $d\gamma(z) = 1/(2\pi)^{dn/2} \exp[-z^2/2]$, $z \in I\!\!R^{dn}$:

$$\tilde{I}_n = \int d\mu_{0;(s)_{n-1}}^{\Lambda,X_n}(\phi_{X_n}) \int_{I\!\!R^{dn}} d\gamma(z) e^{i\sqrt{\delta_m/2} \sum_{k=1}^{n} z_k \cdot \phi(t_k)}. \qquad (4.11)$$

Interchanging integrals one obtains

$$\tilde{I}_n = \int_{I\!\!R^{dn}} d\gamma(z) e^{-\delta_m/4 \sum_{k,k'=1}^{n} z_k z_{k'} G_0^{\Lambda,X_n}(t_k, t_{k'};(s)_{n-1},0)}, \qquad (4.12)$$

where G_0^{Λ,X_n} is defined by (3.12). By construction (3.9)-(3.10) the function $G_0^{\Lambda,X_n}(t_k, t_{k'};(s)_{n-1},0)$ is a convex combination of "diagonalized" covariances (see [21], or [30, Sec.18.2]):

$$G_0^{\Lambda,X_n}(t_k, t_{k'};(s)_{n-1},0) = \sum_i \lambda_i G_0^{(i)}, \quad \sum_i \lambda_i = 1, \qquad (4.13)$$

where coefficients λ_i are monomials of the form $s_{k_1} \cdots s_{k_p}(1-s_{l_1}) \cdots (1-s_{l_{p'}})$ and

$$G_0^{(i)} = \sum_k \mathbb{1}_{Z_k(i)} G_0^T \mathbb{1}_{Z_k(i)}, \quad \bigcup_k Z_k(i) = X_n \quad \text{for all} \quad i. \qquad (4.14)$$

Substituting (4.13) into (4.12), applying then the Hölder inequality, and returning back to the measure $d\mu_{G_0^{(i)}}^{\Lambda,X_n}$, defined by the covariance (4.14), one gets:

$$\tilde{I}_n \leq \prod_i \left(\int_{I\!\!R^{dn}} d\gamma(z) e^{-\delta_m/4 \sum_{k,k'=1}^{n} z_k z_{k'} G_0^{\Lambda,X_n}(t_k, t_{k'})} \right)^{\lambda_i} =$$

$$\prod_i \left(\int d\mu_{G_0^{(i)}}^{\Lambda,X_n}(\phi_{X_n}) \prod_{k=1}^{n} e^{-\delta_m \phi(t_k)^2/4} \right)^{\lambda_i} =: \prod_i K_i^{\lambda_i}. \qquad (4.15)$$

By virtue of (4.14) we have:

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$$K_i = \prod_k \Big(\int d\mu_{G_0^{(i)}}^{\Lambda, Z_k(i)}(\phi_{X_n}) \prod_{Y \in Z_k(i)} e^{-\delta_m \phi(t_Y)^2/4} \Big) =: \prod_k J_k(Z_k(i)). \quad (4.16)$$

By definition (2.23) one gets for $\Lambda = Z_k(i)$ and $F(\phi_{Z_k(i)}) = \prod_{Y \in Z_k(i)} e^{-\delta_m \phi(t_Y)^2/4}$:

$$J_k(Z_k(i)) = C_0(Z_k(i); J)^{-1} \int_{\Omega_{\hat{\beta},\Lambda}} d\omega_{Z_k(i)} e^{-\frac{J}{4} \int_{Z_k(i)}^* dt(\omega(t)-\omega(t_*))^2} \prod_{Y \in Z_k(i)} e^{-\delta_m \omega_Y(\tau_Y)^2/4}.$$
$$(4.17)$$

Here the measure $d\omega_{Z_k(i)}$ is defined by l.h.s. of (2.26), with $J = 0$ in definition of operator B_Λ (2.21) and with $\Lambda_k(i) := Z_k(i) \cap \Lambda$, $S_k(i) := Z_k(i) \cap S_{\hat{\beta}}$:

$$\int_{\Omega_{\hat{\beta},\Lambda}} d\omega_{Z_k(i)}(...) = \int_{\mathbb{R}^{d|\Lambda|}} dx_\Lambda \int_{\Omega_{\hat{\beta},\Lambda}} W_{x_\Lambda; x_\Lambda}^{\hat{\beta}}(d\omega_\Lambda) e^{-\frac{a}{2} \int_0^{\hat{\beta}} d\tau(\omega_\Lambda(\tau), \omega_\Lambda(\tau)) + \frac{d}{2} Tr B_\Lambda}(...).$$

We put also :

$$\int_{Z_k(i)}^* dt(\omega(t) - \omega(t_*))^2 := \sum_{\langle l, l' \rangle \subset \Lambda_k(i)} \int_{S_k(i)} d\tau(\omega_l(\tau) - \omega_{l'}(\tau))^2. \quad (4.18)$$

The normalization factor $C_0 := C_0(Z_k(i); J)$ is the integral (4.17) calculated for $\delta_m = 0$. By the Cauchy-Schwarz inequality we get from integral (4.17) that

$$J_k(Z_k(i)) \le \frac{C_0(Z_k(i); 2J)^{1/2}}{C_0(Z_k(i); J)} \Big(\prod_{Y \in Z_k(i)} \int_{\Omega_{\hat{\beta}}} d\omega_Y e^{-\delta_m \omega_Y(\tau_Y)^2/2} \Big)^{1/2}. \quad (4.19)$$

The factors C_0 for J and $2J$ can be calculated explicitly using (2.27), (2.21). The integral in the product (4.19) can be estimated (for $|Y| = 1$) by

$$\int_{\Omega_{\hat{\beta}}} d\omega_Y e^{-\delta_m \omega(t_Y)^2/2} = C_0(Y, 0) \int d\mu_{G_0^\Lambda}^Y(\phi_Y) e^{-\delta_m \phi_{l_Y/2}(\tau_Y)^2} \Big|_{J=0} =$$

$$C_0(Y, 0) \Big(1 + \frac{\delta_m}{2\sqrt{a}} \cdot \frac{e^{\hat{\beta}\sqrt{a}} + 1}{e^{\hat{\beta}\sqrt{a}} - 1} \Big)^{-d/2} \le e^{\text{const}|Y|} \Big(\frac{2\sqrt{a}}{2\sqrt{a} + \delta_m} \Big)^{d/2}. \quad (4.20)$$

This finishes the proof of the Proposition. $\qquad \square$

Now we return to the proof of Lemma 3.1. Since

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$$\sum_Y \int_Y dt G_0^T(t' - t) = \int_T dt G_0^T(t' - t) = \sum_{j \in \Lambda} \int_0^{\hat{\beta}} d\tau G_{0,ij}^\Lambda(\tau' - \tau) =$$

$$=: \sum_{j \in \Lambda} D_0^\Lambda(i - j) \leq \sum_{j \in \mathbb{Z}^\nu} D_{0,}(j) =: C_G, \qquad (4.21)$$

then taking into account dependence of b_m and δ_m on m (see (2.16)) and definition (3.21), we obtain by summing over *trees* η the estimate:

$$|\mathcal{K}_{\bar{y}}^T(\mathcal{A}_B)| \leq C'(\mathcal{A}_B)\big(16 b \sqrt{a} C_G m^{d/8}\big)^{n-1} \times$$

$$\times \sum_\eta \prod_{p=1}^n d_\eta(p)! \int_0^1 (ds)^{n-1} f(\eta; (s)_{n-1}).$$

The last step of the proof is due to the Battle–Federbush inequality [16,17]:

$$\sum_\eta \prod_{p=1}^n d_\eta(p)! \int_0^1 (ds)^{n-1} f(\eta; (s)_{n-2}) \leq 4^n, \qquad n = |\eta|.$$

This yields (3.28) with

$$\varepsilon(m) = 64 b \sqrt{a} C_G m^{d/8}.$$

Now taking into account (3.30), we obtain the value of the small-mass threshold m_*:

$$m_* = (64 b \sqrt{a} C_G e^c)^{-8/d}. \qquad (4.22)$$

□

Proof of Lemma 3.2. The proof is standard. One can use general theory of polymer-type expansions [42] (or, equivalently, the *Kirkwood–Salsburg* type technique [30, Sec.18.5]), see, also, [49]. □

Proof of Corollary 3.1. If we restrict the sum in the cluster expansion (3.20)–(3.27) in such a way that, for example, $Y_2 \subset B'$, then all terms in this expansion are proportional to $G_0^T(t_1, t_2)$, where $t_1 \in B$, $t_2 \in B'$, then (3.32) follows from the asymptotic behaviour of $G_{0,|i-j|}(\tau)$ in the variable $|i - j|$. Note, also, that in the case of *p.b.c.* (see e.g.[30])

$$G_{0;|i-j|}(\tau \mid p) \leq G_{0;|i-j|}(\tau \mid 0),$$

and it has exponential decay of $G_{0;|i-j|}(\tau \mid 0)$ in $|i - j|$, which is uniform in τ. □

Proof of Lemma 3.4. To prove this lemma one has to check that Propositions 4.1 and 4.2 are true for the *auxiliary potential* (3.38). It is easy to verify that (4.5)-(4.7) are true for $X_\pm = X_\pm(x \mid y) = -b_m e^{-\delta_m(x\pm y)^2/4}$ with $\delta_m/2$ instead of δ_m. So, the estimate (4.4) is the same but with $[e^{-\delta_m(x(t_k)+y(t_k))^2/8} + e^{-\delta_m(x(t_k)-y(t_k))^2/8}]$ instead of $e^{-\delta_m\phi(t_k)^2/4}$. Then in the proof of Proposition 4.2 we have to modify the integral (4.20) to

$$\int d\mu_{G_0^\Lambda}^Y(x_Y) e^{-\frac{\delta_m}{8}(x_{l_Y}(\tau_Y)\pm y_Y(\tau_Y))^2}\Big|_{J=0}, \qquad (4.23)$$

which gives

$$C_0(Y,0)\Big(1 + \frac{\delta_m}{4}G_0^\Lambda(0)\Big)^{-d/2} e^{-\frac{\delta_m y^2}{8(1+\delta_m G_0^\Lambda(0)/4)}}\Big|_{J=0}.$$

Since the last exponent is less than one, the dependence on the configuration y disappears, and this finishes the proof. $\qquad\square$

Proof of Lemma 3.5. We follow the line of reasoning in the proof of Theorem 3.2 [6]. Let $d\tilde{\mu}_0^\Lambda(\phi,\phi' \mid 0) := d\mu_0^\Lambda(\phi \mid 0) \otimes d\mu_0^\Lambda(\phi' \mid 0)$. Then we can rewrite (3.46) as

$$K_{ll'}^\Lambda(\tau,\tau' \mid \eta) = Z(\Lambda \mid \eta)^{-2}\int\int \frac{\phi_l(\tau) - \phi_l'(\tau)}{\sqrt{2}} \cdot \frac{\phi_{l'}(\tau') - \phi_{l'}'(\tau')}{\sqrt{2}}\times$$

$$\times e^{-\hat{V}(\phi_\Lambda)-\hat{V}(\phi_\Lambda')}e^{\frac{J}{2}\sum_{\langle l,l'\rangle}^{\partial\Lambda}\int_0^{\hat{\beta}}d\tau(\phi_l(\tau)-\phi_l'(\tau))\eta_{l'}(\tau)}d\tilde{\mu}_0^\Lambda(\phi,\phi' \mid 0). \qquad (4.24)$$

After change of variables:

$$x_l(\tau) = \frac{\phi_l(\tau) - \phi_l'(\tau)}{\sqrt{2}}, \quad y_l(\tau) = \frac{\phi_l(\tau) + \phi_l'(\tau)}{\sqrt{2}},$$

by definition of the measures it is clear that

$$d\tilde{\mu}_0^\Lambda(\phi,\phi' \mid 0) = d\tilde{\mu}_0^\Lambda(x,y \mid 0). \qquad (4.25)$$

Then, using definition (3.44) we get:

$$K_{ll'}^\Lambda(\tau,\tau' \mid \eta) = Z(\Lambda \mid \eta)^{-2}\int\int x_l(\tau)x_{l'}(\tau')e^{-\tilde{V}(x_\Lambda|y_\Lambda)}\times \qquad (4.26)$$

$$\times e^{\frac{J}{2}\sum_{\langle l,l'\rangle}^{\partial\Lambda}\int_0^{\hat{\beta}}d\tau y_l(\tau)\eta_{l'}(\tau)}d\tilde{\mu}_0^\Lambda(x,x' \mid 0).$$

Define the measure

$$\tilde{\mu}^\Lambda(dx_\Lambda \mid y_\Lambda) = \tilde{Z}(y_\Lambda)^{-1}e^{-\tilde{V}(x_\Lambda|y_\Lambda)}d\mu_0^\Lambda(x_\Lambda \mid 0). \qquad (4.27)$$

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Then

$$K_{ll'}^\Lambda(\tau, \tau' \mid \eta) = Z(\Lambda \mid \eta)^{-2} \int d\mu_0^\Lambda(y_\Lambda \mid 0) \tilde{Z}(y_\Lambda) e^{\frac{J}{2} \sum_{\langle l, l' \rangle}^{\partial \Lambda} \int_0^{\hat{\beta}} d\tau y_l(\tau) \eta_{l'}(\tau)} \times$$

$$\times \langle x_l(\tau) x_{l'}(\tau') \rangle_{\tilde{\mu}^\Lambda(\cdot \mid y_\Lambda)}. \tag{4.28}$$

Now the proof follows from Lemma 3.4, or more explicitly from (3.33) with $\mathcal{A}_{B_1} = x_l$ and $\mathcal{A}_{B_2} = x_{l'}$. $\qquad\square$

5 External field analyticity of the Gibbs state

In the previous sections we have proved uniqueness of the equilibrium state for our model (in the DLR-sense) in the set of tempered EGM, see (3.34). In this section we consider the model (2.1), (2.2) in the presence of external field h:

$$H^\Lambda(h) \;=\; H_0^\Lambda + V^\Lambda + \sum_{j \in \Lambda} h \cdot q_j, \quad h \in I\!\!R^d. \tag{5.1}$$

Here H_0^Λ and V^Λ are defined by (2.1)–(2.3). After rescaling (2.12) one gets:

$$\hat{H}^\Lambda(h) \;=\; \hat{H}_0^\Lambda + \hat{V}^\Lambda + \sum_{j \in \Lambda} \hat{h} \cdot x_j, \quad \hat{h} = m^{-1/4} h. \tag{5.2}$$

Then the following statement is true:

Theorem 5.1 *For the system of quantum particles with Hamiltonian (5.1) there is m_*, such that for any $0 < m < m_*$, $T \geq 0$ and $|h^{(\alpha)}| < h_0(m_*), \alpha = 1, ..., d$, the limit*

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \rho_\Lambda^{\beta, h}(A) \;=\; \rho^{\beta, h}(A), \;\; A \in \mathfrak{A}_0,$$

exists. The limit $\rho^{\beta, h}(A)$ gives a unique state on the algebra $\mathfrak{A}_0$, which can be continuously extended to the algebra $\mathfrak{A}$. Moreover, the functions $\rho^{\beta, h}(A), A \in \mathfrak{A}$ are analytic in the domain $\{h^{(\alpha)} \in \mathbb{C}^d \mid |h^{(\alpha)}| < h_0(m_*)\}$.

Proof. The proof is close to that of Theorem 2.1. The only remark one has to do is that in expansions (3.20)-(3.27) the factor $\exp[-\hat{V}(\phi_{X_n})] \exp[\hat{h} \int_{X_n} \phi(t) dt]$ appears instead $\exp[-\hat{V}(\phi_{X_n})]$ and the operator $\Delta(\eta, \bar{Y})$ in (3.22) acts on the product $\mathcal{A}_B(\phi_B) \exp[-\hat{V}(\phi_{X_n})] \exp[h_m \int_{X_n} \phi(t) dt]$. When the derivatives from $\Delta(\eta, \bar{Y})$ act on the factor $\exp[\hat{h} \int_{X_n} \phi(t) dt]$, the convergence is guaranteed by the small factors $h = m^{1/4} \hat{h}$ from $\{h \in \mathbb{C}^d \mid |h^{(\alpha)}| < h_0(m_*), \alpha = 1, ..., d\}$. In principle this implies a new value of the *light-mass* threshold m_*^h:

$$m_*^h = \min\{m_*, m_*(hC_Ge^{c+1})^{-4}\}, \tag{5.3}$$

where m_* is defined in (4.22), C_G is from the estimate of the product of covariances in $\Delta(\eta, \bar{Y})$, e^c is from the estimate of $F_n^T(\bar{Y})$, and additional e is from the inequality (cf. [21]):

$$\sum_\eta \int_0^1 (ds)^{n-1} f(\eta; (s)_{n-2}) \le e^n, \qquad n = |\eta|.$$

But by virtue of (5.3) there is $h_0(m_*)$ such that $m_*^h = m_*$ for $|h^{(\alpha)}| < h_0(m_*)$, which finishes the proof. $\qquad\square$

As a consequence of the above theorem we obtain the following result (cf. [43]):

Theorem 5.2 *Let Λ_n be the sequence $\Lambda_1 \subset \Lambda_2 \subset ... \subset \Lambda_n...$, $\cup_{n=1}^\infty \Lambda_n = \mathbb{Z}^\nu$, and let $h = e\|h\|$, where e is a unit vector in $\mathbb{R}^d$. By*

$$\sigma(\beta) := \lim_{h\to 0} \lim_{n\to\infty} \rho_\Lambda^{\beta,h}\Big(\frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} q_j \cdot e\Big) \tag{5.4}$$

we define the order parameter for the model (5.1). Then there exists mass $m_* > 0$ such that for any $0 < m < m_*$ and for all temperatures, including $\beta = \infty$, the order parameter is trivial for any tempered boundary condition defined in (2.37), i.e.:

$$\sigma(\beta) = 0. \tag{5.5}$$

Proof. The proof follows directly from the Theorem 5.1 and cluster expansion for the state

$$\hat{\rho}_\Lambda^{\hat{\beta},\hat{h}}\Big(\frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} x_j \cdot e\Big). \tag{5.6}$$

Indeed, if derivatives from $\quad\Delta(\eta, \bar{Y})$ act on the factor

$$\exp\left\{\int_{X_n} \hat{h} \cdot \phi(t)dt\right\},$$

then after thermodynamic limit the corresponding terms are proportional to $h = m^{-1/4}\hat{h}$. If derivatives act on the factor $\phi_l \exp[-\hat{V}(\phi_{X_n})]$, then in the limit we obtain a Gaussian integration with zero mean and with odd numbers of fields. So, taking into account that $\rho^{\beta,h}$ is analytic in h (Theorem 5.1), in the limit $h = 0$ we get zero value (5.5) for the order parameter. $\qquad\square$

6 Convergence of cluster expansions. Uniqueness of the Gibbs state for high temperatures

The problem of the existence and uniqueness of the quantum state (EGM) at high temperatures for the model (2.1)-(2.3) can be solved in the framework of a general approach, developed e.g. in [13,14]. But there exist an independent question of construction of high temperature expansions for quantum state (2.5) and of the proof of its convergence in thermodynamic limit. This question is related to a specificity of the one-site potential (2.3) behaviour at infinity. For the quantum systems with unbounded spins the corresponding expansions were considered in [35,36,45,46,53]. All these results were obtained for the models with the one-site polynomial anharmonic potentials of the form (1.3). Then the convergence is insured by the small parameter $\beta^{(s-1)/2s}$. Since in (2.1)-(2.3) we have $s = 1$, there is no such parameter in our case. We recall that $2s$ is degree of one-site anharmonic polynomial interaction in (1.3).

In the present section we show that approach we developed in Sections 3–5 (the choice of reference measure and the type of cluster expansion) can be adapted to obtain convergent cluster expansions in powers of $\beta > 0$, and to prove analyticity of the state in domain $|\beta - \beta_*/2| < \beta_*/2$ of the complex values of β around the point $\beta_*/2 > 0$.

The construction of cluster expansions is absolutely the same as above, but instead of partition of the interval $[0, \beta] = S_\beta$ into unit intervals we consider it as *one unit* with variable τ running in S_β. In another words, instead of "rods" $\Delta_{j,\hat{\tau}}$ (see (3.5)) we consider for every site $j \in \Lambda$ we consider "rods" $\Delta_{j,\beta} := (j, S_\beta)$. Then the measure $d\mu_{0;(s)_n}^\Lambda$ is constructed in the same way as in Section 3 with help of the covariance $G_{0;j,k}^\Lambda(\tau, \tau'; (s)_n)$, see (3.12), where $X_n = \emptyset$ and $T = \Lambda$. Then following the line of reasoning of Section 3 we obtain:

$$\langle \mathcal{A}_B(\cdot) \rangle_{\mu^\Lambda} = \sum_{n=1}^{n_\Lambda} \sum_{\bar{Y} \subset \Lambda} \mathcal{K}_{\bar{Y}}^\Lambda(\mathcal{A}_B) F_n^\Lambda(\bar{Y}), \tag{6.1}$$

with the same definitions as in (3.24)-(3.30), and with only difference that

$$\Delta_{p,p'}^{(\alpha)} = \int_0^\beta d\tau \int_0^\beta d\tau' G_{0;Y_p,Y_{p'}}^\Lambda(\tau, \tau') \frac{\delta^2}{\delta\phi_{Y_p}^{(\alpha)}(\tau)\delta\phi_{Y_{p'}}^{(\alpha)}(\tau')}, \tag{6.2}$$

and Y_p, $Y_{p'}$ are the sites of the lattice $\mathbb{Z}^\nu \cap \Lambda$. The main result then is the following

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Theorem 6.1 *For the quantum system with Hamiltonian (2.1)–(2.3) there exists a sufficiently small (independent of mass m) value of inverse temperature β_* such that for all $0 < \beta < \beta_*$ the weak limit of the measures*

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \mu^\Lambda = \mu,$$

exists and μ is a unique EGM on $(\Omega_{\beta,\mathbb{Z}^\nu}, \Sigma_{\beta,\mathbb{Z}^\nu})$ in the set of tempered measures. Here μ^Λ as in Theorem 2.1.

Proof. It is very close to the proof of Theorem 2.2. The only difference is that in the estimate (4.20) one should take

$$\left(1 + \frac{\delta_m}{2\sqrt{a}} \cdot \frac{e^{\hat{\beta}\sqrt{a}} + 1}{e^{\hat{\beta}\sqrt{a}} - 1}\right)^{-d/2} \leq \left(\frac{2\hat{\beta}\sqrt{a}}{2\hat{\beta}\sqrt{a} + \delta_m}\right)^{d/2} \leq \left(\frac{2\beta\sqrt{a}}{\delta}\right)^{d/2}. \qquad (6.3)$$

This gives for the small parameter of cluster expansion the value

$$\varepsilon = 64 b \sqrt{a} C_G \beta^{d/2}. \qquad (6.4)$$

Therefore,

$$\beta_* = (64 b \sqrt{a} C_G e^c)^{-2/d}, \qquad (6.5)$$

□

7   Conclusions

In the present paper we construct the Gibbs states for quantum crystal with nonpolynomial anharmonic potential (1.5) and with nearest neighbour harmonic interaction in the *light-mass regime*. These states are constructed on the algebra of quasi-local observables described in Section 2. Our analysis is based on a reformulation of the quantum Gibbs states problem into the study of Euclidean Gibbs Measures (EGM) [2,7,31,33].

The main technical tool we use to get the main result is the *cluster expansions* for EGM. They allow us (Theorem 2.1) to prove the existence and *uniqueness* of EGM in the *light-mass* domain, for all temperatures T, including $T = 0$ ($\beta = \infty$). But for $T = 0$ we can not state that the limit measure satisfies DLR equation.

To prove the existence of a unique translation-invariant measure we start with *periodic* boundary conditions (Section 2). Then we prove convergence of the corresponding cluster expansions and the cluster property of the EGM in the *light-mass* domain for all temperatures T. This gives the *uniqueness*

in the class of translation-invariant measures for small mass $m < m_*$ and all $T \geq 0$. Next we establish that convergence of the EGM cluster expansions and their cluster properties hold also for *zero* boundary conditions. This again implies uniqueness in this case. Finally, following the arguments of [6,11] we prove the uniqueness of EGM on the set of *tempered* configurations (Section 3).

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